HYPERBOLIC ENDS AND CONTINUA

James T. Rogers, Jr.

1. Introduction. Let F be a closed orientable surface of genus greater than one. The Nielson–Thurston theorem states that every homeomorphism of F is isotopic to a homeomorphism of F that (1) has finite order, or (2) is reducible, or (3) is pseudo-Anosov. The last case is the most common and the most interesting.

The behavior of the isotopy class of a pseudo-Anosov homeomorphism is captured in a unique pair of projective classes of measured laminations preserved by the homeomorphism. The underlying geodesic laminations are indecomposable continua with only points and arcs as subcontinua.

Such a geodesic lamination G is best understood by considering its preimage G in the universal covering space of F, hyperbolic 2-space H. In the Poincaré disc model of H, each leaf of G lifts to a (complete) geodesic in H, an arc of a circle in the Euclidean plane.

How can one see that G is not homogeneous? An interesting answer is to show that if G has the micro-transitivity property of Effros, then so does $\tilde{G}$. This is a contradiction, since close to each point x of $\tilde{G}$ is a point y of $\tilde{G}$ such that the geodesic G_x of $\tilde{G}$ containing x and the geodesic G_y of $\tilde{G}$ containing y are ultraparallel, so no bounded homeomorphism can move G_x onto G_y.

This suggests that if X is a homogeneous curve in F and x is a point of its preimage $\tilde{X}$ in H, then it is possible to assign to x a set of points in the circle at ∞ in such a way that this set is a local invariant of $\tilde{X}$ as well as an invariant of the component of x in $\tilde{X}$. How could one do this for an arbitrary homogeneous curve?

If X is a curve with nontrivial shape and Q is the Hilbert cube, then X has an essential embedding into $F\times Q$. Let $p\times 1: H\times Q\to F\times Q$ be the universal cover of $F\times Q$, and let $\tilde{X}$ be the preimage of X. If K is a component of $\tilde{X}$, it will be shown that one can associate with K a certain subset $E(K)$ of the circle at ∞; this will be called the set of ends of K.

THEOREM. *If X is a homogeneous curve, then the set $E(K)$ of ends of K is a local invariant of $\tilde{X}$.*

Given any natural number n, there exists a curve X in $F\times Q$ and a component K of $\tilde{X}$ such that $E(K)$ is an n-point set. The same holds for various infinite subsets of the circle at ∞. For homogeneous curves, however, the topological type of $E(K)$ is quite restricted.

THEOREM. *If X is a homogeneous curve, then $E(K)$ is either a two-point set or a Cantor set.*

Received January 30, 1986.
This research was partially supported by the National Science Foundation.
Michigan Math. J. 34 (1987).

In the proof a certain deck transformation $\phi \times 1$ is defined, and the two-point set is the pair of fixed points of the hyperbolic isometry ϕ. Other hyperbolic isometries and the homogeneity of X are used to show that if $E(K)$ contains more than these two points, then it is a Cantor set.

This invariant set can be used to eliminate a number of possible candidates for homogeneous curves. In particular, it can be used to show that, for homogeneous curves, trivial cohomology implies trivial shape. This follows from the next theorem.

THEOREM. *If X is an acyclic curve (of nontrivial shape), then $E(K)$ does not contain a fixed point of a hyperbolic isometry ϕ such that $\phi \times 1$ is a deck transformation.*

COROLLARY. *A homogeneous, acyclic curve is tree-like.*

The author has proposed a four-part program (Questions 2, 6, 7, and 8 of [6]) to classify homogeneous curves. The above corollary gives an affirmative answer to Question 7.

The author would like to thank Andrew Casson for giving a beautiful lecture at the 1983 USL Mathematics Conference and attracting the author's attention to stable laminations.

2. Homogeneous continua. A *continuum* is a compact, connected, nonvoid metric space. A space is *homogeneous* if its homeomorphism group acts transitively on it.

A metric space X has the *Effros property* if given $\epsilon > 0$, there exists $\delta > 0$ such that whenever y and z are points of X satisfying $d(y, z) < \delta$, there exists a homeomorphism $h: X \to X$ such that $h(y) = z$ and $d(x, h(x)) < \epsilon$, for all x in X. Such a δ is called an Effros δ for ϵ. Effros [2] has shown that each homogeneous continuum has the Effros property.

The standard Cantor set C is a homogeneous subset of the real line. Hence the fact that some of its points are accessible from its complement in $\mathbf{R}$ and some not is a property of the embedding and not a topological property of the Cantor set. Similarly, the fact that some of the arc components of a subcontinuum X of a surface are accessible from the complement and some not does not imply the nonhomogeneity of X.

An interesting example of a homogeneous continuum is the dyadic solenoid S. It is defined as the limit of the inverse sequence (S^1, f) of unit circles S^1 in the complex plane and bonding maps $f_n^{n+1}: S^1 \to S^1$ defined by $f_n^{n+1}(z) = z^2$. Since S^1 is a topological group and f_n^{n+1} a homomorphism of topological groups, it follows that S is a topological group and hence homogeneous.

We may analyze S by constructing a covering group $\tilde{S}$. The space $\tilde{S}$ is the limit of an inverse sequence $(\tilde{X}, \tilde{f})$, where $\tilde{X}_n = \mathbf{R} \times \{2^n\text{th roots of unity}\}$, $q_n: \tilde{X}_n \to S^1$ is the natural covering map, and $\tilde{f}_n^{n+1}$ is the obvious lift of $f_n^{n+1} \circ q_n$.

Formally, we have the following commutative diagram:

$$\begin{array}{ccccc} \tilde{X}_1 & \longleftarrow & \tilde{X}_2 & \longleftarrow & \cdots : \tilde{S} \cong \mathbf{R} \times C \\ \downarrow q_1 & & \downarrow q_2 & & \downarrow q_\infty \\ S^1 & \longleftarrow & S^1 & \longleftarrow & \cdots : S, \end{array}$$

where

$$q_n(r, t) = t \exp(2\pi i r) \quad \text{and} \quad \tilde{f}_n^{n+1}(r, t) = (2r, t^2).$$

Let $q_\infty : \tilde{S} \to S$ be the covering map induced by $\{q_n\}$.

By considering this covering space $\tilde{S}$, we see that (1) the dyadic solenoid S is locally homeomorphic to $\mathbf{R} \times C$, and (2) S is the union of uncountably many arc components or "leaves," each dense in S.

A continuum is indecomposable if it is not the union of two of its proper subcontinua. The following theorem gives a useful condition to determine if a continuum is indecomposable; in particular, it follows that the dyadic solenoid and the perfect laminations described in the next section are indecomposable continua.

THEOREM 2.1. *If each proper, nondegenerate subcontinuum of the continuum X is an arc, then X is an arc, a circle, or an indecomposable continuum.*

3. Geodesic laminations and indecomposable continua. Let H be the interior of the closed unit disk D in the Euclidean plane, and let S^1 be its boundary.

A geodesic in H is the intersection of H and a circle C in the Euclidean plane that intersects S^1 orthogonally (straight lines through the origin are considered circles centered at ∞).

A *reflection* in the geodesic $C \cap H$ is Euclidean inversion in the circle C. An *isometry* of H is a product of reflections in geodesics.

The set H, with this set of isometries, is a surface of constant negative curvature isometric to hyperbolic space. This model of hyperbolic space is called the *Poincaré disc.* The boundary S^1 of H, which is not in H, is called the *circle at* ∞.

Let F be a closed, orientable surface of genus two. The universal cover of F can be chosen to be H, and the group of deck transformations to be a subgroup of the orientation-preserving isometries of H. Each of these deck transformations (except the identity map) is a hyperbolic isometry. The pertinent property for us is that a hyperbolic isometry, when extended to the circle at ∞, has exactly two fixed points in S^1 and none in H.

Let $p : H \to F$ be the universal covering map. The map p can be chosen to be a local isometry.

A *geodesic* in F is the image under p of a geodesic in H. A geodesic is *simple* if it has no transverse intersections. A *simple closed geodesic* is a geodesic that is also a simple closed curve.

A simple closed curve F is *essential* if it does not bound a disk. Each essential simple closed curve in F is isotopic to a unique simple closed geodesic.

A *geodesic lamination* is a closed subset of F that is a disjoint union of simple geodesics (called *leaves*).

Here is a continua theorist's perception of a certain part of the Nielson–Thurston construction. Let C be an essential simple closed curve in F, and let $h: F \to F$ be a pseudo-Anosov homeomorphism. Let $C_n = h^n(C)$, and let $\hat{C}_n$ denote the unique simple closed geodesic isotopic to C_n.

THEOREM 3.1. *In the space of subcontinua $C(F)$ of F (with the Hausdorff metric), some subsequence of $\{\hat{C}_n\}$ converges to a geodesic lamination L with uncountably many leaves.*

A leaf of a lamination is *isolated* if one of its points is accessible by arcs from both sides in F. If we discard the isolated leaves from a lamination, the remaining leaves form a new lamination called the *derived lamination*. A lamination is *perfect* if it has no isolated leaves.

THEOREM 3.2. *If G is the derived lamination of L, then G is a perfect lamination and every leaf of G is dense in G. Furthermore, G is an indecomposable continuum locally homeomorphic to the product of* **R** *and a Cantor set.*

Since G shares so many topological properties with the dyadic solenoid, one might wonder if G is homogeneous. As remarked earlier, the fact that a finite number of leaves of G are accessible from the complement in F is not sufficient reason alone to conclude G is not homogeneous.

Here is an interesting proof that G is not homogeneous. If G were homogeneous, then G would have the Effros property. From a proof similar to that of Theorem 2 of [7], it would follow that $\tilde{G} = p^{-1}(G)$ has the Effros property. The space $\tilde{G}$ is pictured in Figure 1. Close to each point x of $\tilde{G}$ is a point y of $\tilde{G}$ such that the geodesic G_x of G containing x and the geodesic G_y of G containing y are ultraparallel (this means that not only are the geodesics disjoint, but they do not meet even on the circle at ∞). This contradicts the existence of a bounded homeomorphism of $\tilde{G}$ taking x to y.

In the next section, we extend this idea to apply to any curve with nontrivial shape.

The remainder of Section 3 is devoted to applying [8] to the stable geodesic lamination G of a pseudo-Anosov homeomorphism. The reader interested only in homogeneous continua can move directly to Section 4.

Choose $\epsilon > 0$ so that each ϵ-homeomorphism of G lifts to an ϵ-homeomorphism of $\tilde{G}$. Such an ϵ exists, by the proof of [7, Theorem 2], but its existence does not follow from the usual theory of covering spaces, since G is neither path-connected nor locally path-connected.

Let $H(G)$ be the group of homeomorphisms of G with the compact-open topology, and let $B(G)$ be the closed subgroup of $H(G)$ generated by the ϵ-homeomorphisms. The orbit of a point x in G under $B(G)$ is exactly the leaf of G containing x (slide along the product structure to get all the leaf in the orbit; note that no ϵ-homeomorphism can change leaves, since its lift must be bounded).

Hence $B(G)$ is a Polish transformation group acting on the indecomposable continuum G so that the orbits are exactly the leaves of G. The next theorem, therefore, follows immediately from [8, Theorem 3.3] and the Effros' theorem [2].

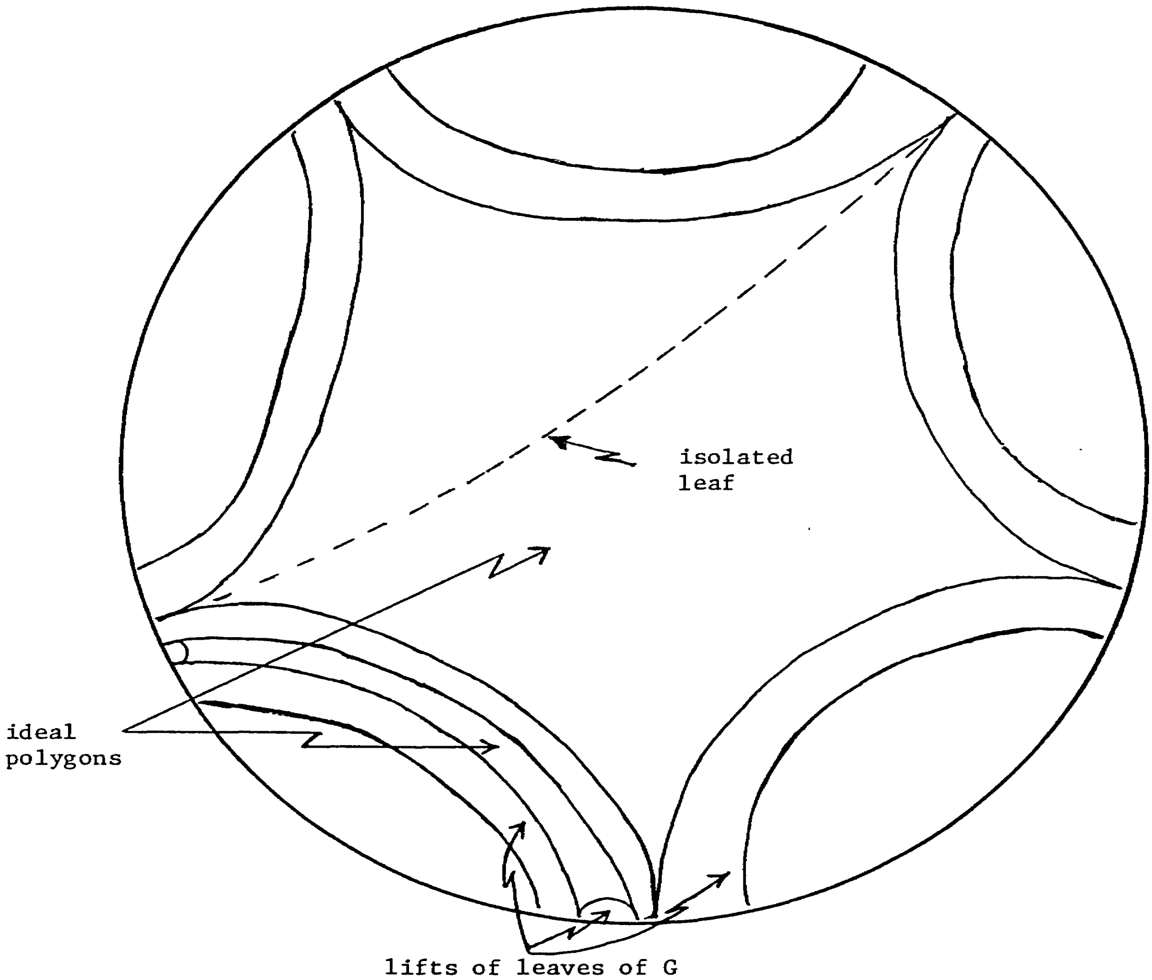

Figure 1 The lift $\tilde{G}$ of a perfect geodesic lamination G

THEOREM 3.3. *Let G be the stable geodesic lamination associated with a pseudo-Anosov homeomorphism of a hyperbolic surface. Then G is an indecomposable continuum with the following properties:*

(1) *G does not have a Borel transversal to its leaves* (*i.e., there does not exist a Borel subset B of G such that B intersects each leaf of G in exactly one point*).

(2) *G admits a nontrivial ergodic Borel measure μ in the sense that* (a) *the μ-measure of each leaf of G is zero, and* (b) *if the measurable set M is the union of some leaves of G, then either M or $G-M$ has μ-measure zero.*

4. The ends of a homogeneous curve of nontrivial shape. Assume the universal covering space (H, p, F) of F is constructed with the following properties. The geodesic $H \cap x$-axis maps to a simple closed geodesic C_1. The geodesic $H \cap y$-axis maps to a simple closed geodesic C_2. Furthermore, the union of C_1 and C_2 is a figure-eight W, and the intersection of C_1 and C_2 is a single point v.

Let $\tilde{W} = p^{-1}(W)$; hence $\tilde{W}$ is the universal cover of a figure eight, the "infinite snowflake" pictured in Figure 2. The set $\mathrm{Cl}(\tilde{W}) \cap S^1$ is a Cantor set; call it Z.

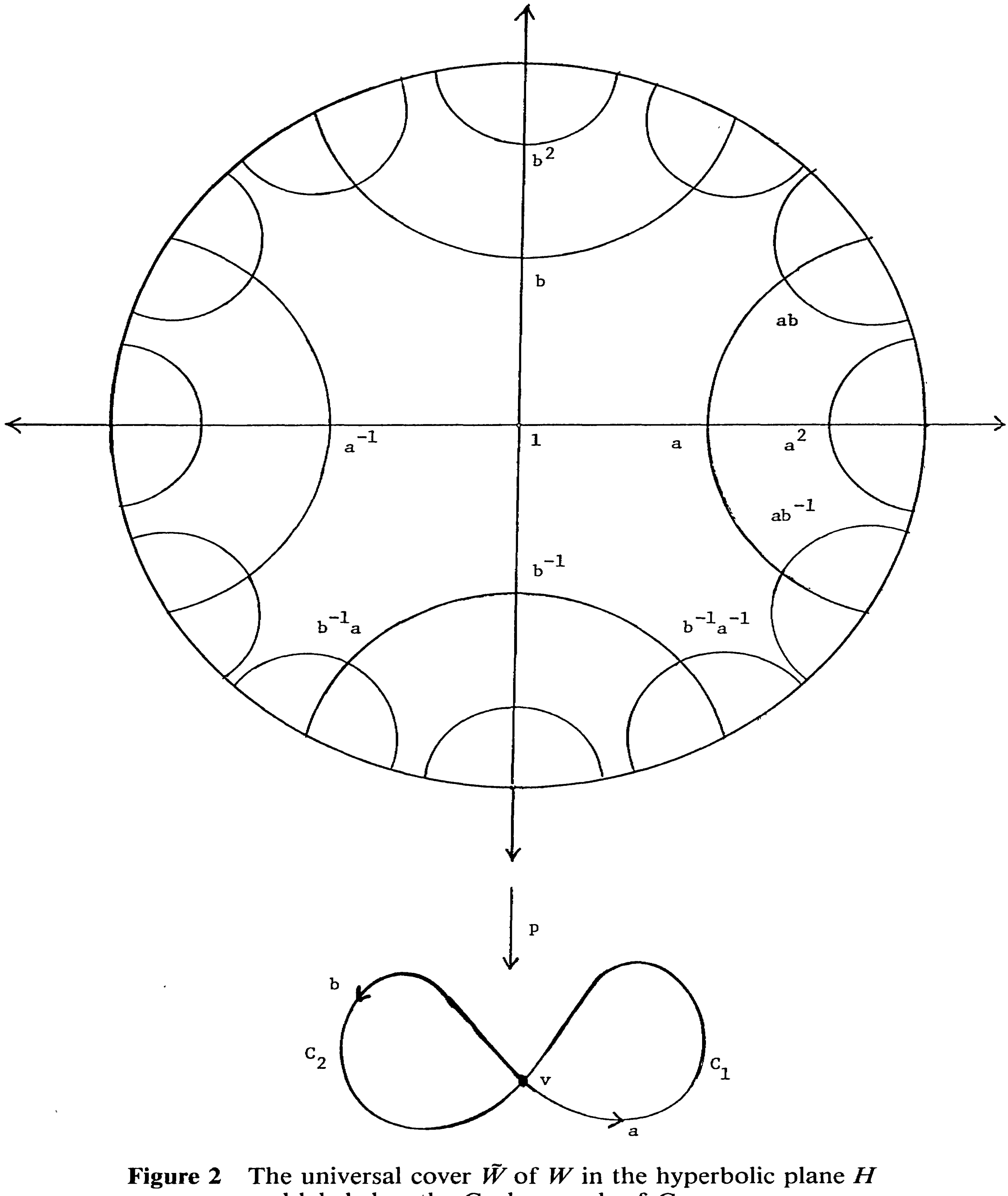

Figure 2 The universal cover $\tilde{W}$ of W in the hyperbolic plane H and labeled as the Cayley graph of G_1

Let Q be a Hilbert cube, and let $p\times 1: H\times Q \to F\times Q$ be the universal covering space of $F\times Q$. Let X be a continuum embedded essentially in $W\times Q$ (this means that the inclusion map is essential; note that this eliminates some continua from consideration). Let $f: X\to W$ be the projection map.

Let $\tilde{X} = (p\times 1)^{-1}(X)$, and let $\tilde{f}: \tilde{X}\to \tilde{W}$ be the projection map. Let K be a component of $\tilde{X}$. The proof of the next theorem is essentially that of [9, Theorem 8].

THEOREM 4.1. *The component K is unbounded in $H \times Q$ (and hence $\tilde{f}(K)$ is unbounded in $\tilde{W}$).*

Define $E(K) = \{z \in S^1 : z$ is a (Euclidean) limit point of $\tilde{f}(K)\}$. Call $E(K)$ *the set of ends of* K. The set $E(K)$ is a closed subset of S^1.

THEOREM 4.2. *If X is homogeneous, then $\tilde{X}$ (and hence K) has the Effros property.*

Theorem 4.2 is essentially proved in [9, Theorem 3]. It follows, therefore, that the set of ends is a local invariant of $\tilde{X}$, which is the content of the next theorem.

THEOREM 4.3. *If X is homogeneous, then there exists a $\delta > 0$ such that if $k \in K$ and $k' \in K'$ and $d(k, k') < \delta$ (where K and K' are components of $\tilde{X}$), then $E(K) = E(K')$.*

Proof. This is an immediate consequence of Theorem 4.2, since each bounded homeomorphism of $\tilde{X}$ preserves ends. □

Assume that each of the geodesics C_1 and C_2 has length greater than one and that 2δ is an Effros δ for $\epsilon = \frac{1}{2}$.

Cover $\{v\} \times Q$ with a finite collection $\mathcal{B}$ of δ-balls. Since K is unbounded and locally compact, there exists a ball B in $\mathcal{B}$ and two lifts $\tilde{B}_0$ and $\tilde{B}_m$ of B and a subcontinuum M of $\tilde{C}$ meeting both $\tilde{B}_0$ and $\tilde{B}_m$.

Let $\phi \times 1 : H \times Q \to H \times Q$ be the deck transformation that maps $\tilde{B}_0$ to $\tilde{B}_m$.

THEOREM 4.4. *If X is homogeneous, then $E(K) = E((\phi \times 1)(K))$.*

Proof. Let m_1 belong to $\tilde{B}_0 \cap M$ and m_2 belong to $\tilde{B}_m \cap M$. Hence

$$d(m_2, (\phi \times 1)(m_1)) < 2\delta.$$

By Theorem 4.3, $E(K) = E((\phi \times 1)(K))$. □

Compactify $H \times Q$ with $S^1 \times Q$. Shrink each set of the form $\{z\} \times Q$, where z is a point of S^1, to a point to obtain another compactification of $H \times Q$. This time the remainder is S^1.

Let $\pi : (H \times Q) \cup S^1 \to H \cup S^1$ be the map of this latter compactification onto the disk D obtained by naturally extending the projection map. Note that the restriction of π to $\tilde{X}$ is just $\tilde{f}$.

Since ϕ is a hyperbolic isometry, it has an attracting point and a repelling point, both on S^1.

THEOREM 4.5. *If X is homogeneous, then the attracting point of ϕ belongs to $E(K)$.*

Proof. Let $K_n = \mathrm{Cl}[(\phi^n \times 1)(K)]$ in $(H \times Q) \cup S^1$. By taking a subsequence if necessary, we may assume that the sequence $\{K_n\}$ of subcontinua of $(H \times Q) \cup S^1$ converges to a subcontinuum P of $(H \times Q) \cup S^1$ (this convergence is in the space of subcontinua with the Hausdorff metric). Therefore $\pi(K_n)$ converges to $\pi(P)$.

Since $\pi(K_n) \cap S^1 = E((\phi^n \times 1)(K)) = E(K)$ for all n, by Theorem 4.4, and since $\pi(K_n) \cap S^1$ converges to $\pi(P) \cap S^1$, it follows that $\pi(P) \cap S^1 = E(K)$. If d belongs to $\pi(K)$ and z is the attracting point of ϕ, then $\phi^n(d)$ converges to z in D. Since $d \in \pi(K)$, it follows that $\phi^n(d) \in \pi((\phi^n \times 1)(K))$, and so $z \in \pi(P)$. Thus $z \in E(K)$. □

THEOREM 4.6. *If X is homogeneous, then the repelling point of ϕ belongs to $E(K)$.*

Proof. The repelling point of ϕ is the attracting point of ϕ^{-1}. Hence the theorem follows from the previous one. □

THEOREM 4.7. *If X is homogeneous, then $E(K)$ is either a two-point set or a Cantor set. Furthermore, $E(K)$ contains a dense subset each point of which is a fixed point of a hyperbolic isometry ϕ such that $\phi \times 1$ is a deck transformation.*

Proof. Suppose $E(K)$ contains more than two points. Since

$$E(K) = E((\phi^n \times 1)(K))$$

for each positive integer n, and since ϕ fixes only two points of S^1, it follows that $E(K)$ is infinite. In fact, there exists a sequence of points $\{z_i\}$ in $E(K)$ that converges to the attracting point z of ϕ.

Furthermore, if $z' \in E(K)$, then arbitrarily close to z' is a point z'' in $E(K)$ such that z'' is the attracting fixed point of a hyperbolic isometry ϕ'' such that $\phi'' \times 1$ is a deck transformation. Such a hyperbolic isometry is constructed in the same manner as was ϕ. In particular, ϕ'' enjoys the same properties as ϕ. It follows that $E(K)$ is perfect and hence a Cantor set. □

We conclude this section by deriving a quick proof of the main result of [9]. A space X is *hereditarily indecomposable* if each continuum in X is indecomposable.

THEOREM 4.8. *Each hereditarily indecomposable, homogeneous continuum is tree-like (i.e., one-dimensional and of trivial shape).*

Proof. It is known [3] that each hereditarily indecomposable continuum that is not tree-like can be embedded essentially in $W \times Q$. In the proof of Theorem 4.4, we find that if X is a homogeneous continuum embedded essentially in $W \times Q$, then $\tilde{X}$ contains the decomposable continuum $M \cup h(\phi \times 1)(M)$. The author [9, Theorem 10] has shown that if X is hereditarily indecomposable, so is $\tilde{X}$. The theorem follows. □

Bing [1] showed that the pseudo-arc is homogeneous. The pseudo-arc is hereditarily indecomposable. Whether there exists another homogeneous, tree-like continuum is an important problem.

5. Acyclic curves of nontrivial shape.

A *curve* is a one-dimensional continuum. A curve X has *trivial shape* (or is *cell-like* or *tree-like*) if each map of X into a figure-eight W is inessential. A continuum X admits an essential map into W if and only if X can be embedded essentially in $W \times Q$, where Q is the Hilbert cube (this means the inclusion map is essential).

A curve X is *acyclic* if each map of X into S^1 is inessential. This is equivalent to $H^1(X)=0$, where $H^1(X)$ is the first Čech cohomology group with integral coefficients.

Consider the following two inverse sequences of groups (i.e., pro-groups):

(1) $Z \xleftarrow{g} Z \xleftarrow{g} \cdots$

(2) $Z \xleftarrow{f} Z \xleftarrow{f} \cdots,$

where Z denotes the integers, $g(z)=0$, and $f(z)=2z$. In both cases, the inverse limit is the trivial group. In the category of pro-groups, however, $(Z,g)=0$, while $(Z,f)\neq 0$ (a pro-group $(G,h)=0$ if, by replacing (G,h) with a subsequence, each bonding map is the zero map). This is a typical example to show that pro-groups contain more information than their inverse limits.

The second pro-group (Z,f) is the pro-homology sequence of the dyadic solenoid S. Hence the Čech homology group $H_1(S)=0$, but pro-$H_1(S)\neq 0$. The dyadic solenoid has nontrivial cohomology; in fact, $H^1(S)$ is isomorphic to the dyadic rational numbers. This example illustrates a theorem due to Lacher [4, Corollary 3.3].

THEOREM 5.1. *If X is a curve, then $H^1(X)=0 \Leftrightarrow$ pro-$H_1(X)=0$.*

The first shape group $\check{\pi}_1(S)=\varprojlim\{S^1, f_\#\}$ of the dyadic solenoid is trivial, since pro-$\pi_1(S)=$ pro-$H_1(S)$. Hence a trivial shape group does not imply a trivial cohomology group. The converse, however, is true for curves.

THEOREM 5.2. *If X is an acyclic curve, then $\check{\pi}_1(X,*)$ is trivial.*

Proof. Let X be the limit of an inverse sequence (X,f) of graphs. With the pointed graph $(X_i,*)$ is associated the Hurewicz homomorphism

$$\phi_i\colon \pi_1(X_i,*)\to H_1(X_i).$$

Since ϕ is a natural transformation, the following diagram commutes.

$$\begin{array}{ccccc}\pi_1(X_1,*) & \xleftarrow{f_\#} & \pi_1(X_2,*) & \longleftarrow & \cdots \\ \downarrow\phi_1 & & \downarrow\phi_2 & & \\ H_1(X_1) & \xleftarrow{f_*} & H_1(X_2) & \longleftarrow & \cdots.\end{array}$$

Since X is acyclic, we can apply Theorem 5.1 and assume, by taking subsequences if necessary, that each f_* is the zero map. It follows that $f_\#(\pi_1(X_i,*))$ is contained in the commutator subgroup of $\pi_1(X_{i-1},*)$.

Let $G_1=\pi_1(X_1,*)$, and let $G_n=(f_1^n)_\#(\pi_1(X_n,*)\subset G_1$. It follows that $G_{n+1}\subset G_1^{(n)}$, the nth derived group of G_1. From Theorem 6.2 of the next section, it follows that $\bigcap_n\{G_n\}$ is trivial. Since the same result holds for each $\pi_1(X_i,*)$, it follows that $\check{\pi}_1(X,*)$ is trivial.

Theorem 5.2 is a one-dimensional phenomenon; the projective plane P^2, for instance, has trivial first cohomology, while $\check{\pi}_1(P^2)=\pi_1(P^2)$ is nontrivial.

6. The ends of a group. Let X be a curve embedded essentially in $W\times Q$. Without loss of generality, we may regard X as the intersection of a decreasing sequence $\{H_n\}$ of cubes-with-handles (i.e., regular neighborhoods of graphs) each of which is embedded essentially in $W\times Q$. For convenience, we assume that $H_1=W\times Q$.

Let K be a component of $\tilde{X}$, and let K_n be the component of $(p\times 1)^{-1}(H_n)$ that contains K. Each K_n has a set of ends $E(K_n)$ contained in Z. The next theorem is obvious.

THEOREM 6.1. $E(K)\subset\bigcap_n\{E(K_n)\}$.

If G_1 is the free group on the two generators a and b, then W is a $K(G_1,1)$ complex and $\tilde{W}$ is the Cayley graph associated with the generators a and b of G_1. To define the ends of G_1, pick finite complexes $M_0\subset M_1\subset\cdots$ whose union is $\tilde{W}$ and define the set of ends $E(G_1)$ of G_1 to be $\varprojlim\{\pi_0(\tilde{W}\backslash M_i)\}$. Clearly $E(G_1)=E(\tilde{W})=Z$.

Thus a point z in S^1 belongs to $E(G_1)$ if and only if there is a sequence $\{g_n\}$ of elements of G_1 such that the sequence $\{g_1, g_1g_2, \ldots, g_1g_2\cdots g_n, \ldots\}$, viewed as vertices of the Cayley graph $\tilde{W}$, converge to z. In this case, of course, we could require that each g_n belong to $\{a^{\pm 1}, b^{\pm 1}\}$.

If L is a subgroup of G_1, then define the ends of L to be $E(L)=\{z\in E(G_1)$: there is a sequence $\{g_n\}$ of elements of L such that $\{g_1, g_1g_2, \ldots, g_1g_2\cdots g_n, \ldots\}$ converges to $z\}$.

If $f_1^n\colon H_n\to H_1$ denotes the inclusion map, then define

$$G_n=(f_1^n)_\#(\pi_1(H_n,*))\subset\pi_1(H_1,*)=G_1.$$

Hence this is a decreasing sequence of subgroups $G_1\supset G_2\supset\cdots$ of G_1. Clearly $E(G_n)=E(H_n)$, for all n.

Let G' denote the commutator subgroup of a group G. Set $G^{(1)}=G'$ and set $G^{(i)}=(G^{(i-1)})'$, the commutator subgroup of $G^{(i-1)}$. The group $G^{(i)}$ is the ith *derived subgroup of G*. The following theorem is well known (see, for example, [5, p. 14]).

THEOREM 6.2. *If G is a free group, then* $\bigcap_i\{G^{(i)}\}$ *is trivial.*

If G is a finitely generated free group, then $E(G')=E(G)$. The next theorem essentially says that quite the opposite is true for the set of ends of a finitely generated subgroup of a derived subgroup of G.

THEOREM 6.3. *Let G_n be a finitely generated subgroup of $G^{(n)}$, the* nth *derived subgroup of the free group G. Choose a fixed finite set of generators of G_n. Each such generator of G_n is a word in the generators of G. If $g\in G^{(n-1)}-G^{(n)}$, then there exists a natural number M such that $m>M$ implies that $g^m\neq wc$, where $w\in G_n$ and c is an initial segment of a generator of G_n.*

Proof. Since $G^{(n-1)}/G^{(n)}$ is free abelian and $g\notin G^{(n)}$, it follows that $g^m\notin G^{(n)}$, for all natural numbers m.

Let c be an initial segment of one of the generators of G_n. If $g^m = wc$, where $w \in G_n$, then this is the only such m, for otherwise $g^{m'} = w'c$ and $m' > m$ imply $g^{m'-m} = w'w^{-1} \in G_n$.

Since there are only a finite number of generators of G_n and hence only a finite number of initial segments of generators, the theorem follows. □

THEOREM 6.4. *If X is acyclic, then $E(K)$ does not contain a fixed point of a hyperbolic isometry ϕ such that $\phi \times 1$ is a deck transformation.*

Proof. Let $G_1 = \pi_1(H_1, *)$, and let $G_n = (f_1^n)_\#(\pi_1(H_n, *)) < G_1$. As in the proof of Theorem 5.2, it follows that $G_{n+1} \subset G_1^{(n)}$.

Let ϕ be a hyperbolic isometry such that $\phi \times 1$ is a deck transformation. A point z in $E(G_1)$ is a fixed point of ϕ if and only if there is an element g in G_1 such that the sequence $\{g^n\}$ converges to z. The theorem now follows from the preceding theorem. □

COROLLARY 6.5. *An acyclic homogeneous curve is tree-like.*

REFERENCES

1. R. H. Bing, *A homogeneous indecomposable plane continuum,* Duke Math. J. 15 (1948), 729–742.
2. E. G. Effros, *Transformation groups and C*-algebras,* Ann. of Math. (2) 81 (1965), 38–55.
3. J. Krasinkiewicz, *Mapping properties of hereditarily indecomposable continua,* Houston J. Math. 8 (1982), 507–516.
4. R. C. Lacher, *Cellularity criteria for maps,* Michigan Math. J. 17 (1970), 385–396.
5. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory,* Springer, Berlin, 1977.
6. J. T. Rogers, Jr., *Homogeneous continua,* Topology Proc. 8 (1983), 213–233.
7. ———, *Homogeneous, separating plane continua are decomposable,* Michigan Math. J. 28 (1981), 317–321.
8. ———, *Borel transversals and ergodic measures on indecomposable continua,* Topology Appl. 24 (1986), 217–227.
9. ———, *Homogeneous hereditarily indecomposable continua are tree-like,* Houston J. Math 8 (1982), 421–428.

Department of Mathematics
Tulane University
New Orleans, LA 70118