AN OBSTRUCTION TO THE RESOLUTION OF HOMOLOGY MANIFOLDS

Frank Quinn

The author's paper [2] asserts that an ENR homology manifold of dimension ≥ 4 has a resolution. Unfortunately there is an error in the proof, and it is currently unknown whether unresolvable homology manifolds exist. The proof does show that there is at most a single integer obstruction, heuristically the index of a 0-dimensional subobject (a "point"). The properties of this obstruction are the subject of this paper.

The obstruction can be roughly described by: let $U \subset X^n$ be an open set with a proper degree 1 map $f: U \to \mathbf{R}^n$. Make f transverse (in a sense to be made precise below) to 0; then the *local index* is defined by $i(X) = \operatorname{index} f^{-1}(0)$. If X is a manifold and we use manifold transversality then $f^{-1}(0)$ can be arranged to be a single point, so $i(X) = 1$. If X is not a manifold then some other form of transversality must be used, and the ones currently available might yield inverses with indices different from 1. This index must be congruent to 1 mod 8, so for example there might be homology manifolds with "points" of index 9.

We discuss the transversality to be used. There is currently no direct transversality construction for homology manifolds, so we pass to Poincaré complexes or chain complexes. In the Poincaré context there is a dimension restriction (≥ 4) for transversality. Therefore we have to raise the dimension, for example by multiplication by CP^2, to apply it to the situation at hand. This leads to an inverse image of positive dimension, which may have a nontrivial index. If chain complexes are used there are no dimension restrictions to transversality, so we can get a 0-dimensional chain complex as the "inverse image" of 0 in $\mathbf{R}^n$. However, 0-dimensional chain complexes can have nontrivial index. In either case the index gives the obstruction.

There are some curious niches in manifold theory for unresolvable homology manifolds. It seems likely that a homology manifold has a canonical topological normal bundle (see Section 5). In this case we could classify normal bundles of homology manifolds by maps to $(B_{\mathrm{TOP}}) \times \mathbf{Z}$, the $\mathbf{Z}$ factor being $(i(X)-1)/8$. Project to B_G as usual, then the fiber is $(G/\mathrm{TOP}) \times \mathbf{Z}$. One of the wonderful things about G/TOP is that it is nearly periodic; there is a natural equivalence $\Omega^4(G/\mathrm{TOP}) \simeq (G/\mathrm{TOP}) \times \mathbf{Z}$. Including homology manifolds in the picture would give an *exactly* periodic fiber. Exact periodicity in the classifying space would lead to (minor) improvements in the formal properties of surgery theory.

In Section 1 we state the main theorem, and indicate corrections to the statements in [2]. Section 2 explains the error in the original proof, and gives counterexamples to easy repairs. In Section 3 we indicate how the correct parts of the

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proof imply Theorem 1.1. Section 4 develops another view of the invariant, using a "transversality" construction for controlled quadratic Poincaré chain complexes. Finally, remarks and questions are given in Section 5.

1. The main result. Recall that a *resolution* of a homology manifold X is a proper map from a manifold $f: M \to X$, such that $f^{-1}(\partial X) = \partial M$ and the point inverses of f are nonempty and contractible in any neighborhood in M.

1.1. THEOREM. *For every non-empty connected ENR homology manifold there is a "local index" $i(X) \in (1+8\mathbf{Z})$ defined, which satisfies*

(a) *if $U \subset X$ is open then $i(U) = i(X)$,*

(b) *$i(\partial X) = i(X)$ if $\partial X \neq \emptyset$, $i(X \times Y) = i(X)i(Y)$, and*

(c) *if $\dim(X) \geq 5$, or $\dim(X) = 4$ and ∂X is a manifold, then there is a resolution if and only if $i(X) = 1$.*

The strong invariance properties allow easy evaluation in many situations. The most useful observation seems to be that if X has an open set which is resolvable (e.g., if it is a polyhedron), then the local index must be 1, so X itself is resolvable.

This theorem replaces the first statement of [2]. Consequently the "characterization of manifolds" should assert that X is a manifold of dimension ≥ 5 if and only if it is an ENR homology manifold of local index 1, and satisfies the disjoint disks property. The statements 1.1–1.3 of [2] should be modified by replacing "homology manifold" by "homology manifold of local index 1."

2. Analysis of the proof. The error in [2] occurs in the paragraph at the top of page 282, after foreshadowing at the bottom of 280. The obstruction is narrowed down to a single integer (it will turn out to be $(1-i(X))/8$). Framed normal maps $M \to Y$ and $Y \to T^{4k}$ are constructed (M a manifold and Y Poincaré). The obstruction is correctly identified as the codimension $4k$ component of the surgery obstruction of $M \to Y$, in $L_{4*}(\mathbf{Z}[\mathbf{Z}^{4k}])$. This component of the surgery obstruction is then asserted to vanish for dimension reasons. This last assertion is not valid.

Given a surgery map $M^{4*} \to N^{4*} \to T^{4k}$ with M and N *both closed manifolds,* we can take the transverse inverse images of a point in T^{4k} to get a surgery map of manifolds of dimension $4(*-k)$. The (simply connected) surgery obstruction of this map is the codimension $4k$ component of the $\mathbf{Z}[\mathbf{Z}^{4k}]$ obstruction. In particular if $k = *$ then these inverse images are points, so the codimension $4k$ component must vanish. Unfortunately in the situation we want to apply this observation, N is only a Poincaré space.

Sylvain Cappell pointed out to me that the implicit presumption (that this argument would work for N Poincaré) is in general false because there are problems with transversality in Poincaré spaces. We give examples to show that in fact the presumption is false in the case we need it.

First we observe that in the case of interest the composition $M \to T^{4k}$ is also a (framed) surgery map. The composition formula implies that the obstruction is the sum of the obstructions of $M \to Y$ and $Y \to T^{4k}$. Further, since M and T^{4k}

are both closed manifolds the argument above applies to show the codimension $4k$ component of the sum is trivial. Therefore the codimension $4k$ component of $M\to Y$ is the negative of that of $Y\to T^{4k}$. This allows us to work directly with Y. In particular, to show the implicit assumption used in [2] is false, it is sufficient to show there is a Poincaré degree 1 normal map $Y^4\to T^4$ whose surgery obstruction has nontrivial codimension 4 component.

Let R denote $\mathbf{Z}[\mathbf{Z}^4]$, and suppose (R^k,λ) is a form representing an element in $L_4(R)$. Let $Y=(Wv^kS^2)\cup_\alpha D^4$, where $T^4=W\cup_\beta D^4$ and the map which collapses the S^2 to points takes α to β. The collapse then defines a degree 1 map $g: Y\to T^4$, which is 2-connected and has $H_3(g;R)=R^k$. By properly choosing α (β plus Whitehead products of π_1 multiples of the S^2) we can arrange the product structure on Y to induce the form λ on H_3. The nonsingularity of λ implies that Y is Poincaré. Since suspensions of Whitehead products are trivial, the attaching map of the 4-cell of Y is stabily trivial. This implies that the normal fibration of Y is trivial, so g is a framed normal map. Finally since arbitrary forms can be realized this way, we can get ones with nontrivial codimension 4 component.

We also observe that it does not help that the Y constructed in [2] has the additional property that it is δ Poincaré over T^{4k}, for small δ. The Y constructed above are r Poincaré for some r. Taking (δ/r)-fold covers in each coordinate in T^4 gives $Y\hat{}\to T^4$ which is δ Poincaré. But the codimension 4 component of the surgery obstruction is invariant under such covers, so can still be nontrivial.

Section 4 of [2] becomes correct (I believe) if references to vanishing are deleted, and the conclusion of the "first reduction" on page 4.1 is changed to: "Then for sufficiently small δ the obstruction of 3.2 is the negative of the codimension $4k$ component of the surgery obstruction of h."

3. Proof of the theorem. We begin by considering surgery obstructions in more detail. Suppose $f: M^m\to Y$ is a surgery map (i.e., degree 1, normal, and $\partial M\to\partial Y$ is a homotopy equivalence). Suppose also that $g: Y\to T^{4n}$ induces an isomorphism of fundamental groups of Y and ∂Y. If $m\geq 6$ (or $m\geq 5$ if $\partial Y=\phi$) then f can be "split" over $T^{n-1}\subset T^n$. This means g can be made transverse to T^{n-1} and f can be made transverse to $g^{-1}(T^{n-1})$ so that the induced map of boundaries is a homotopy equivalence. The fact that the Poincaré space Y can be made transverse is equivalent to the calculation of the surgery group of $\mathbf{Z}[\mathbf{Z}^n]$, since in general there are obstructions to Poincaré transversality [3, §1]. If M is a manifold and manifold transversality is required, then the hard part of the splitting is to get the map on the boundary to be a homotopy equivalence [6; 7, §13A.8]. If M is only required to be Poincaré then we can replace ∂M by the homotopy equivalent ∂Y, and ∂f by the identity. Then ∂M is automatically split, and the nontrivial step is to make the rest of M transverse. As remarked above this requires the calculation of the surgery group. In either case the description of the codimension n component of the obstruction is: split repeatedly over a complete sequence $T^n\supset T^{n-1}\supset\cdots\supset T^0$, and take the simply connected obstruction of the inverse image of T^0. If at some point the dimension gets too low to split, multiply by CP^2 and continue. Multiplication by CP^2 does not change surgery obstructions.

Note that if Y, M are both closed manifolds then the splitting is very easy. There is no boundary to worry about, and no obstruction to transversality. In this case the codimension n obstruction is just obtained by making things transverse to a point in T^n. If Y is Poincaré then transverse images generally may not exist, and are often not unique (even up to bordism) when they do exist. The splitting procedure serves to avoid these problems.

In the case of concern, Y is Poincaré, $f: Y \to T^{4k}$ is the surgery map, and the reference map g is the identity. The prescription is therefore to split down until the dimension restriction stops us, at $f_4: Y^4 \to T^4$, cross with CP^2 to get $Y^4 \times CP^2 \to T^4 \times CP^2 \to T^4$, and continue splitting. The result is $V^4 \to CP^2 \to T^0$, and the obstruction is $(\text{index}(V) - \text{index}(CP^2))/8$.

We see that the error encountered above comes from the dimension restriction in the splitting procedure. If it were not necessary to multiply by CP^2 then we would split down to a 0-dimensional surgery map, which must have obstruction 0. The fact that arbitrary obstructions occur for Poincaré $Y^4 \to T^4$ shows that this multiplication is generally necessary.

We define the invariant $i(X)$, for a homology manifold X. Multiply X by some Euclidean space to make the dimension a multiple of 4, say $4k$. Transfer a piece of X to a Poincaré space $Y \to T^{4k}$ as in [2, §4.1]. Make $Y \times CP^2$ Poincaré transverse to $T^{4k} \times CP^2 \supset T^{4k-1} \times CP^2 \supset \cdots \supset T^0 \times CP^2$, ending with a Poincaré normal map $V \to CP^2$. Then define $i(X) = \text{index}(V)$.

Since $V \to CP^2$ is a normal map $\text{index}(V) \equiv \text{index}(CP^2) \pmod 8$, so $i(X) \in (1 + 8\mathbf{Z})$. Technically this definition depends on choice of a point in X (as in [2, §3.2]), so defines a map $X \to \mathbf{Z}$. The identification as part of a surgery obstruction shows that it is well defined and locally constant. Therefore there is a single value $i(X)$ for connected X. This also makes the restriction property (a) of the theorem clear. Further it is shown above that $(1 - i(X))/8$ is the resolution obstruction of [2, §3.2], so part (c) of the theorem follows from [2, §3.3].

Finally we consider the product formula (b). Suppose all the data specified above is chosen for X and Y, and $i(X) = \text{index}(V)$, $i(Y) = \text{index}(W)$. Taking the product of the data gives data for $X \times Y$, except there is an extra product with CP^2; the end of the splitting is $V \times W \to CP^2 \times CP^2$. Muliplication by CP^2 does not change the surgery obstruction, so $i(X \times Y) = \text{index}(V \times W)$. The formula now follows from the multiplicative property of the index.

4. Another description of the invariant. Ranicki [5] has developed a theory of Poincaré chain complexes which can be used in place of Poincaré spaces in the development. Some simplification is possible in this context because Yamasaki [8] has worked out a controlled version.

Given a (finite complex) Poincaré space Y, the cellular chains of Y can be given the structure of a symmetric Poincaré complex [5, §II.2]. Next suppose $Y \to X$ is a δ Poincaré space over X [2, §2.2]. Choose a triangulation with cells of radius $< \delta$, and consider the cellular chains as a geometric $\mathbf{Z}$ complex over X. This complex can be given a symmetric Poincaré structure with radius $< \delta$, which we denote by $\sigma_\delta(Y)$.

The next lemma is modeled after a transversality remark: suppose $f: M \to \mathbf{R}^n$ is a proper degree 1 normal map, and M is a manifold. Using transversality we can arrange that $f^{-1}(D^n) \to D^n$ is a homeomorphism. By pushing the complement of D^n to ∞ we can get a degree 1 normal bordism of f to a homeomorphism. On the chain level we get a δ quadratic Poincaré structure over $\mathbf{R}^n$ on the relative chains of f. The bordism constructed gives an algebraic bordism of this quadratic complex to the trivial complex. We think of the trivial complex over $\mathbf{R}^n$ as $\sigma_\delta(\mathbf{R}^n) \otimes (0)$, where (0) denotes the trivial 0-dimensional quadratic complex.

LEMMA. *Given $n, \epsilon > 0$ there is $\delta > 0$ such that a strictly n-dimensional quadratic geometric $\mathbf{Z}$-module complex over $\mathbf{R}^n$, which is δ Poincaré over D^n, is ϵ Poincaré bordant over $(D^n)^{-\epsilon}$ to a complex of the form $\sigma_\delta(\mathbf{R}^n) \otimes V$, where V is a 0-dimensional quadratic complex over a point. Further if ϵ is small enough then V is unique up to bordism.*

This follows from a splitting argument. Given δ_n there is δ_{n-1} so that a quadratic complex of radius $< \delta_{n-1}$ is δ_n equivalent to a union of two complexes, one lying over $\mathbf{R}^{n-1} \times [0, \infty)$ and one lying over $\mathbf{R}^{n-1} \times (-\infty, 0]$. This is Yamasaki [8, §2.4], except there a stabilization is required to avoid a controlled Whitehead group problem. The stabilization is not necessary here because the controlled Whitehead group with coefficient ring $\mathbf{Z}$ is trivial ([4, §8]). Denote the common $(n-1)$-dimensional complex in the union (over $\mathbf{R}^{n-1}$) by C^{n-1}, then there is a bordism of the original to $\sigma_\delta(\mathbf{R}) \otimes C^{n-1}$, disjoint union with pieces lying over $\mathbf{R}^{n-1} \times [0, \infty)$ and over $\mathbf{R}^{n-1} \times (-\infty, 0]$. There are null bordisms of these extra pieces constructed by pushing them to $\pm\infty$ in the second coordinate. This gives a bordism of the original to $\sigma_\delta(\mathbf{R}) \otimes C^{n-1}$. If we repeat this argument $n-1$ more times we get the decomposition required in the lemma.

The uniqueness follows in the same way, by splitting bordisms. Slightly more generally, it is useful to observe that a sufficiently small $(n+k)$-dimensional quadratic Poincaré complex over $\mathbf{R}^n$ with boundary of the form $\sigma_\delta(\mathbf{R}^n) \otimes W$, is bordant rel boundary to a complex of the form $\sigma_\delta(\mathbf{R}^n) \otimes V$, where V is a k-dimensional complex with boundary W.

PROPOSITION. *Given n, there is $\epsilon > 0$ such that if X is an ENR homology manifold, $f: X \to \mathbf{R}^n$ is a degree 1 proper normal map, and the relative chain complex of f is bordant as geometric quadratic complexes ϵ Poincaré over D^n to a complex of the form $\sigma_\epsilon(\mathbf{R}^n) \otimes V$, then $i(X) = 1 + \mathrm{index}(V)$.*

We recall that a 0-dimensional quadratic Poincaré complex over $\mathbf{Z}$ is simply an even symmetric bilinear form [5, §I.2], and the "bordism class" is determined by the index.

Proof. The local index $i(X)$ is defined by grafting part of X into a Poincaré space Y over a torus, geometrically splitting $Y \times CP^2$ to get $N \to CP^2$, and taking the index of N. Taking universal covers gives a normal degree 1 bordism of X over $\mathbf{R}^n$ to the universal cover of Y. The radius of this as a controlled Poincaré space is at least bounded since it comes from the universal cover of a compact object. Contraction in $\mathbf{R}^n$ then can be used to get δ control for any $\delta > 0$.

Next apply the chain construction to the geometric splitting, and the argument of Lemma 1 gives a quadratic Poincaré bordism of the chains of $X \times CP^2$ to $\sigma_\delta(\mathbf{R}^n) \otimes C_*(N, CP^2)$. Take an algebraic bordism as specified in the lemma, and tensor with the chains of CP^2. The "slightly more general" version of the uniqueness argument mentioned in the proof of the lemma implies that for sufficiently small ϵ, $C_*(CP^2) \otimes V$ is quadratic Poincaré bordant to $C_*(CP^2, N)$. In particular the indices are the same, proving the proposition. $\square$

COROLLARY. *Given n, there is $\epsilon > 0$ such that if X is an ENR homology n-manifold and $X \to \mathbf{R}^n$ is a proper degree 1 map which is an ϵ homotopy equivalence over D^n, then $i(X) = 1$.*

This is because the relative ϵ chains of an ϵ homotopy equivalence are trivial, so are equal (over D^n) to $\sigma_\epsilon(\mathbf{R}^n) \otimes (0)$. By the proposition $i(X) = 1 + \text{index}(0) = 1$.

The point of the corollary is that ϵ is independent of X, and is measured in D^n. The proof of 1.1 shows that "near resolvability" implies resolvability, in the sense that given X there is $\epsilon > 0$ such that if M is a manifold and $M \to X$ is an ϵ homotopy equivalence then $i(X) = 1$. This would be much more difficult to arrange, since the measuring takes place in X.

5. Remarks. The main question is: is there an unresolvable homology manifold?

We note some geometric properties which imply resolvability. Suppose X has an open set U with a proper degree 1 map $f: U \to \mathbf{R}^n$, and there is a sequence of subspaces $\mathbf{R}^n \supset \mathbf{R}^{n-1} \supset \cdots \supset \mathbf{R}^0$ such that $f^{-1}(\mathbf{R}^i)$ is an i-dimensional homology manifold. Then $i(X) = 1$. This follows from the splitting procedure used in the lemma in Section 4 to give the chain complex description of $i(X)$.

Alternatively if there is U, f with f a δ homotopy equivalence over D^n for sufficiently small δ, then $i(X) = 1$. This is the corollary in Section 4. There are criteria for f to be a δ homotopy equivalence in terms of contractibility of $f^{-1}(A)$ in $f^{-1}(B)$, for appropriate $A \subset B$. It may be possible to construct such f by using the local contractibility of X.

Do homology manifolds have canonical topological normal bundles? A "topological normal bundle" is a topological structure on the normal spherical fibration. Given a structure τ, there is a corresponding degree 1 normal map $M_\tau \to X$. We suggest the following criterion for "canonical": Yamasaki [8] identifies the controlled surgery obstruction as an element $\sigma(\tau) \in H_n^{\ell f}(X; \mathbf{L}_*)$, $\mathbf{L}_*$ the spectrum of quadratic Poincaré chain complexes. On the other hand a topological manifold has a canonical orientation $[M] \in H_n^{\ell f}(X; \mathbf{L}^*)$, $\mathbf{L}^*$ the spectrum of symmetric Poincaré chain complexes. Interpret $(i(X)-1)/8$ as a multiple of the generator in $H_0(pt; \mathbf{L}_*)$, then the pairing $\mathbf{L}^* \wedge \mathbf{L}_* \to \mathbf{L}_*$ interprets the product $[M_\tau](i(X)-1)/8$ as an element of the same group as the surgery obstruction. We say τ is canonical if $\sigma(\tau) = [M_\tau](i(X)-1)/8$. This fits nicely with Section 4 above, and [2, §3.3] where canonical structures are shown to exist locally. If such a structure is unique in the appropriate sense, then global existence follows from this.

If a homology manifold X has a topological normal bundle then there is a $1-LC$ embedding in codimension 3 in a manifold. To see this begin with a degree 1 normal map $M \to X$. Crossing with S^2 gives a degree 1 normal map $M \times S^2 \to X \times S^2$ with trivial controlled surgery obstruction over X. Proceed as in [2, §3.3] to construct $N \to X \times S^2$ which is an ϵ homotopy equivalence over X, for every ϵ. The mapping cylinder of $N \to X$ is then an ENR homology manifold with lots of manifold points and the disjoint disk property, hence is a manifold, and it contains X in the desired way.

This remark suggests trying to construct examples by constructing decompositions of manifolds with decomposition elements the shape of a sphere.

Finally, in contrast to codimension 3, we note that if a homology manifold embeds locally homotopically unknottedly in a manifold in codimension 2, then it is resolvable. There is a mapping cylinder neighborhood ([4], Theorem 1.4), with map $M \to X$ which is an approximate S^1 fibration. Assume (by restriction if necessary) that this is fiber homotopically trivial, and let $W \to X$ be the infinite cyclic cover of M. Then W has a tame locally 1-connected end over X, which has a completion. The new boundary in the completion is a resolution of X.

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Department of Mathematics
Virginia Polytechnic Institute
and State University
Blacksburg, VA 24061