INNER FUNCTIONS AND DIVISION IN DOUGLAS ALGEBRAS

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1. Introduction. Let H^∞ be the space of boundary functions for bounded analytic functions in the open unit disk D, and C be the space of continuous functions on ∂D. It is well known [14] that $H^\infty + C$ is an essentially supremum norm closed subalgebra of L^∞. Guillory and Sarason began the study of division in $H^\infty + C$, and they showed the following.

THEOREM A [6]. *There is a positive integer N with the following property: if I is an inner function and f is a function in $H^\infty + C$ such that $|f| \leq |I|$ on $M(H^\infty + C)$, the maximal ideal space of $H^\infty + C$, then $f^N \bar{I} = f^N / I \in H^\infty + C$.*

Guillory and Sarason posed the problem of finding the best value of N. We solve this problem here. It has been shown [1; 7] that if I is a finite product of interpolating Blaschke products, then we may take $N = 1$. As a consequence, we have the following.

THEOREM B [7]. *If I is an inner function which is not a finite product of interpolating Blaschke products, then we can not take $N = 1$.*

In §2, we show that one may take $N = 2$ in Theorem A. The technique used to prove this is almost the same as that used in [6]. For a function f in $H^\infty + C$, put $Z(f) = \{x \in M(H^\infty + C);\ f(x) = 0\}$. The condition $|f| \leq |I|$ on $M(H^\infty + C)$ implies $Z(I) \subset Z(f)$. The following question is given in [7, p. 5]: If I is an inner function and $f \in H^\infty + C$ with $Z(I) \subset Z(f)$, does there exist a positive integer K, depending on I and f, such that $f^K \bar{I} \in H^\infty + C$? We shall give its negative answer (Theorem 2). Also we shall show that Theorem B can not be extended to general Douglas algebras (Theorem 3). This is a negative answer to another question in [7, p. 5].

In §3, we shall study the factorization of Blaschke products. Let b be an inner function such that $\mathrm{Ord}_b(x)$, the zero's order of b at $x \in Z(b)$, is uniformly bounded. By [12], b is a finite product of interpolating Blaschke products. By [10], there are interpolating Blaschke products $\{b_k\}_{k=1}^n$ and a finite Blaschke product b_0 such that $b = \prod_{k=0}^n b_k$ and $Z(b) = Z(b_1) \supset Z(b_2) \supset \cdots \supset Z(b_n)$ if and only if $Z(b)$ is an interpolation set for H^∞, that is, if $H^\infty \mid Z(b)$ coincides with the space of continuous functions on $Z(b)$. It is known that there exists a finite product of interpolating Blaschke products q such that $Z(q)$ is not an interpolation set for H^∞ (see [11]). We shall show that if I is an inner function such that $\mathrm{Ord}_I(x) = k$ for every $x \in Z(I)$, then there are interpolating Blaschke products $\{b_n\}_{n=1}^k$ and a finite Blaschke product b_0 such that $I = \prod_{n=0}^k b_n$ and $Z(I) = Z(b_n)$ for $1 \leq n \leq k$ (Theorem 4).

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2. Division by inner functions. For a function f in L^∞, put $\operatorname{dist}(f, H^\infty + C) = \inf\{\|f+h\|;\ h \in H^\infty + C\}$ and $\|f\|_1 = \int_{\partial D} |f|\, d\theta/2\pi$.

THEOREM 1. *Let I be an inner function. If f is a function in $H^\infty + C$ such that $|f| \le |I|$ on $M(H^\infty + C)$, then $f^{k+1}\bar{I}^k \in H^\infty + C$ for every $k = 1, 2, \ldots$.*

To prove Theorem 1, we need a lemma.

LEMMA 1 [6, Theorem, p. 176]. *Let I be an inner function and $f \in H^\infty + C$. Then $f\bar{I}^n \in H^\infty + C$ for every positive integer n if and only if*

$$f = 0 \quad on \quad \{x \in M(H^\infty + C);\ |I(x)| < 1\}.$$

Proof of Theorem 1. As in the proof of Theorem [6, pp. 177–179], for a small positive number ϵ there is a system Γ of simple closed rectifiable curves in $\bar{D}$ with the following properties:

(a) the curves in the system Γ have mutually disjoint interiors;

(b) there is an absolute constant K (independent of I and ϵ) between 0 and 1 such that $\{z \in D;\ |I(z)| < \epsilon\}$ is contained in the interior of Γ and $|I| < \epsilon^K$ on Γ;

(c) arc length measure on $\Gamma \cap D$ is a Carleson measure such that $\int_\Gamma |h(z)|\,|dz| \le K_1 \epsilon^{-2} \|h\|_1$ for every $h \in H^\infty$, where K_1 is another absolute constant.

Take a positive integer N such that $NK > 3$. We shall first prove that

$$(1) \qquad (f\bar{I})^m I^N \in H^\infty + C \quad \text{for every } m = 1, 2, \ldots.$$

To see this, take m arbitrarily. Note $|f|^m \le |I|^m$ on $M(H^\infty + C)$ and $\epsilon^m \le |I|^m$ on Γ by (b). Using the corona theorem [5, p. 323], we have $|f|^m/|I|^m < 2$ on $\Gamma \cap \{z \in D;\ |z| > \delta\}$ for a positive number δ with $0 < \delta < 1$. Hence

$$(2) \qquad \frac{|z|^n |f(z)|^m}{|I(z)|^m} < 2 \quad \text{on } \Gamma \text{ for sufficiently large } n.$$

By almost the same argument as the one in [6, pp. 177–179],

$$\begin{aligned}
\operatorname{dist}((f\bar{I})^m I^N, H^\infty + C) &= \inf_n \operatorname{dist}((f\bar{I})^m I^N, \bar{z}^n H^\infty) \\
&= \inf_n \sup\left\{ \left| \frac{1}{2\pi} \int_\Gamma \frac{z^n f(z)^m I(z)^N h(z)}{I(z)^m}\, dz \right|;\ h \in H^\infty,\ \|h\|_1 = 1 \right\} \\
&\le \sup\left\{ \frac{2}{2\pi} \int_\Gamma |I(z)|^N |h(z)|\,|dz|;\ h \in H^\infty,\ \|h\|_1 = 1 \right\} \quad \text{by (2)} \\
&\le \sup\left\{ \frac{\epsilon^{NK}}{\pi} \int_\Gamma |h(z)|\,|dz|;\ h \in H^\infty,\ \|h\|_1 = 1 \right\} \quad \text{by (b)} \\
&\le K_1 \epsilon^{NK-2}/\pi \quad \text{by (c)} \\
&\le K_1 \epsilon/\pi \to 0 \quad (\epsilon \to 0).
\end{aligned}$$

Thus we obtain (1).

We denote by B the Douglas algebra generated by H^∞ and $f\bar{I}$. By (1), we have

$$(3) \qquad I^N B \subset H^\infty + C.$$

By the Chang–Marshall theorem [2; 13], for a fixed positive integer k there are sequences of H^∞ functions $\{h_n\}$ and $\{j_n\}$, with j_n inner, such that $\|f^k\bar{I}^k - \bar{j}_n h_n\| \to 0$ $(n \to \infty)$ and $\bar{j}_n \in B$. By (3), $I^N \bar{j}_n^m \in H^\infty + C$ for every $m, n = 1, 2, \ldots$. By Lemma 1, $I = 0$ on $\{x \in M(H^\infty + C);\ |j_n(x)| < 1\}$. By our assumption,

$$f = 0 \text{ on } \{x \in M(H^\infty + C);\ |j_n(x)| < 1\}.$$

Again by Lemma 1, $f\bar{j}_n \in H^\infty + C$ for every n. Hence

$$\begin{aligned}\operatorname{dist}(f^{k+1}\bar{I}^k, H^\infty + C) &\le \|f^{k+1}\bar{I}^k - f\bar{j}_n h_n\| \\ &\le \|f\|\|f^k\bar{I}^k - \bar{j}_n h_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Thus we obtain $f^{k+1}\bar{I}^k \in H^\infty + C$ for $k = 1, 2, \ldots$. □

THEOREM 2. *There are inner functions I and J such that $Z(I) \subset Z(J)$ and $J^N \bar{I} \notin H^\infty + C$ for every positive integer N.*

To prove the above theorem, we need some notations and lemmas. For a subset F of L^∞, we denote by $[F]$ the closed subalgebra generated by F. We put $X = M(L^\infty)$, the maximal ideal space of L^∞. The Shilov boundary of H^∞ may be identified with X [8, p. 174]. We denote by π the fiber projection from $M(H^\infty + C)$ onto ∂D; $\pi(x) = z(x)$ for $x \in M(H^\infty + C)$. For $\lambda \in \partial D$, we put $X_\lambda = \{x \in X;\ \pi(x) = \lambda\}$ and $M_\lambda(H^\infty + C) = \{x \in M(H^\infty + C);\ \pi(x) = \lambda\}$.

For a Douglas algebra B, the pseudo-hyperbolic distance between x and y in $M(B)$, the maximal ideal space of B, is

$$\rho_B(x, y) = \sup\{|f(y)|;\ f \in B,\ \|f\| \le 1,\ f(x) = 0\}.$$

If $x, y \in D$, then we have $\rho_{H^\infty}(x, y) = |y - x|/|1 - \bar{x}y|$. For $x \in M(B)$, $P_B(x) = \{y \in M(B);\ \rho_B(y, x) < 1\}$ is called the Gleason part of B containing x. If $P_B(x) \neq \{x\}$, $P_B(x)$ is called nontrivial. We abbreviate $\rho = \rho_{H^\infty}$ and $P = P_{H^\infty}$. For $x \in M(B)$, we denote by μ_x the unique representing measure on X for a point x, and denote by $\operatorname{supp} \mu_x$ the closed support set of μ_x.

LEMMA 2. *Let B be a Douglas algebra. If $x \in M(B)$, then $P(x) \subset M(B)$ and $P_B(x) = P(x)$.*

Proof. By the Chang–Marshall theorem,

$$B = [H^\infty, \{\bar{I};\ I \text{ is an inner function with } \bar{I} \in B\}],$$

and

$$M(B) = \{y \in M(H^\infty);\ |I(y)| = 1 \text{ for every inner function } I \text{ with } \bar{I} \in B\}.$$

Hence $M(B) = \{y \in M(H^\infty);\ B \,|\operatorname{supp} \mu_y = H^\infty \,|\operatorname{supp} \mu_y\}$. Let $x \in M(B)$. By [4, p. 143], $\operatorname{supp} \mu_x = \operatorname{supp} \mu_y$ for every $y \in P(x)$. Thus $P(x) \subset M(B)$. Since $H^\infty \subset B$, $P_B(x) \subset P(x)$. To see $P_B(x) = P(x)$, suppose $P_B(x) \subsetneq P(x)$. Then there are $y_0 \in P(x) \setminus P_B(x)$ and a sequence $\{f_n\}$ in B such that

$$f_n(x) = 0, \quad \|f_n\| = 1, \quad \text{and} \quad f_n(y_0) \to 1 \quad \text{as } n \to \infty.$$

Since $\operatorname{supp} \mu_x$ is a weak peak set for H^∞ [8, p. 207], there is a sequence $\{g_n\}$ in H^∞ [4, p. 58] such that

$$g_n(x)=0, \quad \|g_n\|=1 \quad \text{and} \quad g_n(y_0)\to 1 \quad \text{as } n\to\infty.$$

Thus $y_0 \notin P(x)$. This contradiction leads to $P_B(x)=P(x)$.

LEMMA 3 [16]. *Let I be an inner function. Then there is an interpolating Blaschke product b with $[H^\infty, \bar{b}]=[H^\infty, \bar{I}]$.*

For a positive singular measure μ with respect to $d\theta/2\pi$,

$$S(\mu)(z)=\exp\left(-\int_{\partial D}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right) \quad \text{for } z\in D$$

is called a singular inner function.

LEMMA 4. *Let $S(\mu)$ be a singular inner function and let I be a product of finitely many interpolating Blaschke products. Then $I\bar{S}(\mu)\notin H^\infty+C$.*

Proof. Let λ be a point in the support set of μ. Then there is a sequence $\{z_n\}$ in D such that $z_n\to\lambda$ and $S(\mu)(z_n)\to 0$ as $n\to\infty$. We may assume that $\{z_n\}$ is an interpolating sequence. Let x_0 be a cluster point of $\{z_n\}$ in $M(H^\infty)$. Then $x_0\notin D$. By the work of Hoffman [9], $P(x_0)$ is nontrivial, $S(\mu)=0$ on $P(x_0)$, and $I\neq 0$ on $P(x_0)$. To see $I\bar{S}(\mu)\notin H^\infty+C$, suppose $I\bar{S}(\mu)\in H^\infty+C$. Put $h=I\bar{S}(\mu)$. Then $I=hS(\mu)$. Hence $0\neq I\,|\,P(x_0)=hS(\mu)\,|\,P(x_0)=0$. This is a contradiction. □

Proof of Theorem 2. Put $\lambda_n=e^{\pi i/n}\in\partial D$. Then $\lambda_n\to 1$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive numbers with $\sum_{n=1}^\infty na_n<\infty$. Let δ_{λ_n} be the unit point mass at λ_n. Put

$$S_1=S\left(\sum_{n=1}^\infty na_n\delta_{\lambda_n}\right) \quad \text{and} \quad S_2=S\left(\sum_{n=1}^\infty a_n\delta_{\lambda_n}\right).$$

By Lemma 3, there is an interpolating Blaschke product b with zeros $\{z_n\}_{n=1}^\infty$ in D such that

$$[H^\infty,\bar{b}]=[H^\infty,\bar{S}_1]. \tag{1}$$

Then $\pi(Z(b))=\pi(Z(S_1))=\pi(Z(S_2))=\{1,\lambda_n; n=1,2,\dots\}$. We divide $\{z_n\}_{n=1}^\infty$ into countably many disjoint infinite subsets such that

$$\bigcup_{k=1}^\infty\{z_{k,n}; n=1,2,\dots\}=\{z_n\}_{n=1}^\infty, \tag{2}$$

and

$$z_{k,n}\to\lambda_k \ (n\to\infty) \text{ for each } k. \tag{3}$$

Let b_k be an interpolating Blaschke product with zeros $\{z_{k,n}\}_{n=1}^\infty$. By (2) and (3), $b=\prod_{k=1}^\infty b_k$ and $[H^\infty, \bar{S}(a_k\delta_{\lambda_k})]=[H^\infty, \bar{S}(ka_k\delta_{\lambda_k})]=[H^\infty, \bar{b}_k]$. By Lemma 4, $b_k^m\bar{S}(a_k\delta_{\lambda_k})\notin H^\infty+C$. By [14, p. 401], there is $\lambda\in\partial D$ such that $b_k^m\bar{S}(a_k\delta_{\lambda_k})\,|\,X_\lambda\notin H^\infty\,|\,X_\lambda$. Since $\bar{S}(a_k\delta_{\lambda_k})\,|\,X_\lambda$ is constant whenever $\lambda\neq\lambda_k$, we have

$$b_k^m\bar{S}(a_k\delta_{\lambda_k})\,|\,X_{\lambda_k}\notin H^\infty\,|\,X_{\lambda_k} \quad \text{for every } m \text{ and } k. \tag{4}$$

Take a sequence of positive integers $\{m_k\}_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty m_k\sum_{n=1}^\infty(1-|z_{k,n}|)<\infty,$$

$$(5) \qquad m_k \le m_{k+1}, \quad \text{and} \quad m_k \to \infty \ (k \to \infty).$$

Let ψ be a Blaschke product with zeros $\{z_{k,n};\ k,n=1,2,\dots\}$ and corresponding multiplicities $\{m_k;\ k=1,2,\dots\}$, that is, $\psi=\prod_{k=1}^{\infty} b_k^{m_k}$. Put $I=\psi S_1$ and $J=\psi S_2$. We note that $\pi(Z(I))=\pi(Z(J))=\{1,\lambda_n;\ n=1,2,\dots\}$. We shall show that I and J satisfy our assertions.

To see $Z(I)\subset Z(J)$, let $x\in M(H^\infty+C)$ with $I(x)=0$. If $\psi(x)=0$, then evidently $J(x)=0$. Suppose $S_1(x)=0$. There occur two cases.

Case 1. Suppose $\pi(x)=\lambda_n$ for some n. Then $S(na_n\delta_{\lambda_n})(x)=0$. Hence

$$S(a_n\delta_{\lambda_n})(x)=0 \quad \text{and} \quad S_2(x)=0.$$

Thus $J(x)=0$.

Case 2. Suppose $\pi(x)=1$. By (1), $|b(x)|<1$. Then we have

$$\begin{aligned}
|\psi(x)| &= \left|\left(\prod_{k=1}^{\infty} b_k^{m_k}\right)(x)\right| \\
&= \left|\left(\prod_{k=p}^{\infty} b_k^{m_k}\right)(x)\right| && \text{for every } p \text{ by (3)} \\
&\le \left|\left(\prod_{k=p}^{\infty} b_k^{m_p}\right)(x)\right| && \text{by (5)} \\
&= |b(x)|^{m_p} && \text{by (3)} \\
&\to 0 \quad (p\to\infty) && \text{by (5).}
\end{aligned}$$

Hence $\psi(x)=0$. Thus we get $Z(I)\subset Z(J)$.

To finish the proof, let N be a positive integer. Take a positive integer k as $k>N$. Then

$$\begin{aligned}
J^N\bar{I}\,|\,X_{\lambda_k} &= \psi^N S_2^N \bar{\psi}\bar{S}_1\,|\,X_{\lambda_k} \\
&= ab_k^{(N-1)m_k}\bar{S}((k-N)a_k\delta_{\lambda_n})\,|\,X_{\lambda_k} \quad \text{for some constant } a \text{ with } |a|=1 \\
&\notin H^\infty\,|\,X_{\lambda_k} \quad \text{by (4).}
\end{aligned}$$

Hence $J^N\bar{I}\notin H^\infty+C$ for every positive integer N. □

For a Douglas algebra B and $f\in B$, we put $Z_B(f)=\{x\in M(B);\ f(x)=0\}$. For a subset E of ∂D, we put $L_E^\infty=\{f\in L^\infty;\ f$ is continuous at each $x\in E\}$. By [3], $H^\infty+L_E^\infty$ is a Douglas algebra.

THEOREM 3 (cf. [7, Corollary 2]). *Let $B=H^\infty+L^\infty_{\partial D\setminus\{1\}}$. Then there is an inner function I satisfying the following conditions.*

(i) *I vanishes identically on a nontrivial Gleason part of B.*

(ii) *If $g\in B$ with $Z_B(I)\subset Z_B(g)$, then $g\bar{I}\in B$.*

To prove Theorem 3, we need some lemmas.

LEMMA 5 [9, Lemma 4.2]. *Let b be an interpolating Blaschke product with zeros $\{z_n\}_{n=1}^{\infty}$. Then there exist δ $(0<\delta<1)$ and r $(0<r<1)$ satisfying the following properties: The set $\{z\in D;\ |b(z)|<r\}$ is the union of pairwise disjoint domains V_n, $z_n\in V_n$; and $V_n\subset\{z\in D;\ \rho(z,z_n)<\delta\}$. If $|w|<r$, then $b_w(z)=$*

$(b(z)-w)/(1-\bar{w}b(z))$ is an interpolating Blaschke product having one zero in each V_n.

LEMMA 6 [1; 7]. *Let B be a Douglas algebra and I be an interpolating Blaschke product. If $g \in B$ satisfies $Z_B(I) \subset Z_B(g)$, then $g\bar{I} \in B$.*

LEMMA 7 [5, p. 314]. *Let $\{z_n\}$ and $\{w_n\}$ be interpolating sequences in D. If $\inf\{\rho(z_n, w_m);\ n, m = 1, 2, \ldots\} > 0$, then $\{z_n\} \cup \{w_m\}$ is interpolating.*

Proof of Theorem 3. Put $\lambda_n = e^{\pi i/n} \in \partial D$. It is well known that there is an interpolating Blaschke product b with zeros $\{z_n\}_{n=1}^{\infty}$ in D such that $\pi(Z(b)) = \{1, \lambda_n;\ n = 1, 2, \ldots\}$. We divide $\{z_n\}$ into countably many disjoint infinite subsets as

$$\bigcup_{k=1}^{\infty} \{z_{k,n};\ n = 1, 2, \ldots\} = \{z_n\}_{n=1}^{\infty}, \tag{1}$$

and

$$z_{k,n} \to \lambda_k \quad (n \to \infty) \quad \text{for each } k. \tag{2}$$

We may assume

$$\sum_{k=1}^{\infty} k \sum_{n=1}^{\infty} (1 - |z_{k,n}|) < \infty. \tag{3}$$

By Lemma 5, there exist δ and r $(0 < \delta < 1,\ 0 < r < 1)$ and there is a sequence of pairwise disjoint domains $\{V_{k,n};\ k, n = 1, 2, \ldots\}$ such that $z_{k,n} \in V_{k,n}$,

$$V_{k,n} \subset \{z \in D;\ \rho(z, z_{k,n}) < \delta\}, \tag{4}$$

and

$$\text{if } |w| < r, \text{ then } b_w(z) = (b(z) - w)/(1 - \bar{w}b(z)) \text{ is an interpolating Blaschke product having one zero in each } V_{k,n}. \tag{5}$$

Let $\{w_n\}$ be a distinct sequence of complex numbers with $w_1 = 0$, $|w_n| < r$, and $w_n \to 0$ $(n \to \infty)$. For i and k with $1 \le i \le k$, let $\zeta_{k,i,n}$ be the point in $V_{k,n}$ with $b_{w_i}(\zeta_{k,i,n}) = 0$. By (5), $\{\zeta_{k,i,n}\}_{n=1}^{\infty}$ is an interpolating sequence for each k and i with $1 \le i \le k$. Let i and j with $i \ne j$ and $1 \le i,\ j \le k$. Since $b_{w_i}(\zeta_{k,i,n}) = 0$, $b(\zeta_{k,i,n}) = w_i$. Hence for $n, m = 1, 2, \ldots,$

$$\begin{aligned} \rho(\zeta_{k,i,n}, \zeta_{k,j,m}) &\ge |b_{w_j}(\zeta_{k,i,n})| \\ &= \left| \frac{w_i - w_j}{1 - \bar{w}_j w_i} \right| > 0 \quad \text{by (5).} \end{aligned}$$

By Lemma 7, for each fixed k, $\{\zeta_{k,i,n};\ 1 \le i \le k,\ n = 1, 2, \ldots\}$ is an interpolating sequence. Let b_k be the interpolating Blaschke product with zeros $\{\zeta_{k,i,n};\ 1 \le i \le k,\ n = 1, 2, \ldots\}$. Since $\zeta_{k,i,n} \in V_{k,n}$, by (4) we have

$$\begin{aligned} \delta > \rho(\zeta_{k,i,n}, z_{k,n}) &= \left| \frac{z_{k,n} - \zeta_{k,i,n}}{1 - \bar{\zeta}_{k,i,n} z_{k,n}} \right| \\ &\ge \frac{|z_{k,n}| - |\zeta_{k,i,n}|}{1 - |\zeta_{k,i,n}|\,|z_{k,n}|}. \end{aligned}$$

By elementary calculations, $1-|\zeta_{k,i,n}|\leq((1+\delta)/(1-\delta))(1-|z_{k,n}|)$. Then

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum\{1-|\zeta_{k,i,n}|;\ 1\leq i\leq k,\ n=1,2,\dots\}\\
&\quad\leq\frac{1+\delta}{1-\delta}\sum_{k=1}^{\infty}\sum\{1-|z_{k,n}|;\ 1\leq i\leq k,\ n=1,2,\dots\}\\
&\quad=\frac{1+\delta}{1-\delta}\sum_{k=1}^{\infty}k\sum\{1-|z_{k,n}|;\ n=1,2,\dots\}\\
&\quad<\infty\quad\text{by (3).}
\end{aligned}$$

Hence $\prod_{k=1}^{\infty} b_k$ is a Blaschke product. Put $I=\prod_{k=1}^{\infty} b_k$ and we shall show that I is a desired inner function. By (2) and (4), $\pi(Z(I))=\{1,\lambda_n;\ n=1,2,\dots\}$. Since $w_1=0$, $I\bar{b}\in H^{\infty}$, so $Z(b)\subset Z(I)$. Take a sequence $\{x_k\}_{k=1}^{\infty}$ in $Z(b)$ with $\pi(x_k)=\lambda_k$. Then $x_k\in M(B)$. Let x be a cluster point of $\{x_k\}_{k=1}^{\infty}$. Then $x\in Z(b)$ and $\pi(x)=1$. By the work of Hoffman [9], $P(x)$ is a nontrivial Gleason part and $I=0$ on $P(x)$.

To see (i), it is sufficient to prove $x\in M(B)$. Since

$$B=[H^{\infty},\ \{\bar{J};\ J \text{ is an inner function with } Z(J)\subset M_1(H^{\infty}+C)\}],$$

we have $|J(x)|=1$ for every inner function J with $\bar{J}\in B$. Because if $|J(x)|<1$ then $|J(x_k)|<1$ for some k, so $\bar{J}\notin B$. Thus we obtain $x\in M(B)$, hence $P_B(x)=P(x)\subset M(B)$ by Lemma 2.

To see (ii), let $g\in B$ with $Z_B(g)\supset Z_B(I)$. Let $\lambda\in\partial D$ with $\lambda\neq 1$. If $\lambda\neq\lambda_n$ for every n, then $\bar{I}\,|\,X_{\lambda}$ is constant, so $g\bar{I}\,|\,X_{\lambda}\in H^{\infty}\,|\,X_{\lambda}$. Suppose that $\lambda=\lambda_n$ for some n. Then $I\,|\,X_{\lambda_n}=cb_n\,|\,X_{\lambda_n}$ for some constant c with $|c|=1$. Put

$$B_n=\{f\in L^{\infty};\ f\,|\,X_{\lambda_n}\in H^{\infty}\,|\,X_{\lambda_n}\}.$$

Then B_n is a Douglas algebra. Since $Z_{B_n}(g)\supset Z_{B_n}(I)=Z_{B_n}(b_n)$, by Lemma 6 we have $g\bar{b}_n\in B_n$. Hence $g\bar{I}\,|\,X_{\lambda_n}=cg\bar{b}_n\,|\,X_{\lambda_n}\in H^{\infty}\,|\,X_{\lambda_n}$. Thus we obtain $g\bar{I}\in B$, because $B=\{f\in L^{\infty};\ f\,|\,X_{\lambda}\in H^{\infty}\,|\,X_{\lambda}$ for every λ with $\lambda\neq 1\}$. □

In the last part of this section, we shall give some comments. Let B be a Douglas algebra with $B\supset H^{\infty}+C$. An inner function I is called B-interpolating if there is an interpolating Blaschke product b such that $|b|=|I|$ on $M(B)$. Then we have the following.

PROPOSITION. *Let I be an inner function. If I is B-interpolating, then I satisfies the following condition.*

(#) *If J is an inner function with $Z_B(I)\subset Z_B(J)$, then $|J|\leq|I|$ on $M(B)$.*

Proof. Let b be an interpolating Blaschke product with $|b|=|I|$ on $M(B)$. Then $Z_B(b)=Z_B(I)$. Let J be an inner function with $Z_B(I)\subset Z_B(J)$. By Lemma 6, $J\bar{b}\in B$. Put $h=J\bar{b}$; then h is unimodular and $J=bh$. Thus $|J|\leq|b|\,|h|\leq|b|=|I|$ on $M(B)$. □

Theorem 3 says that the converse of this proposition is not true for $B=H^{\infty}+L^{\infty}_{\partial D\setminus\{1\}}$. To see this, let I be an inner function in Theorem 3. By (i) of Theorem 3, I is not B-interpolating. By (ii) of Theorem 3, I satisfies (#).

We do not know whether the converse of the proposition is true or not for $B=H^\infty+C$. We note that by [10, Theorem 1], if I is an $(H^\infty+C)$-interpolating inner function then $I=b_0b_1$, where b_0 is a finite Blaschke product and b_1 is an interpolating Blaschke product.

3. Factorization of Blaschke products. Let $h\in H^\infty$ and $x\in M(H^\infty+C)$ with $h(x)=0$. If $P(x)$ is a nontrivial Gleason part, $P(x)$ carries the structure of an analytic disk, so we can define $\mathrm{Ord}_h(x)$, the order of zero of h at x. If $P(x)$ is trivial, we define $\mathrm{Ord}_h(x)=\infty$.

THEOREM 4. *Let I be an inner function with* $\mathrm{Ord}_I(x)=k$ *for every* $x\in Z(I)$. *Then there are interpolating Blaschke products* $\{b_n\}_{n=1}^k$ *and a finite Blaschke product* b_0 *such that* $I=\prod_{n=0}^k b_n$ *and* $Z(I)=Z(b_n)$ *for* $1\le n\le k$.

To see our theorem, we need two lemmas.

LEMMA 8. *Let b be an inner function, let* $\{x_j\}_{j=1}^\infty$ *be a sequence in* $Z(b)$ *such that* $\mathrm{Ord}_b(x_j)=n$ *for every j, and let x_0 be a cluster point of* $\{x_j\}_{j=1}^\infty$ *in* $M(H^\infty+C)$. *If* $\{y_j\}_{j=1}^\infty$ *is a sequence in* $Z(b)$ *such that* $\{x_j\}_{j=1}^\infty\cap\{y_j\}_{j=1}^\infty=\phi$, $\mathrm{Ord}_b(y_j)=m$ *for every j, and* $\rho(x_j,y_j)\to 0$ *as* $j\to\infty$, *then* $\mathrm{Ord}_b(x_0)\ge n+m$.

Proof. It is enough to prove this for the case $\mathrm{Ord}_b(x_0)<\infty$. Let $k=\mathrm{Ord}_b(x_0)$. By [9, Theorem 5.3], there are interpolating Blaschke products $b_1,\dots,b_k$ and an inner function b_0 such that $b=\prod_{j=0}^k b_j$ and $b_j(x_0)=0$ for $1\le j\le k$. Since x_0 is a cluster point of $\{x_j\}_{j=1}^\infty$, we may assume that $b_0(x_j)\ne 0$ for any j. Moreover, since $\rho(x_j,y_j)\to 0$ as $j\to\infty$, we may assume that $b_0(y_j)\ne 0$ for any j.

For each j, $Z(b_j)$ is an interpolation set for H^∞ [8, p. 205]; thus there exists $\epsilon_j>0$ such that $\rho(x,y)\ge\epsilon_j$ for any $x,y\in Z(b_j)$. Let $\epsilon=\min_{1\le j\le k}\epsilon_j$. Without loss of generality we may assume that $\rho(x_j,y_j)\le\epsilon/2$ for all j. By assumption, b has a zero of order n at x_j. Since $b_0(x_j)\ne 0$, there exist n factors of b other than b_0 which vanish at x_j. None of these factors can vanish at y_j, for $x,y\in Z(b_\ell)$ forces $\rho(x,y)\ge\epsilon_\ell$ and $\rho(x_j,y_j)\le\epsilon/2\le\epsilon_\ell/2$. Hence m factors of b distinct from these and b_0 must vanish at y_j. Therefore $k\ge n+m$. □

LEMMA 9 [10]. *If I is an inner function such that $Z(I)$ is an interpolation set for H^∞, then there is an interpolating Blaschke product b such that $I\bar b\in H^\infty$ and* $Z(I)=Z(b)$.

Proof of Theorem 4. Let I be an inner function with $\mathrm{Ord}_I(x)=k$ for every $x\in Z(I)$. By [12], there are interpolating Blaschke products $\{\psi_1,\psi_2,\dots,\psi_n\}$ and a finite Blaschke product ψ_0 such that $I=\prod_{i=0}^n\psi_i$. By Lemma 8, there is $\epsilon>0$ such that $\rho(x,y)\ge\epsilon$ for every $x,y\in Z(I)$ with $x\ne y$. By Varopoulos [15], $Z(I)=\bigcup_{i=1}^n Z(\psi_i)$ is an interpolation set for H^∞. By Lemma 9, there is an interpolating Blaschke product b_1 such that $I\bar b_1\in H^\infty$ and $Z(b_1)=Z(I)$. Applying Lemma 9 k times, there are interpolating Blaschke products $\{b_1, b_2, \dots, b_k\}$ such that $I\bar b_1\bar b_2\cdots\bar b_j\in H^\infty$ and $Z(b_j)=Z(I)$ for $1\le j\le k$. Since $Z(I\prod_{j=1}^k\bar b_j)=\phi$, $b_0=I\prod_{j=1}^k\bar b_j$ is a finite Blaschke product. This completes the proof. □

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