# WILD SURFACES HAVE SOME NICE PROPERTIES

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This paper was stimulated by hearing in lectures such remarks as the following: "We consider only smoothable surfaces in 3-manifolds so that we can suppose that their intersections with the 2-skeletons of triangulations are nice." This led me to wonder if we might not make the intersections nice even if the surfaces were not smoothable but wild. See Theorem V.3.

I would prefer to give complete proofs of the results in this paper so that the results would be believed even by mathematical agnostics—those who doubt things for which they do not have complete proofs. Complete proofs are given here for the theorems in the main section (§V) of this paper but the proofs of the preliminary theorems come mostly from the literature. Section V deals with the intersection of wild surfaces in triangulated 3-manifolds with other objects in these 3-manifolds, and the preceding sections deal with theorems related to this treatment.

Throughout this paper we deal with 3-manifolds without boundary. *Surfaces* are 2-manifolds imbedded as closed sets in 3-manifolds. Although there are related results about 3-manifolds with boundaries and surfaces without boundaries, they are not treated here. We do not suppose that manifolds and surfaces are compact. Throughout this paper we use $M^3$ to denote a triangulated 3-manifold with metric $\rho$.

**I. Pushing tame sets to polyhedral ones.** One of the important results of the fifties shows that homeomorphisms of 3-manifolds are not wild. A result [10, Theorem 2; 2, Theorem 9] may be stated as follows. We use $M_0^3$ to denote a triangulated 3-manifold (perhaps different from $M^3$).

THEOREM I.1. *Suppose*
*$U$ is an open subset of $M_0^3$,*
*$h$ is a homeomorphism of $U$ into $M^3$, and*
*$\epsilon(x)$ is a positive continuous function defined on $U$.*
*Then there is a homeomorphism $g: U \to h(U)$ such that*
*$g$ is* PL *and*
*$\rho(g(x), h(x)) < \epsilon(x)$ for each $x \in U$.*

At the expense of complicating the statement of Theorem I.1 we could have added that if $C$ is a closed subset of $U$ on which $h$ is locally PL, there is such a $g$ that agrees with $h$ on $C$.

QUESTION. Suppose $U, h$ are as given in Theorem I.1. Is there an isotopy $H_t$ $(0 \le t \le 1)$ of $U$ onto $h(U)$ such that $h = H_0$ and each $H_s$ is polyhedral for $s \in (0, 1]$? It would be especially nice to get an $H_t$ that connects $h$ with a nearby $g$ in

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some canonical way—or even if $g_1, g_2$ are two close approximations to $h$, to isotope $g_1$ to $g_2$ in some canonical way.

Suppose $T$ is a triangulation of $M^3$. A surface $S$ in $M^3$ is *polyhedral* if it is the union of 2-simplexes of some subdivision of $T$. It is *tame* if there is a homeomorphism $h$ of $M^3$ onto itself such that $h(S)$ is polyhedral. We say that $S$ *has a Cartesian product neighborhood* if for some neighborhood $V$ of $S$ there is a homeomorphism $h: (S\times R^1, S\times 0)\to (V, S)$.

A surface $S$ in $M^3$ is called *2-sided* if it separates some connected open subset of $M^3$ containing $S$. It is known that any 2-sphere in $M^3$ is 2-sided and any surface $S$ in $R^3$ is 2-sided in $R^3$.

THEOREM I.2. *Each 2-sided tame surface in $M^3$ has a Cartesian product neighborhood.*

Theorem I.2 has the following converse.

THEOREM I.3. *A surface $S$ in $M^3$ is tame if it has a Cartesian product neighborhood.*

The proof comes from giving $S$ a triangulation $T(S)$ and extending $T(S)$ to a prismatic triangulation $T_0$ of $S\times R^1 = M_0$. Use Theorem I.1 to get a homeomorphism $g$ of $S\times R^1$ onto itself that is PL with respect to $T_0$ on $S\times(0,1)=U$ and the identity on $M_0-U$. Let $f$ be a homeomorphism of $S\times R^1$ onto itself that is fixed outside $S\times(-1,1)$ and shoves $S\times 0$ onto a surface in $S\times(0,1)$ that is polyhedral with respect to $T_0$. The homeomorphism that is $gfg^{-1}$ on $S\times R^1$ and the identity on $M^3-[S\times R^1]$ shows that $S$ is tame. We have pushed $S$ to the side. $\square$

A surface $S$ is *locally polyhedral* at a point $s\in S$ if there is a polyhedral disk $D$ in $S$ with $s\in \operatorname{Int} D$. Also $S$ is *locally tame* at a point $s\in S$ if there is a homeomorphism $H: M^3\to M^3$ such that $h(S)$ is locally polyhedral at $h(s)$.

In our discussion of the proof of Theorem I.3 we could have considered pushing only a part of $S$. The next result follows.

THEOREM I.4. *Suppose*

*$S$ is a 2-sided surface in $M^3$,*

*$U$ is an open subset of $S$ on which $S$ is locally tame, and*

*$\epsilon(x)$ is a continuous non-negative function defined on $M^3$ that is positive on $U$.*

*Then there is an isotopy $H_t$ $(0\le t\le 1)$ of $M^3$ onto itself such that*

*$H_0$ is the identity,*

*each $\rho(x, H_t(x))\le \epsilon(x)$,*

*$H_t(S)$ is locally polyhedral at $H_t(s)$ if $t>0$, $s\in U$, and*

*$H_{t_1}(s_1)\ne H_{t_2}(s_2)$ for $t_1\ne t_2$, $s_1\in U$, and $s_2\in S$.*

We say that a surface $S$ in $M^3$ is *bicollared* if it has a Cartesian product neighborhood. It is *locally bicollared* if for each point $p\in S$ there is a neighborhood $N$

of $p$ in $M^3$ such that $S \cap N$ is bicollared in $N$. Theorems I.2 and I.3 can be extended as follows.

THEOREM I.5. *A surface in $M^3$ is tame if and only if it is locally bicollared.*

QUESTION. If one omits the requirement in Theorem I.4 that $S$ be 2-sided, can one get an $H_t$ satisfying all but the last of the four restrictions?

**II. Recognizing tame surfaces.** Theorem I.5 gives a criteria for recognizing tame surfaces but we consider others. We do not give the proof of Theorem II.1 here but mention that one is found in [3, Theorem 2.2].

THEOREM II.1. *A 2-sphere $S$ in $R^3$ is tame if for each component $U$ of $R^3 - S$ there is a sequence $S_1, S_2, \ldots$ of 2-spheres in $U$ and homeomorphisms $h_1: S \to S_1$, $h_2: S \to S_2, \ldots$ such that each $\rho(x, h_i(x)) < 1/i$.*

We say that the $S$ above can be homeomorphically approximated from both sides.

QUESTION. The following question [9, p. 280] has received considerable attention. Is a 2-sphere $S$ in $R^3$ tame if for each component $U$ of $R^3 - S$ and each positive integer $i$ there is a map $g_i$ of $S$ into $U$ such that $\rho(s, g_i(s)) < 1/i$?

Note that in the above we hypothesized maps rather than homeomorphisms as hypothesized in Theorem II.1. However, it is known that $S$ is tame if it can be homotoped to either side as described in Theorem II.2. See [9, Theorem 1].

THEOREM II.2. *A 2-sphere $S$ in $R^3$ is tame if for each component $U$ of $R^3 - S$ there is a map $H$ of $S \times [0, 1]$ into $R^3$ such that for each $s \in S$*

*$H(s \times 0) = s$, and*

*$H(s \times t) \subset U$ for $t > 0$.*

A set $X$ is 1-ULC if for each $\epsilon > 0$ there is a $\delta > 0$ such that each map of Bd $D^2$ into a subset of $X$ of diameter less than $\delta$ can be extended to send $D^2$ into a subset of $X$ of diameter less than $\epsilon$. It is 1-LC at a point $p \in \bar{X}$ if for each neighborhood $N_1$ of $p$ there is a neighborhood $N_2$ of $p$ such that each map of Bd $D^2$ into $X \cap N_2$ can be extended to map $D^2$ into $X \cap N_1$. A useful result that is proved in [4, Theorem 7] but not here is the following.

THEOREM II.3. *A surface $S$ in $M^3$ is tame if $M^3 - S$ is* 1-LC *at each point of $S$.*

THEOREM II.4. *A 2-sphere $S^2$ in $R^3$ is tame if each component of $R^3 - S$* is 1-ULC [4, Theorem 2].

**III. Approximating surfaces.** The following is a much-used theorem about approximating surfaces. See [1, Theorem 1].

THEOREM III.1. *Suppose $\epsilon(x)$ is a non-negative continuous function defined on a surface $S$ in $M^3$. Then there is a homeomorphism $h$ of $S$ into $M^3$ such that*

*$h(S)$ is locally polyhedral at $h(s)$ if $\epsilon(s) > 0$, and*

*$\rho(s, h(s)) \leq \epsilon(s)$.*

In working with the local structure of a surface $S$ it sometimes suffices to deal with 2-spheres as shown by the following application of Theorem III.1. See [4, Theorem 5].

THEOREM III.2. *If $p$ is a point of a surface $S$ in $M^3$ and $\epsilon > 0$, then there is a disk $D^2$ in $S$ such that*

*$P \in \operatorname{Int} S$, and*

*$D^2$ lies in a 2-sphere $K^2$ in $M^3$ where diameter $K^2 < \epsilon$ and $K^2$ is locally polyhedral on $K^2 - D^2$.*

An extension of Theorem III.1 that has had far-reaching applications is the following.

THEOREM III.3 (Side Approximation Theorem). *Suppose $S$ is a 2-sphere in $R^3$ and $\epsilon > 0$. Then there is a homeomorphism $h$ of $S \times [-1, 1]$ into $R^3$ such that*

*each component of $S - h(S \times [-1, 1])$ has diameter less than $\epsilon$,*

*each $h(S \times t)$ is polyhedral, and*

*each $\rho(s, h(s \times t)) < \epsilon$.*

We say that $h(S \times (-1))$ and $h(S \times 1)$ are approximating $S$ almost from the side — in fact from different sides.

The known proofs of Theorem IV.3 are difficult and will not be given here. One such proof is given in Chapter XIII of [7].

**IV. Tame Sierpinski curves.** A sequence of sets $X_1, X_2, \ldots$ is called a *null sequence* if for each $\epsilon > 0$ there are at most finitely many $X_i$'s with diameters more than $\epsilon$. Let $D_1, D_2, \ldots$ be a null sequence of mutually disjoint disks on a 2-sphere $S^2$ such that their union is dense in $S^2$. A *Sierpinski curve* is any set homeomorphic to $S^2 - \bigcup \operatorname{Int} D_i$. Some people regard it as looking like a thin slice of cheese. A disk $D$ can be changed to a Sierpinski curve by removing from it the interiors of a null sequence of mutually disjoint disks whose union is dense in $D$.

It may be shown that the topology of a Sierpinski curve does not depend on the null sequence of disks whose interiors are removed from $S^2$. A point of a Sierpinski curve $Y$ is called an inaccessible point if it is the image of a point of $S^2 - \bigcup D_i$ under the homeomorphism of $S^2 - \bigcup \operatorname{Int} D_i$ onto $Y$. It follows from Moore's decomposition theorem [11] for a 2-sphere that the set of inaccessible points of $Y$ is homeomorphic to $S^2 - Z$ where $Z$ is a countable dense set of points in $S^2$. We use $I(Y)$ and $A(Y)$ to denote (respectively) the set of inaccessible points and the set of accessible points of $Y$.

A Sierpinski curve $Y$ in $M^3$ is called tame if it lies on a tame 2-sphere in $M^3$. This is not to say that the 2-sphere $S^2$ used to define $Y$ is tame or even that it lies in $M^3$. However, if $Y$ is tame in $M^3$, it lies on a tame 2-sphere $S_0^2$ in $M^3$ and there is a null sequence of mutually disjoint disks $E_1, E_2, \ldots$ on $S_0^2$ such that $S_0^2 - \bigcup \operatorname{Int} E_i = Y$.

An isotopy $H_t$ $(0 \le t \le 1)$ is a $\epsilon$-isotopy if $H_0$ is the identity and each

$$\rho(H_0, H_t) \le \epsilon.$$

The following result is known.

THEOREM IV.1. *Suppose $Y_1, Y_2$ are Sierpinski curves in the same disk $D$ such that each $Y_i$ contains* Bd $D$. *Then there is an isotopy $H_t$ $(0 \leq t \leq 1)$ of $D$ onto itself such that*

*$H_0$ is the identity,*
*each $H_t$ is fixed on* Bd $D$, *and*
$H_1(Y_1) = Y_2$.

*If $\epsilon$ is a positive number such that each component of $D - Y_1$ or of $D - Y_2$ has diameter less than $\epsilon/2$, then we can pick such an $H$ that is an $\epsilon$-isotopy.*

If $C$ is a closed set on the interior of a disk $D$ which does not contain an open subset of $D$, there is a Sierpinski curve $Y$ in $D$ such that Bd $D \subset Y$ and $C$ belongs to the inaccessible part of $Y$. Hence Theorem IV.1 provides us with an engulfing theorem that permits a Sierpinski curve to gobble up certain closed sets. See the Lemma in [6, p. 156].

THEOREM IV.2. *Suppose*

*$D$ is a disk,*
$\epsilon > 0$,
*$Y$ is a Sierpinski curve in $D$ containing* Bd $D$ *such that each component of $D - Y$ has diameter less than $\epsilon$, and*
*$C$ is a nowhere dense closed set in $D$.*

*Then there is an $\epsilon$-isotopy $H_t$ $(0 \leq t \leq 1)$ of $D$ onto itself such that*

*each $H_t$ is fixed on* Bd $D$, *and*
$C \cap$ Int $D$ *lies in the inaccessible part of $H_1(Y)$.*

We shall apply Theorem IV.2 in the case where $D$ is a tame disk in $M^3$. If $N$ is an open subset of $M^3$ containing Int $D$ we can use the Cartesian product structure on Int $D$ to feather $H_t$ $(0 \leq t \leq 1)$ out into $M^3$ so that each of the extended $H_t$'s is fixed off $N$.

We make use of the following result, whose proof is given in [5, Theorem 8.2].

THEOREM IV.3. *A 2-sphere in $M^3$ is tame if it is locally tame mod a tame Sierpinski curve.*

Suppose $Y$ is a tame Sierpinski curve in $M^3$ in a possibly wild 2-sphere $S^2$ in $M^3$ and $\epsilon(x)$ is a continuous function on $S^2$ that is 0 on $Y$ and positive on $S^2 - Y$. A combination of Theorems IV.3 and III.1 enables us to know that there is a homeomorphism $h$ of $S^2$ into $M^3$ such that $h$ is fixed on $Y$, $h(S^2)$ is locally polyhedral mod $Y$, and $\rho(x, h(x)) \leq \epsilon(x)$. The holes in $S - Y$ correspond to those in $h(S) - Y$, and corresponding holes have the same boundaries and are close.

The Side Approximation Theorem (Theorem III.3) can be used to show that even wild 2-spheres in $M^3$ contain tame Sierpinski curves. The result is stated as follows and the proof is given in [5, Theorem 9.1].

THEOREM IV.4. *If $S^2$ is a 2-sphere (possibly wild) in $M^3$, then there is a sequence $Y_1, Y_2, \ldots$ of tame Sierpinski curves in $S^2$ such that*

*each component of $S^2 - Y_i$ has diameter less than $1/i$, and*
*each $Y_i$ lies in the inaccessible part of $Y_{i+1}$.*

Theorems IV.4, IV.3, and III.1 imply that any wild 2-sphere in $M^3$ can be obtained by starting with a tame 2-sphere, getting a Sierpinski curve in it with small holes, and filling in the small holes with small disks (possibly wild).

**V. Moving wild spheres.** A theorem is now given to introduce methods we shall use to ambiently isotope wild surfaces in $M^3$ so as to simplify the intersection of the moved surfaces with other objects.

THEOREM V.1. *Suppose*
*$S^2$ is a 2-sphere (possibly wild) in $R^3$,*
*$P$ is a plane in $R^3$, and*
$\epsilon > 0$.
*Then there is an $\epsilon$-isotopy $H_t$ $(0 \leq t \leq 1)$ of $R^3$ onto itself such that*
*$P \cap H_t(S^2)$ has only a finite number of components with diameters more than $\epsilon$, and*
*each component of $P \cap H_t(S^2)$ with diameter more than $\epsilon$ is a polygonal simply closed curve.*

*Proof.* It follows from Theorem IV.4 that there is a tame Sierpinski curve $Y$ in $S^2$ such that each component of $S^2 - Y$ has diameter less than $\epsilon/n$, where $n$ is to be selected later. Change $S^2$ to a tame 2-sphere $S_0^2$ by using Theorems III.1 and IV.3 to replace the components of $S^2 - Y$ so that the components of $S_0^2 - Y$ have diameters less than $\epsilon/n$.

Since $S_0^2$ is tame, it follows from Theorem I.4 that there is an $(\epsilon/n)$-isotopy $F_t$ $(0 \leq t \leq 1)$ such that $F_1(S_0^2)$ is polyhedral and in general position with respect to $P$. The components of $F_1(S_0^2 - Y)$ have diameters less than $3\epsilon/n$. The components of $P \cap F_1(S_0^2)$ are polygonal simple closed curves.

It follows from Theorem IV.2 that there is a $(3\epsilon/n)$-isotopy $G_t$ $(0 \leq t \leq 1)$ of $F_1(S_0^2)$ onto itself such that $P \cap G_1 F_1(S_0^2)$ lies in the inaccessible part of $G_1 F_1(Y)$. Without changing its name, we suppose $G_t$ $(0 \leq t \leq 1)$ is extended to a $(3\epsilon/n)$-isotopy on $R^3$ that is invariant on $F_1(S_0^2)$. The isotopy $H$ obtained by following $F$ with $G$ satisfies the conditions of the theorem for a suitably chosen $n$.

Now to pick $n$. Since $F$ is an $(\epsilon/n)$-isotopy and $G$ is a $(3\epsilon/n)$-isotopy, $H$ is a $(4\epsilon/n)$-isotopy. Since components of $S^2 - Y$ have diameters less than $\epsilon/n$, the diameters of the images of these components under $H_1$ are less than $9\epsilon/n$. Hence we pick $n = 9$.

The 2-sphere $S$ is the union of the accessible part of $Y$, the inaccessible part of $Y$, and $S - Y$. The image under $H_1$ of the accessible part of $Y$ misses $P$ since $P \cap H_1(S^2)$ lies in the inaccessible part of $H_1(Y)$. The images under $H_1$ of the components of $S - Y$ have diameters less than $\epsilon$. □

We now extend Theorem V.1 by making repeated applications of the techniques of its proof. Theorem V.2 is a variation of Theorem 1 of [6].

THEOREM V.2. *Under the hypotheses of Theorem* V.1 *there is an $\epsilon$-isotopy $H_t$ $(0 \leq t \leq 1)$ of $R^3$ onto itself such that*

*each $H_t$ is fixed outside the $\epsilon$-neighborhood of $P \cap S^2$;*
*each nondegenerate component of $P \cap H_1(S^2)$ is a polygonal simple closed curve;*
*for each $\delta > 0$, at most a finite number of components of $P \cap H_1(S^2)$ have diameters of more than $\delta$;*
*each arc in P that pierces $P \cap H_1(S^2)$ also pierces $H_1(S^2)$; and*
*each point of $P \cap H_1(S^2)$ is a limit point of the union of the nondegenerate components of $P \cap H_1(S^2)$.*

*Proof.* We shall pick a sequence $\epsilon_1, Y_1, g_1 h_1, \epsilon_2, Y_2, g_2 h_2, \epsilon_3, \ldots$, but an element of the sequence is not selected until the previous ones have been used. First we stipulate that $\epsilon_1 = \epsilon/2$ and although we do not pick the other $\epsilon_i$'s now, we remark that they will be selected so that $\sum \epsilon_i < \epsilon$.

Let $Y_1$ be a tame Sierpinski curve $S^2$ such that each component of $S^2 - Y_1$ has diameter less than $\epsilon_1/9$.

Pick $h_1$ and $g_1$ as we picked $F_1$ and $G_1$ in the proof of Theorem V.1, subject to the restrictions that the isotopies leading to $h_1$ and $g_1$ are fixed outside the $\epsilon_1$-neighborhood of $P \cap S^2$. Rather than $g_1 h_1(S_0^2)$ being polyhedral everywhere we only need that it be polyhedral near $P$.

In selecting $\epsilon_2$, we want to start taking precautions to make the limit of the $g_1 h_1$, $g_2 h_2 g_1 h_1$, $g_3 h_3 g_2 h_2 g_1 h_1$, ... be a homeomorphism rather than merely a map. We want to prevent two points from being moved to the same point. We pick $\epsilon_2$ to be less than $\epsilon_1/2$ and also subject to the further restriction that if $p, q$ are points of $R^3$ with $\rho(p,q) > \epsilon_1$, then $\rho(g_1 h_1(p), g_1 h_1(q)) > \epsilon_2$.

Pick $Y_2$ so that the components of $g_1 h_1(S^2 - Y_2)$ have diameters less than $\epsilon_2/9$ and so that $Y_1$ lies in the inaccessible part of $Y_2$. In defining $g_2 h_2$ we use $g_1 h_1(S^2)$ and $g_1 h_1(Y_2)$ instead of $S^2$ and $Y_1$. The accessible part of $Y_1$ was sent by $g_1 h_1$ into the complement of $P$ so we pick $g_2 h_2$ to be the identity on $g_1 h_1(Y_1)$.

We follow this pattern to define $\epsilon_1$, $Y_1$, $g_1 h_1$, $\epsilon_2$, $Y_2$, $g_2 h_2$, $\epsilon_3$, ... and note that $H_1 = \operatorname{limit}(g_1 h_1, g_2 h_2 g_1 h_1, \ldots)$ satisfies most of the requirements of the theorem.

We now examine the condition on piercing. If an arc $A$ in $P$ pierces (in $P$) one of the polygonal arcs in $P \cap g_1 h_1(Y_1)$ at a point $p$, it pierces $g_1 h_1(S_0^2)$ at $p$, where $S_0^2$ is a replacement for $S^2$ that is locally tame near $P$. Since $p$ is an inaccessible point of $g_1 h_1(Y_1)$, there are small simple closed curves in $g_1 h_1(Y_1)$ that circle $p$ in $g_1 h_1(S_0^2)$. Hence they circle $A$. These simple closed curves are not moved further since $H_1 = g_1 h_1$ on $Y_1$. Hence, if $A$ intersects $H_1(S^2)$ only at $p$, $A$ pierces $H_1(S^2)$. Similarly, if an arc in $P$ pierces (in $P$) one of the polygonal arcs in $P \cap g_i h_i \cdots g_2 h_2 g_1 h_1(Y_i)$ at a point $p$, then the arc pierces $H_1(S^2)$ at $p$ if it does not intersect $H_1(S^2)$ at any other point.

If there are points of $P \cap H_1(S^2)$ that are not limit points of the union of the nondegenerate components of $P \cap H_1(S^2)$, $H_1(S)$ may be pushed to the side of $P$ near these points. This push does not alter the piercing. $\square$

We now consider how to move a 2-sphere $S^2$ in $R^3$ to improve its intersection with the 2-skeleton $T^2$ of a triangulation $T$ of $R^3$. To begin with, $S^2$ might inter-

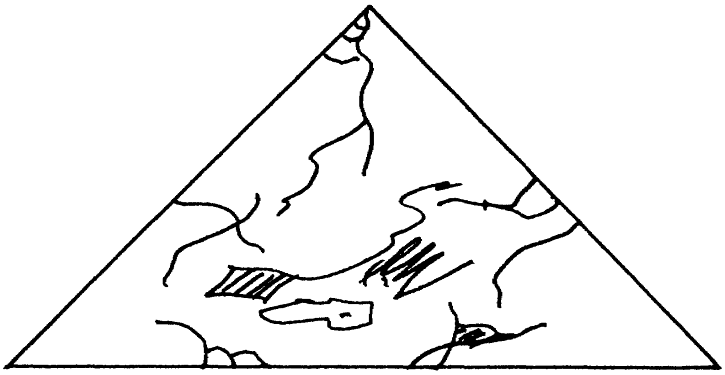

**Figure 1**

sect $T^2$ in a terribly looking set. In fact, if $\Delta^2$ is a 2-simplex of $T$, $T \cap \Delta^2$ might look even worse than shown in Figure 1.

For each $\epsilon > 0$, $S^2$ can be $\epsilon$-approximated by a polyhedral 2-sphere. Hence, there is an $\epsilon$-homeomorphism $g: S^2 \to R^3$ such that $g(S^2)$ is polyhedral. Then $\Delta^2 \cap g(S^2)$ might look like Figure 2.

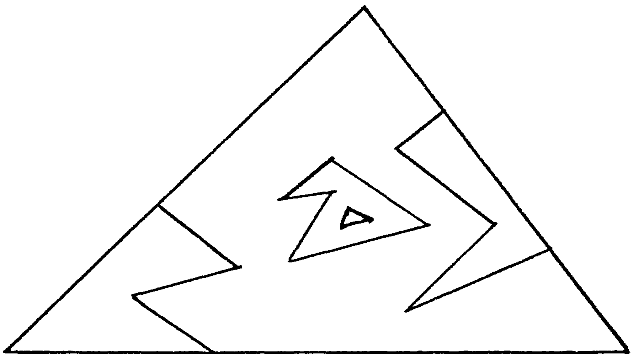

**Figure 2**

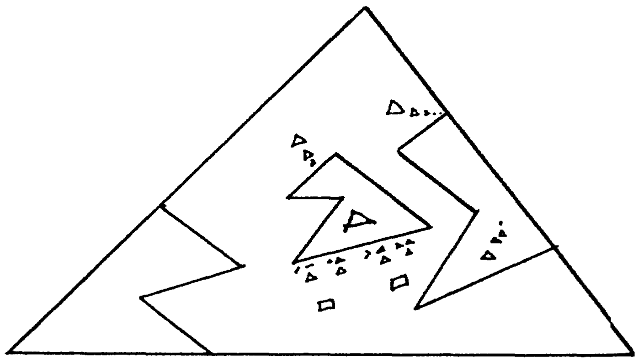

**Figure 3**

However, $g$ is not an ambient homeomorphism and its domain is $S^2$ rather than $R^3$. The following theorem shows that there is an $\epsilon$-homeomorphism $H_1: R^3 \to R^3$ such that $\Delta^2 \cap H_1(S^2)$ might look like Figure 3.

THEOREM V.3. *Suppose*

*$T$ is a triangulation of $R^3$,*

*$S^2$ is a 2-sphere (possibly wild) in $R^3$, and*

$\epsilon > 0$.

*Then there is an $\epsilon$-isotopy $H_t$ $(0 \le t \le 1)$ of $R^3$ onto itself such that:*

*each $H_t$ is fixed outside the $\epsilon$-neighborhood of $S^2 \cap T^2$;*

*$H_1(S^2)$ misses each vertex of $T$;*

*if $\Delta^1$ is a 1-simplex of $T$, then*

*$\Delta^1 \cap H_1(S^2)$ is finite and*

*$\Delta^1$ pierces $H_1(S^2)$ at each point of $\Delta^1 \cap H_1(S^2)$; and*

*if* $\Delta^2$ *is a 2-simplex of* $T$, *then*

*each nondegenerate component of* $\Delta^2 \cap H_1(S^2)$ *is either a polygonal spanning arc of* $\Delta^2$ *or a polygonal simple closed curve in* Int $\Delta^2$,

*any arc in* $\Delta^2$ *that pierces* $\Delta^2 \cap H_1(S^2)$ *in* $\Delta^2$ *also pierces* $H_1(S^2)$,

*for each* $\delta > 0$, *at most finitely many components of* $\Delta^2 \cap H_1(S^2)$ *have diameters more than* $\delta$; *and*

*each degenerate component of* $\Delta^2 \cap H_1(S^2)$ *lies in* Int $\Delta^2$ *and is a limit point of the union of the nondegenerate components.*

*Proof.* It is easy to ambiently move $S^2$ so that the adjusted $S^2$ misses $T^0$, but getting an adjustment that is nice near $T^1$ is harder. If one blindly follows the techniques of the proof of Theorem V.2 and uses $T^2$ instead of $P$, one gets an $H$ such that $T^2 \cap H_1(S^2)$ is somewhat nice except near $T^1$—but there is a problem in making it nice there. Even if one follows [6, Theorem 2] and first gets an $H_t$ $(0 \le t \le 1/2)$ so that $T^0 \cap H_{1/2}(S^2) = 0$ and $T^1 \cap H_{1/2}(S^2)$ is finite and then continues the isotopy following techniques of the proof of Theorem V.2 (but without moving $T_1$ further), one gets an $H$ such that $T^1 \cap H_1(S^2)$ is finite; but there is no assurance that, for some 2-simplex $\Delta$ of $T$, $\Delta \cap H_1(S^2)$ does not contain several arcs whose intersection is a point of Bd $\Delta$.

To avoid such difficulties we proceed as follows. First, suppose without loss of generality that $T^0 \cap S^2 = 0$ and $T^1 \cap S^2$ is of dimension 0. We then model a proof after that of Theorem V.2 by picking $\epsilon_1, Y_1, g_1 h_1, \epsilon_2, \ldots$ but with restrictions. We pick $\epsilon_1$ even smaller than $\epsilon/2$ but leave that detail until later. Then we select $Y_1$, $S_0^2$, and $g_1 h_1$ so that $h_1(S_0^2)$ is locally polyhedral near $T^2$ and in general position with respect to $T^2$. An additional restriction that we impose is that $T^1 \cap h_1(S^2) \subset h_1(I(Y_1))$. Before showing why this restriction is possible, we show how to proceed if it is possible.

Pick $g_1$ to be invariant on $h_1(S_0^2)$ and so that $T_2 \cap g_1 h_1(S_0^2)$ lies in the image under $g_1 h_1$ of the inaccessible part of $Y_1$ while $g_1 h_1(S^2 - Y_1)$ misses $T^1$.

The homeomorphism $g_1 h_1$ defines $H_1 = \text{limit}(g_1 h_1, g_2 h_2 g_1 h_1, \ldots)$ on $Y_1$, and it only remains to define $H_1$ near the components of $g_1 h_1(S^2 - Y_1)$. The closures of each of these components is a disk $D$ such that $T^1 \cap D = \varnothing$ and $T^2 \cap \text{Bd}\, D = \varnothing$. We pick an open set $N(D)$ about Int $D$ so that $T^1 \cap \bar{N}(D) = \varnothing$ and pick the $g_i h_i$'s so that $H_1(D) \subset \bar{N}(D)$.

Finally, we show how to get $h_1$ so that $h_1(S^2 - Y_1)$ does not intersect $T^1$. Let $R_1, R_2, \ldots, R_n$ be a finite number of mutually disjoint rectangular planar disks each of diameter less than $\epsilon/2$, and such that no vertex of $T$ lies on any $R_i$, $T^1 \cap S^2 \subset \bigcup \text{Int}\, R_i$, and $T^1 \cap R_i = p_i q_i$ is a bisector of $R_i$ perpendicular to two edges $E_{i_1}, E_{i_2}$ of $R_i$ (where $S^2 \cap (E_{i_1} \cup E_{i_2}) = \varnothing$). See Figure 4. Let $\delta_i$ be the minimum of the length of $E_{i_1}$ and $\rho(S^2, E_{i_1} \cup E_{i_2})$. Let $\epsilon_1$ be less than any $\delta_i$.

Now pick a preliminary $h_1$ (which we call $k$) so that $k(S_0^2)$ is locally polyhedral and in general position not only near $T^2$ but also near each $R_i$. We pick it in much the same way that we picked $g_1 h_1$ in the proof of Theorem V.2, but the components of $k(S^2 - Y_1)$ have diameters less than $\epsilon_1$ rather than less than $\epsilon/4$. We do not yet suppose that $k(S^2 - Y_1)$ misses $T^1$. When we modify $k$ to get this, we will designate the modified $k$ by $h_1$.

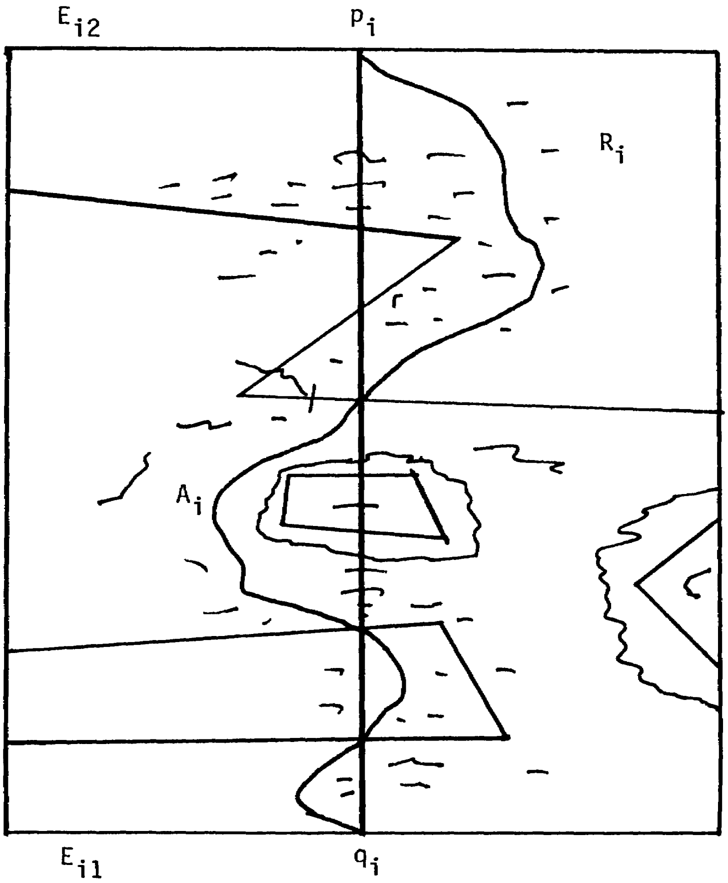


**Figure 4**

Figure 4 shows $R_i \cap k(S_0^2)$ as polygonal and $R_i \cap k(S^2 - Y_1)$ as the union of squiggles. Since these squiggles lie in $k(S^2 - Y_1)$ and $R_i \cap k(S^2)$ misses $k(A(Y_1))$, then if $p_1, p_2, \ldots$ is a sequence of points of $R_i \cap k(S^2 - Y_1)$ converging to a point of $R_i \cap k(S_0^2)$ and $Z_i$ is a squiggle containing $p_i$, the diameters of the $Z_i$'s converge to 0. Consider the decomposition of $R_i$ whose nondegenerate elements are squiggles with holes filled. It follows from [10] that there is a spanning arc $A_i$ of $R_i$ from $p_i$ to $q_i$ that misses each squiggle and each vertex of $R_i \cap k(S_0^2)$ and is such that $A_i \cap k(S_0^2)$ is finite and pierces $k(S_0^2)$ at each point of $A_i \cap k(S_0^2)$. Since there is an isotopy of $R_i$ onto itself that is fixed on $R_i \cap k(S_0^2)$ and makes $A_i$ polygonal, and this isotopy can be feathered into $R^3$ without moving $k(S_0^2)$, we suppose without loss of generality that $A_i$ is polygonal and normal to $R_i \cap k(S_0^2)$ at each point where they intersect.

Let $f_i$ be a PL homeomorphism of $R_i$ onto itself that is fixed on Bd $R_i$, takes $A_i$ to the straight segment from $p_i$ to $q_i$, and is an isometry near each point of $A_i \cap k(S_0^2)$. Let $C_i$ be a small rectangular solid having $R_i$ as a spanning disk. Extend $f_i$ to a PL homeomorphism of $C_i$ onto itself that is fixed on Bd $C_i$, $f_i$ on $R_i$ and an isometry near each point of $A_i \cap k(S_0^2)$. We suppose $T^2$ and the extended $f_i(S_0^2)$ are in general position. The required $h_1$ is $k$ modified by the various extended $f_i$'s. Note that $h_1(S_0^2)$ is locally polyhedral near $T^2$ and $h_1(S^2 - Y_1)$ misses $T^1$. Craggs gave an alternative description of such an $h_1$ in [8, Theorem 6.1]. □

In a similar fashion we could treat a noncompact $S$ and a variable $\epsilon$ to get the following variation of Theorem V.3.

THEOREM V.4. *Suppose*
*$T$ is a triangulation of $M^3$,*
*$S$ is a surface (possibly wild) in $M^3$, and*
*$\epsilon(x)$ is a non-negative continuous function defined on $R^3$ and positive on $S$.*
*Then there is an isotopy $H_t$ $(0 \leq t \leq 1)$ of $M^3$ onto itself such that*
*$H_0$ is the identity,*
*each $\rho(x, H_t(x)) \leq \epsilon(x)$, and*
*$H_1(S)$ satisfies the same restrictions about its intersections with the skeletons of $T$ as did $H_1(S^2)$ in Theorem* V.3.

The following result was stated as Theorem 4 in [6] with the statement that the proof would be given in a later paper. This is that later paper.

THEOREM V.5. *Suppose $S_1, S_2$ are compact surfaces in $M^3$. Then for each $\epsilon > 0$ there is an $\epsilon$-isotopy $H_t$ $(0 \leq t \leq 1)$ of $M^3$ onto itself such that*
*each $H_t$ is fixed outside the $\epsilon$-neighborhood of $S_1 \cap S_2$, and*
*the collection of nondegenerate elements of $S_1 \cap H_1(S_2)$ is a null sequence of tame simple closed curves whose union is dense in $S_1 \cap H_1(S_2)$.*

*Proof.* So that we can speak of Sierpinski curves rather than sets which are locally like Sierpinski curves, we suppose $S_1$ and $S_2$ are 2-spheres. Our proof will be a modification of the preceding three proofs but we avoid epsilonics. We use $Y_i$ $(i=1,2)$ to denote a tame Sierpinski curve in $S_i$ with small holes and $I(Y_i)$, $A(Y_i)$ to denote (respectively) the inaccessible and accessible parts of $Y_i$. Also, $S_{i0}$ denotes a tame 2-sphere obtained by replacing closures of components of $S_i - Y_i$ by nearby closed disks. Let $h_1, h_2 h_1$ be homeomorphisms of $M^3$ onto itself that are the ends of short isotopies starting at the identity and carrying $S_{10}$, $S_{20}$ (respectively) to polyhedral sets, so that $h_1(S_{10}) \cap h_2 h_1(S_{20})$ is the union of a finite number of polyhedral simple closed curves. Then

$$\text{(a)} \qquad h_1(S_{10}) \cap h_2 h_1(S_{20}) = \bigcup J_i.$$

Let $g_1$ be the end of a short isotopy invariant on $h_1(S_{10})$ that makes $h_1(I(Y_1))$ engulf $h_1(S_{10}) \cap h_2 h_1(S_{20})$. Then

$$\text{(b)} \qquad g_1 h_1(I(Y_1)) \supset \bigcup J_i$$

and

$$\text{(c)} \qquad g_1 h_1(A(Y_1)) \cap h_2 h_1(S_{20}) = \varnothing.$$

Also, $g_2$ is the end of a short isotopy invariant on $h_2 h_1(S_{20})$ that makes $h_2 h_1(I(Y_2))$ engulf $h_2 h_1(S_{20}) \cap g_1 h_1(S_1)$. Then

$$\text{(d)} \qquad g_2 h_2 h_1(I(Y_2)) \supset \bigcup J_i$$

and

$$\text{(e)} \qquad g_2 h_2 h_1(A(Y_2)) \cap g_1 h_1(S_1) = \varnothing.$$

Then

$$\begin{aligned}\bigcup(J_i) &= h_1(S_{10})\cap h_2h_1(S_{20}) \quad \text{[from (a)]}\\ &= g_1h_1(S_{10})\cap g_2h_2h_1(S_{20}) \quad \text{[from definitions of } g_1, g_2]\\ &\subset g_1h_1(I(Y_1))\cap g_2h_2h_1(I(Y_2)) \quad \text{[from (b) and (d)]}\\ &\subset g_1h_1(S_{10})\cap g_2h_2h_1(S_{20}) = \bigcup J_i.\end{aligned}$$

Hence,

(f) $$g_1h_1(I(Y_1))\cap g_2h_2h_1(I(Y_2)) = \bigcup J_i.$$

The required isotopy $H_t$ $(0\le t\le 1)$ is defined on $0\le t\le 1/2$, so that

(g) $$H_{1/2} = h_1^{-1}g_1^{-1}g_2h_2h_1 = k.$$

The image of $S_1\cap k(S_2)$ under $g_1h_1$ is

$$g_1h_1(S_1)\cap g_2h_2h_1(I(Y_2)\cup A(Y_2)\cup(S_2-Y_2)).$$

However,

$$g_1h_1(S_1)\cap g_2h_2h_1(A(Y_2)) = \varnothing \quad \text{[by (e)]},$$

so $g_1h_1(S\cap kS_2)$ is the union of the two closed sets $g_1h_1(S_1)\cap g_2h_2h_1(I(Y_2))$ and $g_1h_1(S_1)\cap g_2h_2h_1(S_2-Y_2)$. The components of the second of these sets are small, so we examine the first set:

$$g_1h_1(I(Y_1)\cup A(Y_1)\cup(S_1-Y_1))\cap g_2h_2h_1(I(Y_2)).$$

Note that

$$g_1h_1(A(Y_1))\cap g_2h_2h_1(I(Y_2)) = \varnothing \quad \text{[by (c)]},$$

and components of $g_1h_1(S-Y_1)\cap g_2h_2h_1(I(Y_2))$ are small. We look at

$$g_1h_1(I(Y_1))\cap g_2h_2h_1(I(Y_2)).$$

However, this is $\bigcup(J_i)$ by condition (f). Then the components of $S_1\cap k(S_2)$ that are not small are simple closed curves in $h_1^{-1}g_1^{-1}(\bigcup J_i)$.

We continue the description of $H_t$ $(0\le t\le 1)$ as in the proof of Theorem V.2, but with one exception. Recall in the proof of Theorem V.2 that as a final move we pushed isolated points of $P\cap H_1(S^2)$ to one side of $P$. We might encounter difficulty in pushing isolated points off $S_1\cap H_1(S_2)$ at the end so if the possibility of an isolated point developing as $t$ approaches 1 occurs, we change the $S_{10}$ and $S_{20}$ at later stages to introduce small simple closed curves in the intersection of their respective images so that each point of $S_1\cap H_1(S_2)$ is a limit point of the union of the images of the $J_i$'s. □

Just as we varied Theorem V.3 to consider noncompact surfaces and variable $\epsilon$'s, a variation of Theorem V.5 could be given.

## REFERENCES

1. R. H. Bing, *Approximating surfaces with polyhedral ones,* Ann. of Math. (2) 65 (1957), 456–483.

2. ———, *An alternative proof that* 3-*manifolds can be triangulated,* Ann. of Math. (2) 69 (1959), 37–65.
3. ———, *Conditions under which a surface in* $E^3$ *is tame,* Fund. Math. 47 (1959), 105–139.
4. ———, *A surface is tame if its complement is* 1-ULC, Trans. Amer. Math. Soc. 101 (1961), 294–305.
5. ———, *Pushing a* 2-*sphere into its complement,* Michigan Math. J. 11 (1964), 33–45.
6. ———, *Improving the intersections of lines and surfaces,* Michigan Math. J. 14 (1967), 155–159.
7. ———, *The geometric topology of* 3-*manifolds,* Amer. Math. Soc., Providence, R.I., 1983.
8. R. Craggs, *Improving the intersection of polyhedra in* 3-*manifolds,* Illinois J. Math. 12 (1968), 567–586.
9. J. Hempel, *A surface in* $S^3$ *is tame if it can be deformed into each complementary domain,* Trans. Amer. Math. Soc. 111 (1964), 273–287.
10. E. E. Moise, *Affine structures in* 3-*manifolds,* V. *The triangulation theorem and Hauptvermutung,* Ann. of Math. (2) 56 (1952), 96–114.
11. R. L. Moore, *Concerning upper semicontinuous collections of continua,* Trans. Amer. Math. Soc. 27 (1925), 416–429.

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