PERTURBATIONS OF MATRIX ALGEBRAS

Man-Duen Choi and Kenneth R. Davidson

Two operator algebras acting on the same Hilbert space are said to be close if their unit balls are close in the Hausdorff metric. We are interested in algebraic and spatial characteristics of an algebra which persist under small perturbations. Even in low dimensions there is much pathology, and several examples are given to demonstrate this. On the other hand, certain classes of algebras behave very well. Two such classes are considered here: semi-simple algebras and reflexive algebras with distributive lattices. In these two cases, any algebra sufficiently close to one of these algebras is similar to it via an operator close to the identity.

The study of perturbations of operator algebras was initiated by Kadison and Kastler [12] for von Neumann algebras. This has stimulated a lot of further work in W^* and C^* algebras [3, 4, 5, 6, 7, 11, 17, 18, 19]. There has also been some work on nonself-adjoint algebras, notably Lance's work [14] on nest algebras. There is a strong connection between perturbation results and classification of algebras up to similarity. This comes out strongly in [15, 16] and [9] where the similarity theory for nest algebras is obtained. Some examples related to ours are obtained in [13]. More generally still, various authors have considered perturbations of arbitrary Banach algebras [10, 20].

In the self-adjoint case, it often turns out that close algebras are unitarily equivalent via a unitary close to the identity. In [12], it was shown that close von Neumann algebras can be decomposed into summands of various types, preserving closeness. Then in [4, 17] it was shown that close type I von Neumann algebras are unitarily equivalent. No counterexample to "close implies unitarily equivalent" is known among von Neumann algebras or separable C^* algebras. However, there is a counterexample [3] among larger C^*-algebras. Furthermore, there is some strange behaviour known about C^* algebras almost contained in others [7, 11].

For nests, it is not possible to use unitaries to get all close algebras. However, Lance [14] has shown that close nests yield close algebras via a similarity close to the identity. One of the motivations of this paper was an attempt to generalize the result to a larger class of nonself-adjoint algebras—the reflexive algebras with commutative subspace lattices. The results in this paper deal only with the finite-dimensional case, although the remarks of Sections 4 and 5 indicate some of the possibilities in the infinite-dimensional case. These ideas will be pursued elsewhere [22].

1. Preliminaries. In this paper, $\mathcal{H}$ will always be a finite-dimensional Hilbert space. The algebra of $n \times n$ matrices is denoted by $\mathfrak{M}_n$, and $\mathcal{L}(\mathcal{H})$ denotes this

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algebra acting on the n-dimensional space $\mathcal{H}$. Also, $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Given an algebra $\mathcal{A}$, Lat $\mathcal{A}$ denotes the lattice of subspaces invariant for each operator in $\mathcal{A}$. Dually, if $\mathcal{L}$ is a lattice, then Alg $\mathcal{L}$ denotes the algebra of all operators leaving each element of $\mathcal{L}$ invariant. An algebra $\mathcal{A}$ is reflexive if it equals Alg Lat $\mathcal{A}$. Dually, $\mathcal{L}$ is reflexive if $\mathcal{L}$ equals Lat Alg $\mathcal{L}$. This notion is extended to subspaces of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ by saying that $\mathcal{S}$ is reflexive if it contains every linear map T such that $Tx \in \mathcal{S}x$ for every vector x in $\mathcal{H}_1$.

If M is a subspace, $P(M)$ denotes the orthogonal projection onto M. The word "projection" will always mean self-adjoint idempotent.

If M and N are two subspaces of a normed linear space, the distance between them is given by

$$d(M,N) = \max\left\{ \sup_{m \in M} \frac{\|m+N\|}{\|m\|},\ \sup_{n \in N} \frac{\|n+M\|}{\|n\|} \right\},$$

which is equivalent to (but slightly different from) the Hausdorff distance between the two unit balls. For subspaces of a Hilbert space, it is easy to check that $d(M,N) = \|P(M) - P(N)\|$. More often, this distance will be applied to subalgebras of $\mathcal{M}_n$.

Two algebras $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ are said to be similar if there is an invertible operator $T \in \mathcal{L}(\mathcal{H})$ such that $T^{-1}\mathcal{A}T = \mathcal{B}$, and T implements the similarity.

2. Examples. Let $\mathcal{S}$ be a subspace of $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, and form two algebras on $\mathcal{H}_1 \oplus \mathcal{H}_2$ as follows:

$$\mathcal{A}(\mathcal{S}) = \left\{ \begin{bmatrix} \lambda I & S \\ 0 & \lambda I \end{bmatrix} : \lambda \in \mathbf{C},\ S \in \mathcal{S} \right\}, \quad \text{and}$$

$$\mathcal{B}(\mathcal{S}) = \left\{ \begin{bmatrix} \lambda I & S \\ 0 & \mu I \end{bmatrix} : \lambda, \mu \in \mathbf{C},\ S \in \mathcal{S} \right\}.$$

PROPOSITION 2.1.
(a) $\mathcal{B}(\mathcal{S})$ *is reflexive if and only if* $\mathcal{S}$ *is reflexive.*
(b) $\mathcal{A}(\mathcal{S})$ *is reflexive if and only if* $\mathcal{S}$ *is reflexive and is a proper subspace of* $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$.

Proof. Part (a) is easy [13]. For part (b), note that

$$\begin{aligned} \text{Lat}\, \mathcal{A}(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)) &= \{M \oplus 0 : M \subseteq \mathcal{H}_1\} \cup \{\mathcal{H}_1 \oplus N : N \subseteq \mathcal{H}_2\} \\ &= \text{Lat}\, \mathcal{B}(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)), \end{aligned}$$

so $\mathcal{A}(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1))$ cannot be reflexive. Suppose $\mathcal{A}(\mathcal{S})$ is reflexive. Then the following implication holds.

$$\begin{aligned} & Tx \in \mathcal{S}x \quad \text{for all } x \in \mathcal{H}_2 \\ & \Rightarrow \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} Tx \\ 0 \end{bmatrix} \in \mathcal{A}(\mathcal{S}) \left(\begin{bmatrix} y \\ x \end{bmatrix} \right) \\ & \Rightarrow \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \in \text{Alg Lat}\, \mathcal{A}(\mathcal{S}) = \mathcal{A}(\mathcal{S}) \\ & \Rightarrow T \in \mathcal{S}. \end{aligned}$$

This shows that $\mathcal{S}$ is reflexive.

Conversely, suppose $\mathcal{S}$ is a proper reflexive subspace of $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. By part (a), $\mathcal{B}(\mathcal{S})$ is reflexive and thus Alg Lat $\mathcal{A}(\mathcal{S}) \subseteq \mathcal{B}(\mathcal{S})$. Since $\mathcal{S}$ is proper, there must exist a non-zero vector x_0 so that $\mathcal{S}x_0$ is not all of $\mathcal{H}_1$. Choose a vector y_0 in $\mathcal{H}_1$ which is not in $\mathcal{S}x_0$. Set $M = \{(z+\lambda y_0)\oplus\lambda x_0 : \lambda \in \mathbf{C}, z \in \mathcal{S}x_0\}$. It is readily apparent that M is invariant for $\mathcal{A}(\mathcal{S})$. Now if

$$T = \begin{bmatrix} \lambda I & S \\ 0 & \mu I \end{bmatrix}$$

belongs to $\mathcal{B}(\mathcal{S})$ and leaves M invariant, then $T(y_0 \oplus x_0) = (Sx_0 + \lambda y_0)\oplus \mu x_0$ belongs to M. So $\mu = \lambda$, T belongs to $\mathcal{A}(\mathcal{S})$, and $\mathcal{A}(\mathcal{S})$ is reflexive. □

EXAMPLE 2.2. For t in $\mathbf{C}$, let

$$\mathcal{A}_t = \left\{ \begin{bmatrix} \lambda & a \\ 0 & \lambda + at \end{bmatrix} : \lambda, a \in \mathbf{C} \right\}.$$

By Proposition 2.1, $\mathcal{A}_0$ is not reflexive. For $t \neq 0$, let

$$T_t = \begin{bmatrix} 1 & t^{-1} \\ 0 & 1 \end{bmatrix}.$$

Then

$$T_t^{-1}\mathcal{A}_t T_t = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} : \lambda, \mu \in \mathbf{C} \right\}.$$

Thus $\{\mathcal{A}_t, t \neq 0\}$ is a continuous family of similar, reflexive, abelian, semi-simple algebras. However, $\mathcal{A}_0$ is neither reflexive nor semi-simple. Clearly, $d(\mathcal{A}_s, \mathcal{A}_t) \leq |s-t|$. So as t tends to zero, $\mathcal{A}_0$ is the limit of a "homotopy" of reflexive, semi-simple algebras. □

EXAMPLE 2.3. A slight modification of Example 2.2 provides a limit algebra $\mathcal{A}_0$ which is reflexive and is the limit of reflexive algebras $\mathcal{A}_t$, $t \neq 0$, which are algebraically quite different from $\mathcal{A}_0$. For t in $\mathbf{C}$, let

$$\mathcal{A}_t = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda + at & 0 \\ 0 & 0 & \lambda \end{bmatrix} : \lambda, a \in \mathbf{C} \right\} \quad \text{and} \quad T_t = \begin{bmatrix} 1 & t^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then as before, $d(\mathcal{A}_s, \mathcal{A}_t) \leq |s-t|$, and for $t \neq 0$,

$$(T_t)^{-1}\mathcal{A}_t T_t = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix} : \lambda, \mu \in \mathbf{C} \right\}.$$

This time, Proposition 2.1 shows that $\mathcal{A}_0$ is reflexive. For $t \neq 0$, $\mathcal{A}_t$ is a continuous family of similar, reflexive, semi-simple algebras. As t tends to 0, one obtains $\mathcal{A}_0$ as a limit which is reflexive, but not semi-simple. □

EXAMPLE 2.4. A variation on this theme produces a net $\mathcal{A}_t$ of non-reflexive algebras converging to a reflexive algebra $\mathcal{A}_0$. Furthermore, there is a continuous family of isomorphisms $\{\phi_t : \mathcal{A}_0 \to \mathcal{A}_t : t \in \mathbf{C}\}$. Define

$$\mathcal{S}_t = \left\{ \begin{bmatrix} a & b \\ c & ta \end{bmatrix} : a, b, c \in \mathbf{C} \right\}, \qquad \mathcal{A}_t = \left\{ \begin{bmatrix} \lambda I & S \\ 0 & \lambda I \end{bmatrix} : S \in \mathcal{S}_t \right\} \subseteq \mathfrak{M}_4.$$

For $t \neq 0$, $\mathcal{S}_t v = \mathbf{C}^2$ for every non-zero vector v. As $\mathcal{S}_t$ is a proper subspace of $\mathcal{L}(\mathbf{C}^2, \mathbf{C}^2)$, it is not reflexive. Clearly, $\mathcal{S}_0$ is reflexive and proper. Hence, by Proposition 2.1, $\mathcal{A}_0$ is reflexive, but $\mathcal{A}_t$ is not reflexive if $t \neq 0$. The map $\phi_t : \mathcal{A}_0 \to \mathcal{A}_t$, defined by

$$\phi_t\left(\left[\begin{array}{cc|cc} \lambda & 0 & a & b \\ 0 & \lambda & c & 0 \\ \hline 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{array} \right] \right) = \left[\begin{array}{cc|cc} \lambda & 0 & a & b \\ 0 & \lambda & c & ta \\ \hline 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{array} \right],$$

is readily seen to form a continuous family of algebraic isomorphisms for t in $\mathbf{C}$. In particular, $\lim_{t \to 0} \|\phi_t - \mathrm{id}\| = 0$. Since reflexivity is a similarity invariant, it follows that $\mathcal{A}_0$ is not similar to $\mathcal{A}_t$ for $t \neq 0$.

Also, let $T_r = \mathrm{diag}\{1, 1, 1, r\}$ be the 4×4 diagonal matrix. Then it is readily verified that $T_r^{-1} \mathcal{A}_t T_r = \mathcal{A}_{rt}$. Thus $\mathcal{A}_t$ $(t \neq 0)$ are all similar, and for $|s - t|$ small, $\mathcal{A}_s$ and $\mathcal{A}_t$ are close and the similarity can be taken close to the identity. □

EXAMPLE 2.5. Next we give an example akin to the previous one of a continuous family $\mathcal{A}_t$ of isomorphic, abelian, reflexive algebras, no two of which are similar. Furthermore, we will exhibit close, similar algebras which are not similar via a similarity near the identity.

For t in $\mathbf{C}$, let

$$\mathcal{S}_t = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a+b & 0 \\ 0 & 0 & 0 & a+tb \end{bmatrix} : a, b \in \mathbf{C} \right\}, \quad \text{and}$$

$$\mathcal{A}_t = \left\{ \begin{bmatrix} \lambda I & S \\ 0 & \lambda I \end{bmatrix} : \lambda \in \mathbf{C},\ S \in \mathcal{S}_t \right\} \subseteq \mathfrak{M}_8.$$

It is easy to verify that $\mathcal{S}_t$ is reflexive, so $\mathcal{A}_t$ is reflexive for every t by Proposition 2.1. The canonical map $\psi_t : \mathcal{S}_0 \to \mathcal{S}_t$ by

$$\psi_t\left(\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a+b & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \right) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a+b & 0 \\ 0 & 0 & 0 & a+tb \end{bmatrix}$$

induces an algebraic isomorphism $\phi_t : \mathcal{A}_0 \to \mathcal{A}_t$ by

$$\phi_t\left(\begin{bmatrix} \lambda I & S \\ 0 & \lambda I \end{bmatrix} \right) = \begin{bmatrix} \lambda I & \psi_t(S) \\ 0 & \lambda I \end{bmatrix}.$$

Furthermore, ϕ_t is isometric if $0 \leq t \leq 1$. It is readily checked that $\|\phi_t - \phi_s\| \leq |t - s|$, so $\mathcal{A}_t$ is a continuous family of algebras.

It will be shown that the algebras $\mathfrak{A}_s$ are similar to $\mathfrak{A}_t$ only for s belonging to $\{t, t^{-1}, 1-t, (1-t)^{-1}, 1-t^{-1}, t(t-1)^{-1}\}$. Once this is established, one can see that $C = \{t : |t-\frac{1}{2}| \le 1, \operatorname{Im} t > 0\} \cup [0, \frac{1}{2}]$ gives a family of mutually nonsimilar algebras. Note that $\mathcal{S}_t$ is spanned by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S_t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{bmatrix}.$$

$\mathcal{S}_t$ contains a rank-two element if and only if t equals 0 or 1. Also for the case $t \neq 0$ and $t \neq 1$, $\mathcal{S}_t$ contains P, S_t, $P - S_t$, and $tP - S_t$ as the only rank-three elements (up to scalar multiples). These four rank-three elements satisfy

(t) $$T_3 = T_1 - T_2 \quad \text{and} \quad T_4 = tT_1 - T_2.$$

Now any isomorphism implemented by a similarity preserves rank, and preserves equation (t). So the only possibilities for relating $\mathfrak{A}_s$ and $\mathfrak{A}_t$ via a similarity are obtained by permuting the four lines $\mathbf{C}T_i$, $1 \le i \le 4$. A routine but somewhat tedious calculation shows that the permutation determines the constant s, and that $s = t, t^{-1}, 1-t, (1-t)^{-1}, 1-t^{-1}$, and $t(t-1)^{-1}$ are the only possible values. (Indeed, the permutations give an action of S_4 on this family. The four even permutations of order 2 fix $\mathfrak{A}_t$, and the quotient group is isomorphic to S_3. See Added in Proof, remark 3.)

Next, we verify that $\mathfrak{A}_{t^{-1}}$ and $\mathfrak{A}_{1-t}$ are actually similar to $\mathfrak{A}_t$. Let

$$V_t = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{bmatrix} \quad \text{and} \quad U^{\pm} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $V_1 \mathcal{S}_t = \mathcal{S}_{t^{-1}} V_t$ and $U^+ \mathcal{S}_t = \mathcal{S}_{1-t} U^-$. So let

$$T_t = \begin{bmatrix} V_1 & 0 \\ 0 & V_t \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} U^+ & 0 \\ 0 & U^- \end{bmatrix}.$$

Then $\mathfrak{A}_{t^{-1}} = T_t \mathfrak{A}_t T_t^{-1}$ (if $t \neq 0$) and $\mathfrak{A}_{1-t} = W_t \mathfrak{A}_t W_t^{-1}$. The transformations $t \mapsto t^{-1}$ and $t \mapsto 1-t$ generate the group S_3, so $\mathfrak{A}_s$ is similar to $\mathfrak{A}_t$ if and only if s belongs to $\{t, t^{-1}, 1-t, (1-t)^{-1}, 1-t^{-1}, t(t-1)^{-1}\}$.

Now notice that as t approaches 1, $\operatorname{dist}(\mathfrak{A}_t, \mathfrak{A}_{t^{-1}})$ tends to zero, and these algebras are similar. Suppose the four lines of rank-three matrices on $\mathcal{S}_t$ are permuted by a similarity W so that equation (t^{-1}) is satisfied. There are four possible permutations, but in each case a routine computation shows that $\|W - I\| \ge 1$. (In fact, only two of the four are implemented by similarities at all.) So $\mathfrak{A}_t$ and $\mathfrak{A}_{t^{-1}}$ can be very close and similar, yet the similarity is far from the identity. □

3. Semi-simple algebras. In the remainder of this paper, we show that good perturbation results hold for special classes of algebras. The main theorem of this section deals with semi-simple algebras, and follows by analogy to the self-adjoint case [4]. We begin with a lemma which is likely known to some readers. A proof is included for completeness.

LEMMA 3.1. *Let $\mathcal{B}$ be a Banach algebra with unit I. Then for each B in the unit ball of $\mathcal{B}$, there is an idempotent F satisfying $\|B-F\| \leq 8\|B-B^2\|$. When $\|B-B^2\| \leq 1/8$, the constant 8 can be reduced to 6.*

Proof. If $\|B-B^2\| \geq 1/8$, then let $F=0$ and we are done. Now suppose that $\|B-B^2\| = \epsilon < 1/8$. Define

$$T=[I-4(B-B^2)]^{-1/2}=I+\sum_{n=1}^{\infty}\binom{2n}{n}(B-B^2)^n.$$

This series converges absolutely, and

$$\|T-I\| \leq \sum_{n=1}^{\infty}\binom{2n}{n}\epsilon^n \leq \sum_{n=1}^{\infty}\frac{(4\epsilon)^n}{2}=\frac{2\epsilon}{1-4\epsilon}\leq 4\epsilon.$$

Furthermore, T commutes with B and $(I-2B)^2T^2=I$. Define F by $I-2F=(I-2B)T$. Squaring yields the identity $F=F^2$. Furthermore,

$$\begin{aligned}\|F-B\| &= \tfrac{1}{2}\|(I-2B)-(I-2F)\| \\ &= \tfrac{1}{2}\|(I-2B)(I-T)\| \leq \tfrac{1}{2}(3)(4\epsilon)=6\epsilon.\end{aligned}$$

□

THEOREM 3.2. *If $\mathcal{A}$ is a semi-simple subalgebra of $\mathfrak{M}_n$ then there is an $\epsilon>0$ and a constant $C<\infty$ so that, if $\mathcal{B}$ is any algebra such that $d(\mathcal{A},\mathcal{B})<\epsilon$, then $\mathcal{B}$ is similar to $\mathcal{A}$ via an invertible operator S such that $\|S-I\|<Cd(\mathcal{A},\mathcal{B})$.*

Proof. By the Wedderburn theory, $\mathcal{A}$ is similar to a self-adjoint algebra

$$\bigoplus\sum_{i=1}^{s}\mathfrak{M}_{k_i}\otimes I_{d_i}, \quad \text{where } \sum_{i=1}^{s}k_i d_i \leq n.$$

So we may assume that $\mathcal{A}$ has this form already and assume that the Hausdorff distance between the unit balls of $\mathcal{A}$ and $\mathcal{B}$ is less than $\epsilon \leq 1/40$. Next one see that $\mathcal{B}$ is semi-simple. For if $\mathcal{B}$ has non-trivial radical, then it contains an element B of norm one such that $B\mathcal{B}B=0$. Choose A in $\mathcal{A}$ with $\|A-B\|<\epsilon$ and $\|A\|\leq 1$. Then choose C in $\mathcal{B}$ such that $\|A^*-C\|<\epsilon$ and $\|C\|\leq 1$. So

$$\begin{aligned}\|AA^*A\| &= \|AA^*A-BCB\| \\ &\leq \|(A-B)A^*A\|+\|B(A^*-C)A\|+\|BC(A-B)\| \\ &< 3\epsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|AA^*A\|^2 &= \|(AA^*A)^*(AA^*A)\| \\ &= \|(A^*A)^3\| = \|A\|^6 > (1-\epsilon)^6.\end{aligned}$$

This forces $\epsilon > 1/6$, which is a contradiction.

Next, let E be a central idempotent in $\mathcal{A}$. Choose B in $\mathcal{B}$ with $\|B-E\|<\epsilon$ and $\|B\|\leq 1$. Then

$$\|B-B^2\| = \|(B-E)(I-E)-B(B-E)\| < 2\epsilon \leq 1/8.$$

By Lemma 3.1, there is an idempotent F in $\mathcal{B}$ with

$$\|F-E\| \leq \|F-B\|+\|B-E\| < 6(2\epsilon)+\epsilon = 13\epsilon.$$

This idempotent must be central in $\mathcal{B}$. For if $C=(I-F)CF$ belongs to $\mathcal{B}$ and has norm one, then choose A in $\mathcal{A}$ such that $\|C-A\|<\epsilon$ and $\|A\|\leq 1$. Thus the conditions $FC=0$, $CF=C$, and $EA=AE$ yield

$$C=C-(I-E)AE=(E-F)C+(I-E)C(F-E)+(I-E)(C-A)E$$

and

$$1=\|C\|<27\epsilon<1.$$

This contradiction shows that $BF=FBF$ for all B in $\mathcal{B}$. Similarly, one obtains that $FB=FBF$ for all B in $\mathcal{B}$, so F is central.

Next, if E_i $(1\leq i\leq s)$ are the minimal central idempotents of $\mathcal{A}$, let F_i be central idempotents in $\mathcal{B}$ with $\|E_i-F_i\|<13\epsilon$. Since F_iF_j is an idempotent, it is either zero or has norm at least one. An easy estimate shows that for $i\neq j$,

$$\|F_iF_j\|=\|F_iF_j-E_iE_j\|=\|(F_i-E_i)F_j+E_i(F_j-E_j)\|<39\epsilon<1$$

and hence $F_iF_j=0$.

Let $\mathfrak{F}$ be the Boolean algebra

$$\left\{F_\sigma=\sum_{i\in\sigma}F_i\text{: for all subsets } \sigma \text{ of } \{1,\ldots,s\}\right\},$$

and let $\mathcal{E}$ be the corresponding Boolean algebra formed from the E_i's. For each E_σ, the previous argument produces a central idempotent F in $\mathcal{B}$ with $\|F-E_\sigma\|<13\epsilon$. For j in σ,

$$\begin{aligned}\|F_j-FF_j\|&=\|(I-E_\sigma)(F_j-E_j)+(E_\sigma-F)F_j\|\\&<13\epsilon+13\epsilon(1+13\epsilon)<39\epsilon<1.\end{aligned}$$

However, F_j-FF_j is an idempotent, and hence $FF_j=F_j$. Similarly, for j in the complement of σ, $FF_j=0$. Hence $F=F_\sigma$, and $\|F_\sigma-E_\sigma\|<13\epsilon$ for all subsets σ. If $\mathcal{A}$ and $\mathcal{B}$ do not contain the identity operator, let

$$E_0=I-\sum_{i=1}^{s}E_i \quad\text{and}\quad F_0=I-\sum_{i=1}^{s}F_i.$$

Extend $\mathcal{E}$ and $\mathfrak{F}$ to $\mathcal{E}_0$ and $\mathfrak{F}_0$, the Boolean algebras generated by $\{E_0,\ldots,E_s\}$ and $\{F_0,\ldots,F_s\}$, and extend the definition of E_σ and F_σ. Note that $\|E_\sigma-F_\sigma\|<13\epsilon$ is still valid.

Now, one readily computes that

$$\sum E_\sigma=2^sI \quad\text{and}\quad \sum F_\sigma=2^sI,$$

$$\begin{aligned}\sum F_\sigma E_\sigma&=2^s\sum_{i=0}^{s}F_iE_i+2^{s-1}\sum_{i\neq j}^{s}F_iE_j\\&=2^{s-1}\left(\sum_{i=0}^{s}F_iE_i+\left(\sum_{i=0}^{s}F_i\right)\left(\sum_{j=0}^{s}E_j\right)\right)\\&=2^{s-1}\left(\sum_{i=0}^{s}F_iE_i+I\right).\end{aligned}$$

Let $T=\sum_{i=0}^{s} F_i E_i$; then

$$2^s(T-I)=2\sum F_\sigma E_\sigma-\sum E_\sigma-\sum F_\sigma$$
$$=\sum (F_\sigma-E_\sigma)(2E_\sigma-I).$$

Hence

$$\|T-I\|\le 2\max\|F_\sigma-E_\sigma\|\|2E_\sigma-I\|<26\epsilon.$$

Furthermore, $F_iT=TE_i$ for $0\le i\le s$, so $T^{-1}F_iT=E_i$. Thus $\mathcal{B}'=T^{-1}\mathcal{B}T$ is similar to $\mathcal{B}$ via an invertible operator T satisfying $\|T-I\|<26\epsilon$. Now, $\mathcal{A}$ and $\mathcal{B}'$ have a common centre.

This reduces the problem to the case $\mathcal{A}=\mathfrak{M}_k\otimes I_d$. Let E_{ij} be matrix units for $\mathcal{A}$. Proceeding along standard lines, one can find matrix units F_{ij} in $\mathcal{B}'$ close to E_{ij}. As above, one argues that $T'=\sum_{i=1}^{k} E_{ii}F_{ii}$ implements the similarity, and is close to I. □

REMARK. The argument of the last paragraph modified along the lines of [5] yields universal constants C and ϵ_0 so that if $\mathcal{A}$ is a C^*-subalgebra of any matrix algebra, then given any algebra $\mathcal{B}$ with $d(\mathcal{A},\mathcal{B})=\epsilon<\epsilon_0$, there is an invertible matrix T with $\|T-I\|\le C\epsilon$ and $S\mathcal{B}S^{-1}=\mathcal{A}$.

4. Distributive lattices. In [1], Arveson considers algebras which are reflexive and contain a maximal abelian von Neumann algebra. This is equivalent to requiring the lattice $\mathcal{L}$ of invariant subspace projections to be commutative. As such, it generates a σ-complete Boolean algebra of projections $\mathcal{E}$. Conversely, if $\mathcal{L}$ is a commutative lattice, then Alg $\mathcal{L}$ contains $\mathcal{L}'=\mathcal{E}'$, which is a von Neumann algebra with abelian commutant $\mathcal{E}''$. Unfortunately, this class is not closed under similarity. Thus we propose that this class should be enlarged to the similarity invariant class of reflexive algebras which contain a σ-complete bounded Boolean algebra. It turns out that each of these algebras is similar to one in Arveson's class [22]. In this paper, we restrict ourselves to the finite-dimensional considerations, where this is equivalent to being reflexive with a distributive lattice.

This first lemma is a special case of a theorem of Wermer [21] which orthogonalizes a bounded σ-complete Boolean algebra. A proof is included for convenience.

LEMMA 4.1. *Let $\mathcal{G}$ be a finite group of operators on a Hilbert space. Then there is an invertible positive operator T such that $T\mathcal{G}T^{-1}$ is a group of unitaries, and*

$$\max\{\|T-I\|,\|T^{-1}-I\|\}\le\max\{\|G\|-1:G\in\mathcal{G}\}.$$

Proof. Define

$$T=\left(\frac{1}{|\mathcal{G}|}\sum_{G\in\mathcal{G}} G^*G\right)^{1/2}.$$

If H belongs to $\mathcal{G}$, then

$$T^2H=\frac{1}{|\mathcal{G}|}\sum G^*GH=\frac{1}{|\mathcal{G}|}\sum (GH^{-1})^*G=(H^{-1})^*T^2.$$

Hence

$$THT^{-1}=T^{-1}(H^{-1})^*T=((THT^{-1})^{-1})^*.$$

Thus THT^{-1} is unitary for each H in $\mathcal{G}$. Furthermore, let $b=\max\{\|G\|: G\in\mathcal{G}\}$. Then for each $G\in\mathcal{G}$, $b^2I\geq G^*G$ and $b^2I\geq GG^*$; thus $(G^{-1})^*G^{-1}\geq b^{-2}I$. Hence $b^{-2}I\leq T^2\leq b^2I$, and thus $T\leq bI$ and $T^{-1}\leq bI$. Therefore

$$\max\{\|T-I\|, \|T^{-1}-I\|\}\leq b-1. \qquad \square$$

COROLLARY 4.2. *Let $\mathcal{E}$ be a finite Boolean algebra of idempotents on a Hilbert space $\mathcal{H}$ bounded by $1+\epsilon$, and $I\in\mathcal{E}$. Then there is an invertible positive operator T such that $T\mathcal{E}T^{-1}$ is a Boolean algebra of (self-adjoint) projections. If $\epsilon\leq 1$, then*

$$\max\{\|T-I\|, \|T^{-1}-I\|\}\leq 2\sqrt{3\epsilon}.$$

Proof. Consider the group $\mathcal{G}=\{2E-I: E\in\mathcal{E}\}$ and apply Lemma 4.1. To estimate the bound of $\mathcal{G}$, let E belong to $\mathcal{E}$ and decompose it as

$$E=\begin{pmatrix}1 & A\\ 0 & 0\end{pmatrix}$$

on $\mathcal{H}=E\mathcal{H}\oplus(E\mathcal{H})^{\perp}$. Then $\|E\|^2=1+\|A\|^2$, so

$$\|A\|^2=\|E\|^2-1\leq(1+\epsilon)^2-1\leq 3\epsilon.$$

Also,

$$2E-I=\begin{bmatrix}1 & 2A\\ 0 & -1\end{bmatrix}$$

is of norm less than $1+2\|A\|\leq 1+2\sqrt{3\epsilon}$. Thus

$$\max\{\|T-I\|, \|T^{-1}-I\|\}\leq 2\sqrt{3\epsilon}. \qquad \square$$

THEOREM 4.3. *For a reflexive subalgebra $\mathcal{A}$ of $\mathcal{M}_n$, the following are equivalent:*

(1) *$\mathcal{A}$ contains a Boolean algebra $\mathcal{E}$ of idempotents such that, for each $L\in\operatorname{Lat}\mathcal{A}$, there exists $E\in\mathcal{E}$ with $P(E)=L$.*
(2) *$\operatorname{Lat}\mathcal{A}$ is distributive.*
(3) *$\mathcal{A}$ contains n nonzero, commuting idempotents $E_1,\dots,E_n$ such that $\sum_{i=1}^n E_i=I$.*
(4) *$\mathcal{A}$ is similar to an algebra with commutative subspace lattice.*

Furthermore, if the Boolean algebra $\mathcal{E}$ is bounded by $1+\epsilon\leq 2$, then the similarity T in (4) may be taken to satisfy $\max\{\|T-I\|, \|T^{-1}-I\|\}\leq 2\sqrt{3\epsilon}$.

Proof. First, we show that (1) implies (4). The similarity is provided by Corollary 4.2. Furthermore, the similarity T satisfies $\max\{\|T-I\|, \|T^{-1}-I\|\}<2\sqrt{3\epsilon}$ for $\epsilon\leq 1$. For (4) implies (3), note that (3) is invariant under similarity; so it may be assumed that $\mathcal{A}$ has a commutative subspace lattice. Hence $\mathcal{A}$ contains the self-adjoint algebra $\mathcal{L}'$ which contains the abelian self-adjoint algebra $\mathcal{L}''$. There is a maximal abelian self-adjoint algebra $\mathcal{M}$ with $\mathcal{L}''\subset\mathcal{M}\subset\mathcal{L}'$. $\mathcal{M}$ consists of all $n\times n$ diagonal matrices with respect to some orthonormal basis. The minimal projections E_i, $1\leq i\leq n$, in $\mathcal{M}$ will suffice.

Given (3), one knows that $\mathcal{A}$ contains the algebra $\mathcal{B}$ spanned by the E_i's. Hence $\operatorname{Lat}\mathcal{A}$ is contained in $\operatorname{Lat}\mathcal{B}$. The set E of idempotents in $\mathcal{B}$ is the Boolean algebra of idempotents spanned by the E_i's; the ranges of these idempotents are precisely the elements of Lat $\mathcal{B}$. This proves (1). Also, (2) follows since a sublattice of a Boolean algebra is always distributive.

Finally it remains to show (2) implies (1). Suppose $\mathcal{L} = \operatorname{Lat} \mathcal{A}$ is distributive. Choose a maximal chain of subspaces

$$0 = N_0 \subset N_1 \subset \cdots \subset N_k = I$$

in $\mathcal{L}$. Define a map θ from $\mathcal{L}$ into 2^k by $\theta(L) = \{i : L \vee N_{i-1} \geq N_i\}$. Also, for each $1 \leq i \leq k$, let $L_i = \bigwedge \{L \in \mathcal{L} : i \in \theta(L)\}$. Since $\mathcal{L}$ is distributive,

$$N_{i-1} \vee L_i = \bigwedge \{N_{i-1} \vee L : i \in \theta(L)\} = N_i.$$

By induction on k, it will be shown that there are subspaces M_i $(1 \leq i \leq k)$ such that $\mathcal{H} = M_1 \dotplus M_2 \dotplus \cdots \dotplus M_k$ is an algebraic direct sum, and such that

$$L = \bigvee \{L_i : i \in \theta(L)\} = \bigvee \{M_i : i \in \theta(L)\}.$$

This is obvious for $k = 1$. Suppose that it holds for $k-1$. In particular, $\mathcal{L}' = \{L \in \mathcal{L} : L \leq N_{k-1}\}$ (as a lattice acting on N_{k-1}) satisfies the hypotheses, so M_i, $1 \leq i \leq k-1$, exist as described. Let $M_k = L_k \ominus (N_{k-1} \wedge L_k)$. By construction, $M_k \wedge N_{k-1} = 0$ and $M_k \vee N_{k-1} = I$, so

$$\mathcal{H} = N_{k-1} \dotplus M_k = M_1 \dotplus \cdots \dotplus M_{k-1} \dotplus M_k.$$

Now if L belongs to $\mathcal{L}$ and $L \leq N_{k-1}$, the formula holds. Otherwise $L \not\leq N_{k-1}$, whence $L \vee N_{k-1} = I$ and k belongs to $\theta(L)$. First, notice that $\theta(L \wedge N_{k-1}) = \theta(L) \setminus \{k\}$, for if $i \in \theta(L) \setminus \{k\}$ then

$$\begin{aligned}(L \wedge N_{k-1}) \vee N_{i-1} &= (L \vee N_{i-1}) \wedge (N_{k-1} \vee N_{i-1}) \\ &\geq N_i \wedge N_{k-1} = N_i,\end{aligned}$$

and so $i \in \theta(L \wedge N_{k-1})$. The other inclusion is trivial. So by the distributive law and the induction hypothesis,

$$\begin{aligned}\bigvee \{L_i : i \in \theta(L)\} &= \bigvee \{L_i : i \in \theta(L \wedge N_{k-1})\} \vee L_k \\ &= (L \wedge N_{k-1}) \vee L_k \\ &= (L \vee L_k) \wedge (N_{k-1} \vee L_k) \\ &= L \wedge I = L.\end{aligned}$$

Also, $\bigvee \{M_i : i \in \theta(L)\} = (L \wedge N_{k-1}) \vee M_k$. Since the lattice of subspaces of a finite-dimensional space is modular, and $M_k \leq L$, one obtains

$$\begin{aligned}(L \wedge N_{k-1}) \vee M_k &= L \wedge (N_{k-1} \vee M_k) \\ &= L \wedge I = L.\end{aligned}$$

Now let E_i be the idempotent with range M_i and kernel $\sum_{j \neq i} M_j$. Then E_i are a commuting family of idempotents which leave each M_j invariant. Thus, E_i leave $\mathcal{L}$ invariant and belong to $\mathcal{A}$. They generate a Boolean algebra $\mathcal{E}$, and $\sum_{i \in \theta(L)} E_j$ is clearly an idempotent with range L. This shows that (2) implies (1). □

THEOREM 4.4. *Let $\mathcal{A} = \operatorname{Alg} \mathcal{L}$ be a subalgebra of $\mathfrak{M}_n$ with a commutative subspace lattice $\mathcal{L}$. Let $\mathcal{B}$ be another subalgebra such that $d(\mathcal{A}, \mathcal{B}) < \epsilon \leq .01$. Then there is a lattice isomorphism ϕ of $\mathcal{L}$ onto $\mathfrak{M} = \operatorname{Lat} \mathcal{B}$ such that $\|\phi - \mathrm{id}\| < 4\epsilon$.*

Proof. Fix M in $\mathfrak{M}$, and let $P = P(M)$ be the orthogonal projection onto M. For each operator A in the self-adjoint algebra $\mathcal{L}'$ (which is a subalgebra of $\mathcal{A}$), pick B and C in $\mathcal{B}$ such that $\|B - A\| < \epsilon\|A\|$ and $\|C - A^*\| < \epsilon\|A\|$. Then

$$\begin{aligned}\|PA - AP\| &= \|PA(I-P) - (I-P)AP\| \\ &\leq \max\{\|P(A - C^*)(I-P)\|, \|(I-P)(A-B)P\|\} < \epsilon\|A\|.\end{aligned}$$

This means that the derivation $\delta_P \mid_{\mathcal{L}'}$ has norm at most ϵ. Since $\mathcal{L}''$ is abelian, it follows from [6] that

$$d(P, \mathcal{L}'') < 2\|\delta_P \mid_{\mathcal{L}'}\| < 2\epsilon.$$

Choose $R = R^*$ in $\mathcal{L}''$ such that $\|P - R\| < 2\epsilon$. Thus $\sigma(R)$ is contained in

$$(-2\epsilon, 2\epsilon) \cup (1 - 2\epsilon, 1 + 2\epsilon).$$

By the functional calculus, there is a projection Q in $\mathcal{L}''$ with $\|Q - R\| < 2\epsilon$, and thus $\|P - Q\| < 4\epsilon$. This projection Q belongs to $\mathcal{L}$. For if this is not the case, then $\mathcal{A}$ contains an operator $A = Q^\perp A Q$ of norm one. Choose B in $\mathcal{B}$ with $\|A - B\| < \epsilon$. Then

$$\begin{aligned}\|A\| &= \|(I-Q)AQ - (I-P)BP\| \\ &\leq \|(P-Q)AQ\| + \|(I-P)A(Q-P)\| + \|(I-P)(A-B)P\| \\ &< 4\epsilon + 4\epsilon + \epsilon = 9\epsilon < 1.\end{aligned}$$

This contradiction establishes the claim. This projection is unique because if Q_1 is any other projection in $\mathcal{L}''$, then $\|Q - Q_1\| = 1$. Denote this projection Q by $\phi(M)$.

Conversely, if Q belongs to $\mathcal{L}$, let B belong to $\mathcal{B}$ with $\|Q - B\| < \epsilon$. Then

$$\|B - B^2\| \leq \|(I-Q)(B-Q)\| + \|(Q-B)B\| < \epsilon(2+\epsilon) < 1/8.$$

By Lemma 3.1, there is an idempotent F in $\mathcal{B}$ with $\|B - F\| < 6\epsilon(2+\epsilon) < 13\epsilon$. Thus $\|Q - F\| < 14\epsilon$. As in the previous paragraph, it follows that $M = F\mathcal{H}$ is invariant for $\mathcal{B}$. An easy estimate provides $\|Q - P(M)\| < 28\epsilon$, and hence $\|Q - \phi(M)\| < 32\epsilon < 1$. So $\phi(M) = Q$. Thus ϕ is surjective.

Next, we show that ϕ is injective. Let M belong to Lat $\mathcal{B}$. Let $Q = \phi(M)$, and let F be the idempotent obtained above so that $\phi(F\mathcal{H}) = Q$. It suffices to show that $F\mathcal{H} = M$. So decompose $\mathcal{H} = M \oplus M^\perp$, and write F as a matrix with respect to this decomposition:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}.$$

Since F is idempotent, so are both F_{11} and F_{22}. Also,

$$\|F - P(M)\| \leq \|F - \phi(M)\| + \|\phi(M) - P(M)\| < 18\epsilon < 1.$$

Hence $F_{22} = 0$ and $F_{11} = I$, and so $F\mathcal{H} = M$. Thus ϕ is a bijection.

To see that ϕ is a lattice isomorphism, suppose M and N belong to Lat $\mathcal{B}$. Let F_M and F_N be idempotents in $\mathcal{B}$ with range M and N (respectively) satisfying

$$\|\phi(M)-F_M\|<18\epsilon \quad \text{and} \quad \|\phi(N)-F_N\|<18\epsilon.$$

The operator $F_M F_N$ is an idempotent in $\mathcal{B}$ with range $M\wedge N$ since

$$(F_M F_N)^2 = F_M(F_N F_M F_N) = F_M(F_M F_N) = F_M F_N.$$

Similarly, $F_M+F_N-F_M F_N$ is an idempotent in $\mathcal{B}$ with range $M\vee N$. A computation gives

$$\begin{aligned}\|\phi(M)\phi(N)-F_M F_N\| &\le \|\phi(M)(\phi(N)-F_N)\| + \|(\phi(M)-F_M)F_N\| \\ &< 18\epsilon(2+18\epsilon)<40\epsilon.\end{aligned}$$

Thus

$$\|\phi(M)\phi(N)-P(M\wedge N)\|<80\epsilon \quad \text{and} \quad \|\phi(M)\phi(N)-\phi(M\wedge N)\|<84\epsilon<1.$$

So

$$\phi(M\wedge N)=\phi(M)\phi(N)=\phi(M)\wedge\phi(N).$$

Similarly,

$$\phi(M\vee N)=\phi(M)+\phi(N)-\phi(M)\phi(N).$$

So ϕ is a lattice isomorphism, and $\|\phi-\mathrm{id}\|<4\epsilon$. □

COROLLARY 4.5. *Let $\mathcal{A}=\operatorname{Alg}\mathcal{L}$ be a subalgebra of $\mathfrak{M}_n$ with commutative subspace lattice $\mathcal{L}$. Let k be the length of a maximal chain in $\mathcal{L}$. Let $\mathcal{B}$ be another subalgebra of $\mathfrak{M}_n$ with $d(\mathcal{A},\mathcal{B})=\epsilon<.01$. Then $\mathcal{B}$ is similar to $\mathcal{A}$. Furthermore, if $\epsilon<(8\sqrt{k})^{-1}$, then the similarity can be implemented by an operator T such that $\|T-I\|<8\sqrt{k}\epsilon$.*

Proof. Let ϕ be the lattice isomorphism of Lat $\mathcal{B}$ onto $\mathcal{L}$ given by Theorem 4.4. Since $\|\phi-\mathrm{id}\|\le 4\epsilon<1$, ϕ preserves rank. It follows from the structure of distributive lattices (cf. Theorem 4.3) that ϕ is implemented by a similarity T. So $T^{-1}\mathcal{B}T$ has lattice $\mathcal{L}$. Also, since $d(\mathcal{A},\mathcal{B})<1$, $\mathcal{A}$ and $\mathcal{B}$ have the same dimension. Thus $T^{-1}\mathcal{B}T=\mathcal{A}$.

To get a norm estimate for $\|T-I\|$, we use the ideas from the proof of Theorem 4.3. The subspaces $M_i=L_i\ominus(N_{i-1}\wedge L_i)$ of that proof form the ranges and kernels of a Boolean algebra $\mathcal{F}$ in $\mathcal{B}$. Let E_i be the corresponding atoms of the Boolean algebra generated by $\mathcal{L}$. Since $\|\phi-\mathrm{id}\|\le 4\epsilon$, it follows that $d(E_i,M_i)\le 8\epsilon$. From this, it is immediate that $\|P(E_i)-P(M_i)\|\le 8\epsilon$. So let

$$T=\sum_{i=1}^{k} P(M_i)P(E_i).$$

Since $P(E_i)$ are pairwise orthogonal, it follows that

$$\begin{aligned}\|T-I\|^2 &= \|(T-I)(T-I)^*\| \\ &= \|\textstyle\sum (P(M_i)-P(E_i))P(E_i)(P(M_i)-P(E_i))\| \\ &\le k64\epsilon^2.\end{aligned}$$

Thus $\|T-I\| \le 8\sqrt{k}\epsilon < 1$, and hence T is invertible.

Since TE_i is contained in M_i for each i, TE_i equals M_i. Hence $F_i = TP(E_i)T^{-1}$ is the idempotent with range M_i and kernel $\sum_{j\neq i} M_j$. From the proof of Theorem 4.3, it follows that the Boolean algebra of idempotents generated by the F_i's contains Lat $\mathcal{B}$. So conjugation by T carries Lat $\mathcal{B}$ onto $\mathcal{L}$, and $T^{-1}\mathcal{B}T = \mathcal{A}$. □

Putting these results together with the results of Theorem 4.3, we obtain the following.

THEOREM 4.6. *Let $\mathcal{L}$ be a distributive subspace lattice on a finite-dimensional space. Then the following are equivalent.*

(1) *$\mathcal{B}$ is an algebra close to* Alg $\mathcal{L}$.

(2) *$\mathcal{B}$ is reflexive, and* Lat $\mathcal{B}$ *is isomorphic to $\mathcal{L}$ via ϕ close to the identity.*

(3) *$\mathcal{B}$ is similar to* Alg $\mathcal{L}$ *via an operator close to the identity.*

5. Closing remarks. The reader may have noticed that some of the results of the previous section (in particular Theorem 4.4) go through in infinite dimensions (see [22]). Indeed, it was an attempt to extend the similarity results of [9] and [16] and the perturbation results of [14] to Arveson's class of commutative subspace lattices that led to the results of this paper. However, the constant k in Corollary 4.5 precludes the use of this result in the infinite-dimensional case. This leads one to ask the following question.

QUESTION 5.1. Suppose $\mathcal{B}$ is a norm (or weakly) closed subalgebra of $\mathcal{B}(\mathcal{H})$. Suppose that $\mathcal{E}$ is a finite self-adjoint Boolean algebra of projections, and

$$\sup\{d(E, \mathcal{B}) : E \in \mathcal{E}\} = \epsilon$$

is small. Is $\mathcal{E}$ similar to a Boolean subalgebra of $\mathcal{B}$ via an invertible close to the identity?

Naturally, the import of this question relies on the interpretation that "close" should be independent of the size of $\mathcal{E}$. If $\mathcal{B}$ is a C^* algebra, the answer is affirmative. For span $\mathcal{E}$ is isomorphic to the algebra $l^\infty_{(n)}$, and the result is due to Christensen [7]. Following his technique, it is possible to find a linear map of $\mathcal{E}$ into $\mathcal{B}$ which is close to the identity. However, the use of complete positivity precludes using this proof from that point.

We also mention some other questions remaining in the finite-dimensional situation.

QUESTION 5.2. For which subalgebras $\mathcal{A}$ of $\mathcal{M}_n$ are there constants $\epsilon_0 > 0$ and $C < \infty$ so that if $\mathcal{B}$ is an algebra satisfying $d(\mathcal{A}, \mathcal{B}) = \epsilon < \epsilon_0$, then $\mathcal{B}$ is similar to $\mathcal{A}$ via an operator T such that $\|T-I\| < C\epsilon$. (These algebras might be called stable.)

QUESTION 5.3. Is there a subalgebra $\mathcal{A}$ of $\mathcal{M}_n$ and a sequence $\mathcal{B}_k$ of algebras similar to $\mathcal{A}$ with $d(\mathcal{A}, \mathcal{B}_k)$ tending to zero, yet any similarity from $\mathcal{A}$ to any $\mathcal{B}_k$ is uniformly bounded away from the identity? (Compare with Example 2.5).

Although it has not been used explicitly in this paper, it is important to be able to compute the distance to a reflexive algebra using the lattice. It is easy to show that for any operator T and lattice $\mathcal{L}$,

$$\alpha(T)=\sup\{\|P(L)^{\perp}TP(L)\| : L\in \mathcal{L}\}$$

is at most $d(T, \operatorname{alg} \mathcal{L})$. Alg $\mathcal{L}$ is called *hyper-reflexive* if there is a constant K such that for all operators T, $d(T, \operatorname{Alg} \mathcal{L})\leq K\alpha(T)$. In [2], Arveson shows that $K=1$ suffices for nest algebras. In [6], it is shown that many von Neumann algebras are hyper-reflexive. Both these results have proven to be extremely useful tools for perturbation problems. However, it is shown in [23] that there are CSL algebras which are not hyper-reflexive. This in turn yields algebras with nearby lattices which are not similar. (See [22] for further discussion of these ideas.)

ADDED IN PROOF.

1. John Phillips has informed us that he was aware of Theorem 3.2. His approach: If $d(\mathfrak{A}, \mathfrak{B})$ is small, then the linear projection P of $\mathfrak{B}$ onto $\mathfrak{A}$ is close to the identity. So $m(a_1, a_2)=P(P^{-1}(a_1)P^{-1}(a_2))$ defines an associative multiplication on $\mathfrak{A}$ close to the original multiplication. Now apply Theorem 3 of [20] to obtain the similarity.

2. Ken Harrison and Bill Longstaff [24] have some results on automorphic images of commutative subspace lattices, which are related to Theorem 4.3.

3. A further comment on Example 2.5: The four lines $T_1, \ldots, T_4$ span a two-dimensional (complex) plane. So they can be thought of as four points in the projective plane. The action of $\mathrm{GL}(2, \mathbf{C})$ on $\mathbf{C}^2$ induces an action on the projective plane by fractional linear (Möbius) maps. It is well known that this acts transitively on triples of distinct points, but such a choice is unique. Thus given four points, there are at most 24 possible maps of four points onto a given four. This limits the number of possible algebras $\mathfrak{A}_s$ similar to $\mathfrak{A}_t$ to at most 24 by considering equation (t). Since the even permutations fix $\mathfrak{A}_t$, there are at most 6 possibilities.

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Department of Mathematics
University of Toronto
Toronto, Ontario
Canada M5S 1A1

and

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1