CELLULAR-INDECOMPOSABLE OPERATORS AND BEURLING'S THEOREM

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1. Introduction. Let H be a Hilbert space with norm $\| \ \|$ consisting of functions analytic on the open unit disk Δ. We will assume that H has the following properties.

(1) The polynomials are dense in H.

(2) Multiplication by z, T_z, is a bounded linear operator on H.

(3) If $zg \in H$ for some function g analytic on Δ, then $g \in H$.

(4) For each point $b \in \Delta$, the linear functional of evaluation at b, λ_b, is continuous with respect to the norm of H.

Requiring that H have property (3) is actually equivalent to requiring that T_z be bounded below (see Proposition 2). In this paper we will be concerned primarily with the operator T_z on H.

For $-\infty < \alpha < \infty$, let D_α represent the Hilbert space of analytic functions f on Δ satisfying $\|f\|_\alpha < \infty$, where if $f = \sum_{n=0}^{\infty} a_n z^n$, $\|f\|_\alpha^2 = \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2$. It is not difficult to verify that the spaces D_α with norm $\| \ \|_\alpha$ satisfy properties (1)–(4) above. Note that D_{-1}, D_0, and D_1 are the Bergman, Hardy, and Dirichlet spaces respectively. Also note that the operator T_z on D_α corresponds to the unilateral weighted shift with weight sequence

$$\left\{ \left(\frac{n+2}{n+1} \right)^{\alpha/2} \right\}_{n=0}^{\infty}$$

relative to the orthonormal basis

$$\left\{ \left(\frac{1}{n+1} \right)^{\alpha/2} z^n \right\}_{n=0}^{\infty} \text{ of } D_\alpha$$

(cf. [9]).

We say that a closed subspace $M \subset H$ is invariant if it is invariant under the operator T_z; that is, a subspace M is invariant if it is closed and $zM \subset M$. For a function $f \in H$, define $[f] = H$-closure of $\{pf : p \text{ is a polynomial}\}$. We say that an invariant subspace M of H is cyclic provided there is some function $f \in M$ such that $M = [f]$. We denote by $N_1 \ominus N_2$ the orthogonal complement of N_2 in N_1 for closed subspaces $N_2 \subset N_1 \subset H$; and by $M_1 \vee M_2$, the closed linear span of the subspaces M_1 and M_2 of H. For M_1 and M_2 invariant subspaces of H, observe that $M_1 \cap M_2$ and $M_1 \vee M_2$ are invariant.

Adopting terminology established in [8], we say that the operator T_z on H is cellular-indecomposable if $M_1 \cap M_2 \neq \{0\}$ for any two nonzero invariant subspaces M_1 and M_2 of H. The following proposition applies to T_z on D_0 since each function in the Hardy space is the quotient of H^∞ functions, and since for ϕ a multiplier of D_α and $f \in D_\alpha$, $\phi[f] \subset [f]$ (cf. [3, Proposition 7]).

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PROPOSITION 1. *Let $M(H)$ denote the set of multipliers of H. If for each nonzero $f \in H$ we have $[f] \cap M(H) \neq \{0\}$ and $M(H)[f] \subset [f]$, then the operator T_z on H is cellular-indecomposable.*

Proof. If for nonzero f and g in H we have $f_1 \in [f]$ and $g_1 \in [g]$, where f_1 and g_1 are nonzero multipliers of H, then $f_1 g_1 \in [f] \cap [g]$. □

For $\alpha > 1$, each function of D_α is a multiplier of D_α so that Proposition 1 applies to T_z on these spaces. Similarly, the operator T_z on H is cellular-indecomposable whenever it is similar to a strictly cyclic shift (cf. [9, pp. 92–101]).

Some necessary conditions for a subnormal operator to be cellular-indecomposable are given by Olin and Thomson in [8]. Our main result is the following theorem; its proof is presented in Section 3.

THEOREM 1. *If the operator T_z on H is cellular-indecomposable, then each nonzero invariant subspace M of H satisfies* $\dim(M \ominus (z-c)M) = 1$ *for any $c \in \Delta$ with $|c| < r_1(T_z)$.*

Here $r_1(T_z)$ represents a positive constant associated with the operator T_z on H. Let $c \in \Delta$ satisfy $|c| < r_1(T_z)$. In Section 2, we define $r_1(T_z)$ and show that $(z-c)M$ is closed for any closed subspace M of H. We remark that the dimension of $M \ominus (z-c)M$ must be at least 1 for any nonzero invariant subspace M of H. This follows since $M = (z-c)M$ implies $M = \bigcap_{n \geq 0} (z-c)^n M = 0$.

The referee has pointed out to the author that Theorem 1 is actually a consequence of Lemma 4 of [8], a general result concerning cellular-decomposable semi-Fredholm operators. Our setting allows for a different, more elementary approach to the proof of Theorem 1; moreover, some of the propositions in the sequel which are useful in our approach have independent interest.

Whether or not the converse of Theorem 1 is true is an open question. We do, however, have the following result. Define

$$\operatorname{dist}(M_1, M_2) = \inf\{\|g-f\| : \|g\| = \|f\| = 1, g \in M_1, f \in M_2\}$$

for any nonzero subspaces M_1 and M_2 of H.

THEOREM 2. *If each nonzero invariant subspace M of H satisfies*

$$\dim(M \ominus zM) = 1,$$

then $\operatorname{dist}(M_1, M_2) = 0$ *for any two nonzero invariant subspaces M_1 and M_2 of H.*

In Section 4, we show how Beurling's Theorem follows from our results. Beurling's Theorem states that every invariant subspace of D_0 is cyclic and that each invariant subspace has an inner function as a cyclic vector; in other words, if $M \subset D_0$ is invariant, there is an inner function w such that $M = [w]$. We conclude this introduction with a few remarks concerning a cellular-decomposable operator.

The operator T_z of D_{-1}, the Bergman shift, is cellular-decomposable. Horowitz [6] has exhibited nonzero invariant subspaces M_1 and M_2 of D_{-1} such that $M_1 \cap M_2 = \{0\}$. More recently, Apostol, Bercovici, Foiaş, and Pearcy [1] have shown that the lattice of invariant subspaces for the Bergman shift contains a lattice that is isomorphic to the lattice of all subspaces of D_{-1}. The point is that the Bergman shift belongs to the class $\mathbf{A}_{\aleph_0}$ of universal dilations introduced by Bercovici, Foiaş, and Pearcy in [2]. Their results concerning $\mathbf{A}_{\aleph_0}$ have many interesting applications to T_z on D_{-1}. For example, Corollary 4.3 of [2] asserts the existence of an invariant subspace M of D_{-1} such that $\dim(M \ominus zM) = \infty$.

2. Preliminaries. Let H be a Hilbert space of functions analytic on Δ satisfying properties (1)–(4) above. Recall that an operator A on H is bounded below if $\|Af\| \geq \delta\|f\|$ for some $\delta > 0$ and all $f \in H$. It is not difficult to verify that A is bounded below if and only if $\operatorname{Ker} A = \{0\}$ and the range of A is closed.

PROPOSITION 2. *Let $c \in \Delta$. The operator of multiplication by $(z-c)$ on H, T_{z-c}, is bounded below if and only if $(z-c)g \in H$ (for some function g analytic on Δ) implies $g \in H$.*

Proof. Let $(z-c)g \in H$ and suppose T_{z-c} is bounded below. By property (1) of H there is a sequence $\{p_n\}$ of polynomials such that $\|p_n - (z-c)g\| \to 0$. By property (4) of H, $p_n(c) \to 0$. Therefore,

$$\left\|(z-c)\left(\frac{p_n - p_n(c)}{z-c}\right) - (z-c)g\right\| \to 0.$$

Since $\operatorname{Ran} T_{z-c}$ is closed, $g \in H$.

Conversely, if $(z-c)g \in H$ implies $g \in H$, then the range of T_{z-c} is closed by property (4) of H. Since $\operatorname{Ker} T_{z-c} = \{0\}$, T_{z-c} is bounded below. □

We see that property (3) of H implies that the operator T_z on H is bounded below. Let $m(T_z) = \inf\{\|T_z f\| : \|f\| = 1\}$ be the lower bound of T_z on H, and define

$$r_1(T_z) = \sup_{n \geq 1}[m(T_z^n)]^{1/n} = \lim_{n \to \infty}[m(T_z^n)]^{1/n}$$

(cf. [9, pp. 68, 69]).

It's not difficult to show that T_{z-c} is bounded below whenever $|c| < r_1(T_z)$ (cf. [9, Proposition 13]). Let M be any closed subspace of H. We now observe that $(z-c)M$ is closed for $|c| < r_1(T_z)$. For the spaces D_α, $r_1(T_z) = 1$ (cf. [9, Proposition 15]).

Throughout the remainder of this paper the letter c will denote a complex number in Δ whose modulus is less than $r_1(T_z)$. Also, the letter k will denote a nonnegative integer.

As in [7], we will say that a subspace M of H has property (L) for $(z-c)$ provided

(L) M is invariant, and $(z-c)g \in M$ (for some function g analytic on Δ) implies $g \in M$.

Note that H has property (L) for $(z-c)$. This follows from Proposition 2 since $|c| < r_1(T_z)$ implies that T_{z-c} is bounded below. Thus in considering whether or not an invariant subspace M of H has property (L), we may assume that $g \in H$. Also note that if $M \neq \{0\}$ has property (L) for $(z-c)$, then M contains some function f such that $f(c) \neq 0$.

We now establish several easy propositions for future reference. Proposition 3 is essentially Proposition 1 of [7]; we include a (somewhat easier) proof here for completeness.

PROPOSITION 3. *If a nonzero subspace* $M \subset H$ *has property* (L) *for* $(z-c)$, *then* $\dim(M \ominus (z-c)M) = 1$.

Proof. Let g_1 and g_2 be nonzero elements of $M \ominus (z-c)M$. Since M has property (L) for $(z-c)$, $g_1(c) \neq 0$ and $g_2(c) \neq 0$. Choose the constant β such that $g_1(c) - \beta g_2(c) = 0$. Then $g_1 - \beta g_2 \in (M \ominus (z-c)M) \cap (z-c)M$ (since M has property (L) for $(z-c)$). Hence, $g_1 = \beta g_2$. □

PROPOSITION 4. *Let* $M \subset H$ *be invariant. If* $\dim(M \ominus (z-c)M) = 1$ *and if* M *contains a vector* f *such that* $f(c) \neq 0$, *then* M *has property* (L) for $(z-c)$.

Proof. Let w span $M \ominus (z-c)M$. $w(c) \neq 0$ since not all functions in M vanish at c. Now, if $(z-c)g \in M$ for some analytic function g, then $(z-c)g = \beta w + (z-c)g_1$ for some constant β and some $g_1 \in M$. But $\beta = 0$ since $\beta w(c) = 0$. Hence, $g = g_1 \in M$. □

PROPOSITION 5. *Let* $M \subset H$ *be invariant. If* $\dim(M \ominus (z-c)M) = 1$, *then* $\dim((z-c)^k M \ominus (z-c)^{k+1} M) = 1$.

Proof. Let w span $M \ominus (z-c)M$. $(z-c)^k w$ may be written uniquely as

$$(z-c)^k w = (z-c)^k w_1 + (z-c)^{k+1} w_2,$$

where $w_1, w_2 \in M$ and where $(z-c)^k w_1$ is in $(z-c)^k M \ominus (z-c)^{k+1} M$. Let $f \in (z-c)^k M \ominus (z-c)^{k+1} M$ be arbitrary. $f = (z-c)^k f_1$ for some $f_1 \in M$. Now, $f_1 = \beta w + (z-c) f_2$ for some constant β and some function $f_2 \in M$. Hence

$$\begin{aligned} f = (z-c)^k f_1 &= \beta(z-c)^k w + (z-c)^{k+1} f_2 \\ &= \beta(z-c)^k w_1 + (z-c)^{k+1}(\beta w_2 + f_2). \end{aligned}$$

Since f, $(z-c)^k w_1 \in (z-c)^k M \ominus (z-c)^{k+1} M$, we have $(z-c)^{k+1}(\beta w_2 + f_2) = 0$ so that $f = \beta(z-c)^k w_1$. □

We remark that Proposition 5 may be proved using elementary properties of the Fredholm index.

Let $b \in \Delta$. By the Riesz representation theorem, there is a function $K_b \in H$ such that $f(b) = \lambda_b(f) = \langle f, K_b \rangle$ for each $f \in H$. For a closed subspace $M \subset H$, let P_M denote the orthogonal projection of H onto M. The following proposition is self-evident.

PROPOSITION 6. *Let* M *be an invariant subspace of* H *containing a function* f *such that* $f(c) \neq 0$. *If* $\dim(M \ominus (z-c)M) = 1$, *then* $P_M(K_c)$ *spans* $M \ominus (z-c)M$.

PROPOSITION 7. *If a nonzero invariant subspace* M *of* H *is cyclic, then* $\dim(M \ominus (z-c)M) = 1$.

Proof. Let $f \in M$ satisfy $[f] = M$. $f = w + (z-c)v$ for some $w \in M \ominus (z-c)M$ and some $v \in M$. Let $g \in M \ominus (z-c)M$ be arbitrary. Since f is cyclic for M, there is a sequence $\{p_n\}$ of polynomials such that $\|p_n f - g\| \to 0$. Splitting $p_n f - g$ into the part in $M \ominus (z-c)M$ and the part in $(z-c)M$, we have

$$\|p_n(c)w - g + ((p_n - p_n(c))w + (z-c)vp_n)\| = \|p_n f - g\| \to 0$$

so that $\|p_n(c)w - g\| \to 0$. Hence, $g = \beta w$ for some constant β. □

For the Hardy space D_0, the converse of Proposition 7 is valid.

PROPOSITION 8. *If* $M \subset D_0$ *is invariant and* $\dim(M \ominus zM) = 1$, *then* M *is cyclic.*

Proof. Let w span $M \ominus zM$. Claim $[w] = M$. To prove this we show that $g \in M \ominus [w]$ implies $g \equiv 0$. Let $g \in M \ominus [w]$.

We have $\langle g, z^n w \rangle = 0$ for all $n \geq 0$. Since $\langle g, w \rangle = 0$, we have $g = zg_1$ for some $g_1 \in M$. However, $\langle g_1, w \rangle = \langle zg_1, zw \rangle = \langle g, zw \rangle = 0$, so that $g_1 = zg_2$ for some $g_2 \in M$. Continuing this way we see that g has a zero of infinite order at $z = 0$; hence, $g \equiv 0$ and we must have $[w] = M$. $\square$

REMARK. Let M be a nonzero invariant subspace of D_0. That $\dim(M \ominus zM) = 1$ and that M is cyclic are well known (cf. [4, problem 123]).

PROPOSITION 9. *If* $f \in H$ *satisfies* $f(c) \neq 0$, *then* $[f]$ *has property* (L) *for* $(z - c)$.

Proof. This follows from Propositions 4 and 7 (or may be proved directly). $\square$

The following proposition is an easy consequence of Proposition 9.

PROPOSITION 10. *For* $f, g \in H$ *with* $f(c) \neq 0$ *and* $g(c) \neq 0$, $[f] \cap [g]$ *has property* (L) *for* $(z - c)$.

3. Proofs of Theorems 1 and 2. We are now in a position to prove Theorems 1 and 2 stated in Section 1. Recall that c represents a complex number in Δ of modulus less than $r_1(T_z)$.

Proof of Theorem 1. Let $M \neq \{0\}$ be an arbitrary invariant subspace for T_z on H. We assume that any two nonzero invariant subspaces of H have nonzero intersection and prove that $\dim(M \ominus (z - c)M) = 1$. Without loss of generality, we may assume that M contains a function f such that $f(c) \neq 0$. (If not, $M = (z - c)^k M_0$ for some invariant subspace M_0 containing a function not vanishing at c. By Proposition 5, $\dim(M_0 \ominus (z - c)M_0) = 1$ implies $\dim(M \ominus (z - c)M) = 1$.) We prove Theorem 1 by showing that M has property (L) for $(z - c)$.

Suppose $(z - c)g \in M$ for g analytic on Δ and suppose that $g(c) \neq 0$. We show that $g \in [(z - c)g] \vee [f] \subset M$. By assumption, $[g] \cap [f] \neq \{0\}$. Since any nonzero subspace having property (L) for $(z - c)$ must contain a function not vanishing at c, there is a function $w \in [g] \cap [f]$ such that $w(c) \neq 0$ (Proposition 10). Now, let $\{q_n\}$ and $\{p_n\}$ be sequences of polynomials such that

$$\left\| q_n g + \frac{g(c)}{w(c)} w \right\| \to 0 \quad \text{and} \quad \left\| p_n f - \frac{g(c)}{w(c)} w \right\| \to 0.$$

Note that $q_n(c) g(c) \to -g(c)$ by property (4) of H so that $q_n(c) \to -1$. We have

$$\begin{aligned} \left\| \left(\frac{q_n - q_n(c)}{z - c} \right) (z - c) g + p_n f - g \right\| &\leq \| q_n g + p_n f \| + \| (-1 - q_n(c)) g \| \\ &\leq \left\| q_n g + \frac{g(c)}{w(c)} w \right\| + \left\| p_n f - \frac{g(c)}{w(c)} w \right\| \\ &\quad + |q_n(c) + 1| \, \| g \|. \end{aligned}$$

The last expression goes to zero and it follows that $g \in [(z-c)g] \vee [f] \subset M$.

If we drop the assumption that $g(c) \neq 0$, then $g = (z-c)^k g_1$ for some g_1 such that $g_1(c) \neq 0$. By the argument of the preceding paragraph, we have

$$g_1 \in [(z-c)g_1] \vee [f].$$

It follows by the continuity of T_{z-c} on H that

$$\begin{aligned} g = (z-c)^k g_1 &\in [(z-c)^{k+1} g_1] \vee [(z-c)^k f] \\ &= [(z-c)g] \vee [(z-c)^k f] \subset M, \end{aligned}$$

and we have completed the proof that M has property (L) for $(z-c)$. □

EXAMPLE. Consider the cellular-indecomposable operator T_z on D_0. By Theorem 1, each invariant subspace M of D_0 satisfies $\dim(M \ominus (z-c)M) = 1$ for $|c| < r_1(T_z) = 1$. We must have $|c|$ strictly less than $r_1(T_z)$ since, for example,

$$\dim(D_0 \ominus ((z-1)D_0)^-) = 0.$$

(If $f \in D_0 \ominus ((z-1)D_0)^-$ then $\hat{f}(0) = \hat{f}(n)$ for all $n \geq 0$, so that $f \equiv 0$.)

REMARK. It is easy to see that the following generalization of Theorem 1 is valid. Let $\{c_i\}_{i=1}^k \subset \Delta$ satisfy $|c_i| < r_1(T_z)$ for $i = 1, 2, \ldots, k$. If the operator T_z on H is cellular-indecomposable then each nonzero invariant subspace M of H satisfies $\dim(M \ominus (z-c_1)(z-c_2)\cdots(z-c_k)M) = k$.

Proof of Theorem 2. We assume that each nonzero invariant subspace M of H satisfies $\dim(M \ominus zM) = 1$ and show that $\operatorname{dist}(M_1, M_2) = 0$ for any two nonzero invariant subspaces M_1 and M_2 of H.

Let M_1 and M_2 be nonzero invariant subspaces of H. Without loss of generality, we may assume that there are functions $f \in M_1$ and $g \in M_2$ such that $f(0) \neq 0$ and $g(0) \neq 0$. By assumption, $N = [f] \vee [zg]$ satisfies $\dim(N \ominus zN) = 1$. By Proposition 4, N has property (L) for z, so there exist sequences $\{q_n\}$ and $\{p_n\}$ of polynomials such that $\|p_n f + z q_n g - g\| \to 0$. We have sequences $\{p_n f\} \subset M_1$ and $\{(1 - z q_n) g\} \subset M_2$ such that $\|p_n f - (1 - z q_n) g\| \to 0$.

Now, $\inf\{\|(1 - z q_n) g\|\} > 0$; otherwise, $\|(1 - z q_{n_j}) g\| \to 0$ for some subsequence $\{q_{n_j}\}$ of $\{q_n\}$ and this contradicts the fact that $g(0) \neq 0$. It follows that there is a constant R such that $\inf_{n \geq R}\{\|p_n f\|\} > 0$. We have

$$\left\| \frac{p_n f}{\|p_n f\|} - \frac{(1 - z q_n) g}{\|(1 - z q_n) g\|} \right\| \to 0$$

as $n \geq R$ goes to ∞. Hence, $\operatorname{dist}(M_1, M_2) = 0$. □

4. Beurling's Theorem. By Theorem 1 and Proposition 8, each invariant subspace M of the Hardy space D_0 is cyclic. Also, by the proof of Proposition 8, if w spans $M \ominus zM$ then $[w] = M$. Now w has constant modulus on the unit circle, since w is orthogonal to $z^n w$ for $n \geq 1$. Hence, there is a constant β such that $\beta w = F$ is inner. $M = [F]$ and we have proved Beurling's Theorem.

We note that since $\dim(FD_0 \ominus zFD_0) = 1$ and since $F \in FD_0 \ominus zFD_0$, we must have $[F] = FD_0$ by the proof of Proposition 8.

Finally, we remark that if M contains a function not vanishing at zero, then $P_M(1)$ spans $M \ominus zM$. It follows that $P_M(1)$ is a constant multiple of the inner function F such that $M = FD_0$. It was Beurling's observation that $P_{FD_0}(1) = \overline{F(0)}F$ that led to the well-known proof of his theorem by Helson and Lowdenslager (cf. e.g. [5, p. 99]).

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