BRANCHED IMMERSIONS OF SURFACES

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Introduction. Let M and N be surfaces and $f: \partial M \to N$ an immersion. When can f be extended to an immersion or polymersion $F: M \to N$? (That is, locally $F(z) = z^n$, $n > 0$.) In this paper, we consider the case where M is a compact connected surface of genus k with n boundary components and $N = \mathbf{R}^2$. In addition, f will be required to be a *normal immersion,* that is, it has finitely many intersections and these are transverse double points. We also determine how many different extensions there are.

DEFINITION. Let $f: \partial M \to N$ be an immersion and let $F_1, F_2: M \to N$ be two extensions of f. F_1 and F_2 are *equivalent* if there is a diffeomorphism $h: M \to M$ such that $h|_{\partial M} = \mathrm{Id}$ and $F_1 = F_2 \circ h$.

C. Titus [8] solved the existence problem for $F: D^2 \to \mathbf{R}^2$. S. Blank [3] gave a different and elegant solution to both the existence and equivalence problems by associating to an immersion a natural 'word.' He showed that equivalence classes of immersions which extend f are in one-to-one correspondence with 'groupings' of the word. (These groupings are combinatorial structures on the word.) Blank's method will be generalized in this paper.

Later, K. Bailey [1], M. Barall [2], C. Ezell [4], G. Francis [5], M. Marx [7], and S. Troyer [9] modified some of Blank's techniques to deal with various other cases. The algorithms given in this paper require only the same concise conditions that were present in the simplest case.

ROTATION NUMBER OF f. Let $f: S^1 \to \mathbf{R}^2$. The rotation number (or tangent winding number) of f, $R(f)$, is the index of the map

$$x \mapsto \frac{f_x'(1)}{|f_x'(1)|}: S^1 \to S^1.$$

Given $f: \partial M \to \mathbf{R}^2$, we have $f = f_1 \cup \cdots \cup f_n$, where $f_i: S^1 \to \mathbf{R}^2$ is f restricted to the ith boundary component. Define the *total rotation number* of f to be $\tau(f) = R(f_1) + \cdots + R(f_n)$. By a result of A. Haefliger [6], if the compact surface M is immersed in the plane then $\tau(f) = X(M) = 2 - n - 2k$. However, this condition is not sufficient for f to extend, as can be seen from the example of Figure 1. This and subsequent figures represent images in $\mathbf{R}^2$.

THE BLANK WORD. Let $P_1, \ldots, P_m$ be the bounded connected components of $\mathbf{R}^2 - \mathrm{Im}(f)$. Choose $p_i \in P_i$. For each p_i construct a ray $a_i: [0, \infty) \to \mathbf{R}^2$ which is an embedding such that:

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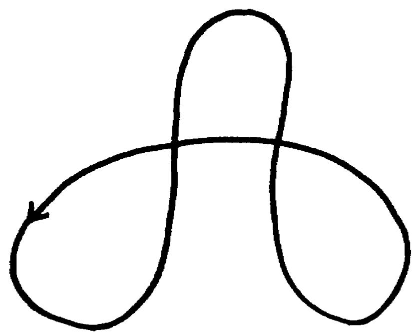

Figure 1

(i) $a_i(0) = p_i$,
(ii) $\hat{a}_i$ is unbounded,
(iii) $\hat{a}_i \cap \hat{a}_j = \varnothing$ for $i \neq j$,
(iv) $\hat{a}_i$ does not intersect f in a double point, and
(v) $\hat{a}_i$ is transverse to f.

Note that we refer to $\mathrm{Im}(a_i)$ as $\hat{a}_i$. The set $\{\hat{a}_i\}$ is called the ray system for f. Intersection points of $\hat{a}_i$ and $\mathrm{Im}(f)$ will be denoted a_i^{+1} or a_i^{-1}. The sign is positive if f crosses the ray from right to left and negative otherwise. The natural ordering on $\hat{a}$ will be denoted by $<$.

The Blank word of f_j, $w(f_j)$, is the sequence of signed letters representing the intersection points of $\{\hat{a}_i\}$ and $\mathrm{Im}(f)$ obtained in one circuit of the curve in the direction indicated by the orientation. This word is defined up to cyclic permutation and may be written uniquely upon the choice of a basepoint of S^1. All cyclic permutations of $w(f_j)$ will be considered equivalent. The word $w(f_j)$ may be thought of as an element of the free group on m letters. It is often useful to consider the intersection points $a_i^{\pm 1}$ to be on the domain of f_j. Note that if f_j had been the sole boundary immersion, the word obtained from it would be a subword of $w(f_j)$.

The word $w(f_j)$ is said to be *reduced* if it contains no subsequence of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$. Since $w(f_j)$ is defined only up to cyclic permutation, reduced also means that the word is not of the form $a_i^{\pm 1} \cdots a_i^{\mp 1}$. If there is a non-reduced occurrence along $\hat{a}_i$ the ray may be altered slightly so that $w(f_j)$ is reduced: Change a_i in a neighborhood of $(a_i^{\pm 1}, a_i^{\mp 1})$ to parallel and run close to $f \mid_{a_i^{\pm 1} a_i^{\mp 1}}$. It can be easily verified [3] that this modified ray system has the effect of cancelling $a_i^{\pm 1} a_i^{\mp 1}$ and perhaps other such consecutive pairs. Henceforth, $w(f_j)$ is assumed to be reduced.

Given $f = f_1 \cup \cdots \cup f_n$, we define the *Blank word of f*, $w(f)$, to be the set of the n Blank words, $\{w(f_i)\}$.

DEFINITION. Two pairs, $(a_i^{\pm 1}, a_i^{\mp 1})$ and $(a_j^{\pm 1}, a_j^{\mp 1})$, of a component w of a word are *linked* if

$$w = \cdots a_i^{\pm 1} \cdots a_j^{\pm 1} \cdots a_i^{\mp 1} \cdots a_j^{\mp 1} \cdots .$$

OPERATIONS ON $w(f)$.

(1) *Join*—Let w_1 be a component of a word that is of the form $Aa_i^{\pm 1}$, where A is a subsequence of w_1. Similarly, assume that w_2 can be written as $a_i^{\mp 1}B$. Define the *join* of w_1 and w_2 to be the replacement of these two words by the single component AB.

This algebraic operation corresponds to the following geometric construction on M: Join the points a_i and a_i^{-1} on ∂M with an embedded interval in M transverse to ∂M. Cutting along this arc segment yields a surface of genus k with $n-1$ boundary components. If we follow this new boundary component we obtain the word AB.

(2) *Assemblage*—Let w be a component of a word with linked pairs $(a_i^{\pm 1}, a_i^{\mp 1})$ and $(a_j^{\pm 1}, a_j^{\mp 1})$. That is, $w = Aa_i^{\pm 1}Ba_j^{\pm 1}Ca_i^{\mp 1}Da_j^{\mp 1}$. An *assemblage operation* consists of replacing w with the word $ADCB$.

This corresponds to the following operation on M: If M has genus > 0, an arc segment joining a_i and a_i^{-1} may be constructed on M so that M is not disconnected. Cut along this arc segment to obtain a surface of genus one less with one additional boundary component. Because the pairs had been linked, a_j and a_j^{-1} will now be on different boundary components. Join these two components using a_j and a_j^{-1} as above. We now have the original number of boundary components again. The word obtained by following around the new boundary component will be $ADCB$.

(3) *Grouping*—A component of a word is said to *group* if each negative letter can be paired with a positive letter from the same ray so that the pairs are disjoint and no two pairs are linked.

When M is D^2, the property of grouping is equivalent to being able to construct pairwise disjoint arc segments $[a_i^{-1}, a_i]$ on M for all negative letters.

DEFINITION. A word $w(f)$ is said to have a *k-grouping* if the components $w(f_1), \ldots, w(f_n)$ can be successively joined into a single component so that the resulting word admits a sequence of k assemblage operations after which the word groups.

In the example of Figure 2, M has two boundary components and genus one. Note that if M has a k-grouping, then the arcs on M connecting the pairs of the k-grouping divide M into disks.

DEFINITION. Two k-groupings of $w(f)$ are *equivalent* if each negative letter of $w(f)$ is paired to the same positive letter by both k-groupings.

The example in Figure 2 has two non-equivalent k-groupings, depending on the choices along the ray $\hat{a}$.

THEOREM 1. *Let M be a compact connected orientable surface of genus k with n boundary components. A normal immersion $f: \partial M \to \mathbf{R}^2$ can be extended to an immersion $F: M \to \mathbf{R}^2$ if and only if $\tau(f) = 2 - n - 2k$ and $w(f)$ has a k-grouping. The number of non-equivalent extensions will be equal to the number of non-equivalent k-groupings.*

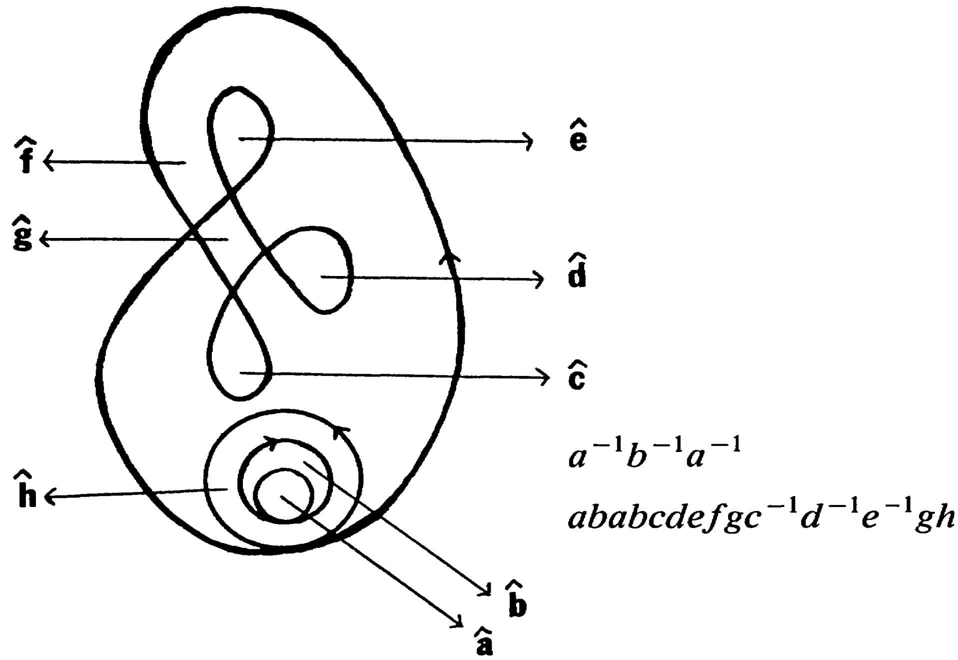

Figure 2

Proof of Theorem 1. First we show how a k-grouping yields an extension of f. As mentioned previously, we want to consider a letter of $w(f)$ to be on both the intersection of $\text{Im}(f)$ with a ray and the corresponding point of ∂M. For each pair (a_i, a_i^{-1}) of the k-grouping, the points a_i and a_i^{-1} on ∂M are to be joined with an embedded interval in M transverse to ∂M. The intervals are to be constructed in the order indicated by the definition of a k-grouping, that is, joins first, then assemblage operations, then the grouping. In addition, the intervals are to be mutually disjoint. The set of embedded intervals will then divide M up into disks. Label these disks D_i.

We now extend f to $\partial M \cup \{\text{added intervals}\}$ by mapping the interval from a_i to a_i^{-1} in M to the interval (a_i, a_i^{-1}) in the ray a_i. Call this extension f^*. We would like to define f_i as f^* restricted to the boundary of D_i, but we want the f_i terms to be immersions in general position.

Extend f^* to embeddings of tubular neighborhoods of the intervals. Change the boundary of each disk D_i in the following way: For each interval on the boundary of D_i, round off the corners so that no new intersections are introduced. This will make f_i an immersion.

DEFINITION. A pair (a_i^{+1}, a_i^{-1}) is called *positive* if a_i^{+1} is farther out on the ray than a_i^{-1} and is *negative* otherwise.

Now we change f_i so that it is in general position. This is done by pulling ∂D_i away from the intervals one interval at a time. Those corresponding to positive pairs are to be done first, so that for each ray there will be a tubular neighborhood that contains all intervals from negative pairs and none from positive pairs. See Figure 4.

We will eventually show that there can be no negative pairs and that each f_i is an embedding. Therefore, by the Riemann mapping theorem, each f_i extends to D_i and f^* extends to M. Note that the extension class is independent of the embeddings of the intervals between the pairs, for any two embeddings could be made to agree by a diffeomorphism of M fixing ∂M.

In the works of some of the other authors [1, 2, 4, 5, 7, 9], the letters of the word are endowed with additional structure in order to detect positive pairs. The lemmas that follow show that this structure and analysis is not required for the generalization of Blank's theorem.

DEFINITION. The parts of $\mathrm{Im}(f_i)$ which correspond to these pulled away intervals are called *positive* or *negative incidences* (depending on the type of pair they came from). The parts of $\mathrm{Im}(f_i)$ that are restrictions of f are called *arcs*.

CONSTRUCTION. Let $g: S^1 \to \mathbf{R}^2$ be a normal immersion. Then $\mathrm{Im}(g)$ can be broken up into embedded circles, $\{C_j\}$, of which there are P with $R(C)=1$ and N with $R(C)=-1$ such that $\sum_j R(C_j)=P-N=R(g)$.

Proof. Consider a self-intersection of g. Perturb g near this point so that the two arcs containing it are tangent. See Figure 3. This will have no effect on the rotation number. Now reparameterize $\mathrm{Im}(g)$ as shown so that there are two tangent immersions whose rotation numbers add up to $R(g)$. Continue applying this construction to these two immersions until all are embeddings C_j (with $R(C_j)=\pm 1$), with $\sum_j R(C_j)=R(g)$. □

Figure 3

Consider f_i, the immersions defined on ∂D_i. Let Q_i be the number of negative incidences in ∂D_i. Break up f_i into embeddings so that $R(f_i)=P_i-N_i$. Note that $\sum_i Q_i=2c$, where c is the total number of negative pairs in the k-grouping. Henceforth, we will suppress the subscript i when it is clear that we are working within a single D_i.

LEMMA 1. *In a break-up of f_i, there cannot be a negative circle composed solely of arcs.*

Proof. No ray can intersect a negative circle, for that would give an unpaired a^{-1} in $w(f)$. Therefore, none of the regions defined by $\mathrm{Im}(f)$ can be contained entirely within a negative circle. □

The next two lemmas are due to S. Blank [3]. They were originally proved with $M=D^2$ but are shown here to hold for any M we consider.

LEMMA 2. $N \leq Q$.

Proof. Let C be a negative circle obtained in a break-up of f_i. By Lemma 1, C must contain at least one incidence. Suppose it is a positive incidence. Then C must also have a negative incidence along the same ray such that the interval of the negative incidence contains that of the positive. The inside of C must be between the positive incidence and the ray but cannot cross the ray, by Lemma 1. So there must be a negative incidence between the positive incidence and the ray.

Thus every negative circle has at least one negative incidence. We now show that one negative incidence cannot be shared by two negative circles.

Look at a tubular neighborhood T of $\hat{a}$ which contains only the negative incidences. Let m be the number of such incidences. For each negative incidence there is one arc which enters T and one which leaves T. There may also be arcs of f_i crossing the ray. Circles which contain arcs crossing a ray must be positive. Discounting these arcs, we have m arcs entering and leaving T. These can yield at most m different circles. □

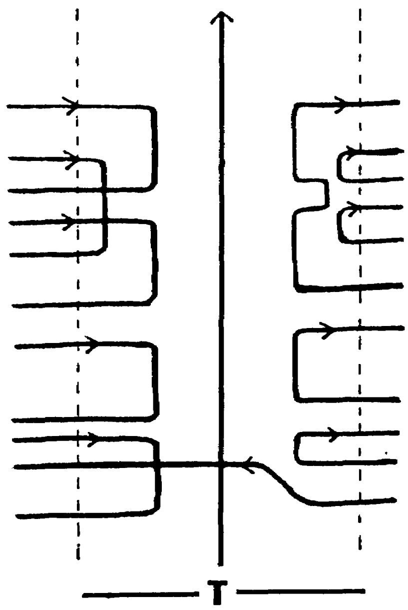

Figure 4

DEFINITION. In a break-up of f_i, a *key circle* is a negative circle with two negative incidences or a positive circle.

LEMMA 3. *For each f_i, there is at least one key circle. Equivalently,*

$$P - N + Q \geq 1 \quad (P > 0 \text{ or } N < Q).$$

Proof. We work toward a proof by contradiction. Suppose $P = 0$ and $N = Q$. Let a^{-1} be the point of ∂D which is on a negative incidence and which is the largest such point on $\hat{a}$. We assume that a^{-1} implies the arc is leaving T. (If the arc is entering T the following argument still holds, but f_i must be traced in the opposite direction.)

Let C be the circle in the break-up of f_i that contains a^{-1}. Let a be the first point on C after a^{-1} which is on an incidence from $\hat{a}$. We now prove a stronger fact than the lemma: There is a key circle along $f_i \mid_{a^{-1}a}$ (or $f_i \mid_{aa^{-1}}$ if the arc is leaving T).

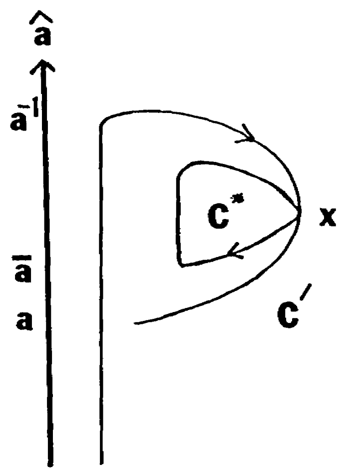

Figure 5

Let $C' = C \mid_{a^{-1}a}$. There cannot be an incidence on C' since C can have at most one negative incidence (we are assuming that $N = Q$). So if $C' = f_i \mid_{a^{-1}a}$ then $C' = f \mid_{a^{-1}a}$ and $w(f)$ would not be reduced. Therefore there must be a circle C^*, tangent to C' at a point x. $R(C^*) = -1$ (since $P = 0$ by assumption), so the interiors of C and C^* intersect in a neighborhood of x. C^* must have an incidence along $\hat{a}$ or else there would be a region defined by $\mathrm{Im}(f)$ contained in the intersection of the two interiors. Thus C^* meets $\hat{a}$. Let $\bar{a}$ be the first point of intersection of C^* with $\hat{a}$ after x. Restore the original parameterization at the point of tangency to obtain an arc C'' from a^{-1} to $\bar{a}$. But the same argument can be applied to C''. Therefore we obtain a contradiction by infinite repetition. □

The proof of Lemma 3 yields the following stronger statement.

LEMMA 4. *Let C be a circle in a break-up of f_i with the following property: C contains a^{-1}, a letter from a negative incidence such that there is a positive letter a^{+1} on C with $a^{+1} < a^{-1}$. Then $f_i \mid_{a^{-1}a^{+1}}$ (or $f_i \mid_{a^{+1}a^{-1}}$) contains a key circle.*

LEMMA 5. *Let $\bar{M}$ be the result of cutting and smoothing M along one of the pair intervals (a_i, a_i^{-1}). Let g be the resulting immersion on $\partial\bar{M}$. Then if the pair is positive, $\tau(g) = \tau(f) + 1$. Otherwise $\tau(g) = \tau(f) - 1$.*

Proof. The proof is immediate from Figure 6.

LEMMA 6. *Let w be the number of pairs in the k-grouping. Then M is divided up into $(2 - n - 2k + w)$ disks by the intervals connecting the pairs.*

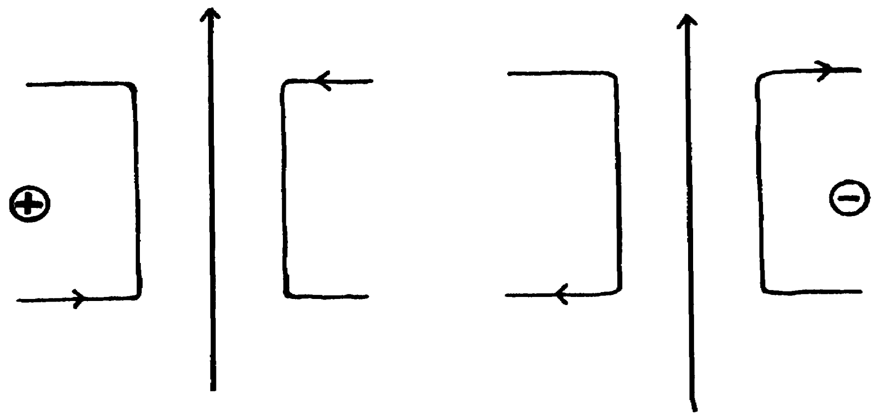

Figure 6

Proof. After performing the required $(n-1)$ joins and k assemblage operations, M has been reduced to a single disk. Each of the remaining $(w-n+1-2k)$ pairs create an additional D_i-type region. □

LEMMA 7. *For each f_i, there is exactly one key circle. Equivalently,*

$$P-N+Q=1.$$

That is, either $(P=0$ and $N=Q-1)$ or $(P=1$ and $N=Q)$.

Proof. Let w be the total number of pairs. Let c be the number negative pairs and d the number of positive pairs. For each f_i we have $R(f_i)=P_i-N_i\geq 1-Q_i$ be Lemma 3. Then

$$\sum_{i=1}^{2-n-2k+w} R(f_i)\geq (2-n-2k+w)-2c$$
$$=2-n-2k+d-c.$$

We could have obtained this sum of the $R(f_i)$'s in another way. The f_i's are obtained from f by cutting along d positive pairs and c negative pairs. By Lemma 5 we have

$$\sum_i R(f_i)=\tau(f)+d-c$$
$$=2-n-2k+d-c.$$

Therefore, the previous inequality must actually have been an equality. This could only have occurred if $P_i-N_i=1-Q_i$ for each region D_i.

COROLLARY 1. *Let $f\colon \partial M\to \mathbf{R}^2$ where M is a compact connected orientable surface with n boundary components and genus k. If f has a k-grouping then $\tau(f)\geq 2-n-2k$.*

Proof. Follow the same procedure as in the proof of Lemma 7 but make no assumption on $\tau(f)$. □

The next three lemmas were proved by S. Blank [3] for $M=D^2$ and are now shown to hold for any M.

LEMMA 8. *If D is a negative circle obtained in a break-up of f_i, then no part of D can come from a positive incidence.*

LEMMA 9. *In a break-up of f_i, no positive circle can contain any part of a negative incidence.*

LEMMA 10. *There are no negative pairs* $(Q=0)$.

The proofs of Lemmas 8 and 9 will be given after the proof of Lemma 10.

Proof of Lemma 10. Let (a, a^{-1}) be a negative pair. Let D_i and D_j be the two D_i-type regions containing the two sides of the incidence from (a, a^{-1}). Note that we may have $D_i = D_j$. Let P be the region defined by $\mathrm{Im}(f)$ that contains the point a^{-1} on its boundary and intersects the interval (a, a^{-1}) (on the ray). Break up f_i and f_j and consider the circles which contain the negative incidences (a, a^{-1}). By Lemma 9 these circles are negative. Suppose P is not contained in the union of the interiors of these circles. This means that P "overflows" beyond the two circles through some incidences. By Lemma 8 these incidences must be negative. Apply the same technique on the circles on the other side of these new incidences. Eventually we contain P in a union of negative circles. This means that P cannot be the unbounded region defined by $\mathrm{Im}(f)$, since all incidences are in the interior of P and the boundary is only arcs. Therefore P must contain the origin of a ray. But this contradicts the fact that no ray can originate in the interior of a negative circle. So there are no negative incidences. □

Proof of Lemma 8. Let a be the largest point on $\hat{a}$ corresponding to a point on a positive incidence in D. By the proof of Lemma 2 there is a point $\underline{a}^{-1}$ on $\hat{a}$ that corresponds to the end of a negative incidence on D with $\underline{a}^{-1} > a$. Similarly, if a^{-1} is the smallest point on $\hat{a}$ on a positive incidence in D then there is a point $\underline{a}$ with $\underline{a} \le a^{-1}$ such that $\underline{a}$ is the end of a negative incidence on D. We first consider the case where $\underline{a}^{-1} > a$ and $\underline{a} < a^{-1}$.

Consider the two restrictions $f_i|\widehat{a^{-1}\underline{a}}$ and $f_i|\widehat{\underline{a}^{-1}a}$. In a break-up of f_i there is exactly one key circle. Therefore at least one of the two restrictions is without a key circle, contradicting Lemma 4.

Now look at the case where the positive and negative incidences on D terminate at the same point on $\hat{a}$. This point must correspond to a negative intersection of $\mathrm{Im}(f)$ with $\hat{a}$. Therefore, the positive incidence must extend beyond that point. So there must be another circle, D', tangent to D at the largest point of the positive incidence on D. If D' is a positive circle we are done since the remainder of f_i would not have a key circle. If D' is a negative circle, substitute D' for one of the restrictions of f_i in the first case and perform the same analysis. □

Proof of Lemma 9. First consider a circle C which travels along a negative incidence and crosses the ray $\hat{a}$. (The same argument holds for a circle which first crosses a ray and then goes along a negative incidence.) Since no ray can intersect a negative circle, $R(C) = +1$. The intersection of C with $\hat{a}$ must be positive and so cannot correspond to the end of the negative incidence. Therefore, the incidence continues farther along the ray. By Lemma 7 there can be no more than one positive circle in a break-up of f_i, so this negative incidence must be on a negative circle. But then f_i restricted to the arc defined by this circle is without a key circle. So no such C can exist.

Let T be the tubular neighborhood of $\hat{a}$ which contains only the negative incidences. By the previous paragraph, all of these incidences are on circles which enter and leave T without crossing $\hat{a}$. If there are m incidences, there can be at most m negative circles. If one of the circles were positive there would be strictly less than m negative circles. There would then be two key circles, a violation of Lemma 7. □

Now we have $P-N+Q=1$ and $N=Q=0$. Therefore, $R(f_i)=P=1$ and f_i is an embedding and can be extended to D_i. We now have an extension $F\colon \to M \to \mathbf{R}^2$. We have already mentioned that the extension class of F is independent of the intervals. Note that the constructed intervals are part of the inverse image of the rays $F^{-1}\{\hat{a}_i\}$. Therefore, any two different k-groupings of $w(f)$ would result in extensions that could not be factored through each other. So we get one extension class for each k-grouping.

We now complete the proof of Theorem 1 by showing that if f extends to F, then $w(f)$ has a k-grouping and $\tau(f)=2-n-2k$.

As mentioned earlier, $\tau(f)=2-n-2k$ by a result of A. Haefliger [6]. Let $\{\hat{a}_i\}$ be a ray system for $f=F\mid_{\partial M}$. Look at the inverse image of all ray segments in $\operatorname{Im}(f)$. Each a^{-1} is paired with the a^{+1} which is the other boundary point of the interval in $F^{-1}(\hat{a})$ which contains a^{-1}. We claim that this natural pairing determines a k-grouping.

We first show that all boundary components can be joined. That is, no proper subset of $\{f_i\}$ has the property that all negative letters are paired with positive letters from the same subset.

Assume that m such mutually exclusive subsets of $\{f_i\}$ do exist. Let g_i ($i=1,\dots,m$) be f restricted to the ith subset. Let n_i be the corresponding number of boundary components.

We now consider $w(g_i)$. We can assume that the n_i components of $w(g_i)$ may be completely joined. If $w(g_i)$ does not group then there must be linked pairs (a_i, a_i^{-1}) and (a_j, a_j^{-1}) such that $w(g_i)=Aa_i^{\pm 1}Ba_j^{\pm 1}Ca_i^{\mp 1}Da_j^{\mp 1}$. Since the two intervals connecting these pairs are disjoint, cutting along them reduces the genus by one. The resulting word $ADCB$ must either group or have two more linked pairs. Continue cutting along linked pairs until the resulting word groups.

Do this for all g_i's. Note that the total number of all linked pair steps for all g_i's must be $\le k$ since otherwise the intervals could not be pairwise disjoint. Let k_i be the number of linked pair steps in the process for g_i.

Since $w(g_i)$ has a k_i-grouping, by Corollary 1 $\tau(g_i)\ge 2-n_i-2k_i$. Therefore,

$$\tau(f)=\sum_{i=1}^{m}\tau(g_i)\ge\sum_{i=1}^{m}(2-n_i-2k_i)=2m-n-2\sum k_i$$
$$\ge 2m-n-2k.$$

But since $\tau(f)=2-n-2k$, we must have $m=1$. Note also that for the above to be an equality, there must be exactly k steps of linked pairs. Therefore $w(f)$ has a k-grouping. □

BRANCHED IMMERSIONS. We now consider extensions with branch points. A *polymersion* $F: M \to \mathbf{R}^2$ is a map locally of the form $F(z) = z^m$, $m > 0$ and z a complex variable. We seek necessary and sufficient conditions for extending a normal immersion $f: \partial M \to \mathbf{R}^2$ to a polymersion. As in the case of extensions to immersions, these conditions are presented in terms of $w(f)$ and operations that modify the word. We define an *appendage* to $w(f)$ to be a collection of L words ($L \geq 0$), each of the form $(a^{-1})^p$, where a is a positive letter in $w(f)$ and $p > 1$. The *valence,* val, of an appendage is the sum $\sum_{j=1}^{L} p(j)$. We denote the word $w(f)$ together with an appendage by $w(f)^*$.

THEOREM 2. *An immersion $f: \partial M \to \mathbf{R}^2$ extends to a polymersion from M if and only if $w(f)$, the word of f, admits an appendage satisfying*
(i) $\tau(f) = X(M) + \text{val} - L$ *and*
(ii) $w(f)^*$ *has a k-grouping.*

Proof of Theorem 2. Assume F is a polymersion extending f. Since M is compact there are only a finite number of branch points of F, that is, points in M where F is locally of the form $F(z) = z^m$ ($m > 1$); $m-1$ is the multiplicity of the branch point. Furthermore, we can assume that each component in $\mathbf{R}^2$ of the complement of the image of f contains at most one image of a branch point, and that no such image point is on the image of f. In addition, we can assume that the ray for a component containing a branch value starts at that value. Let $v_1, \ldots, v_s$ denote the values of branch points and for each v_i let b_i^j, $j = 1, \ldots, \sigma(i)$, be the branch points with value v_i. Let $m(i,j) - 1$ be the multiplicity of b_i^j. Choose a family of circles S_i^j embedded in M which satisfy: each circle bounds a disk containing one branch point, namely b_i^j; the images of the circles under F are mutually disjoint; and these images are in a standard form, namely each is a normal immersion with $m(i,j) - 1$ double points as illustrated in Figure 7. Furthermore, if b_i^j and b_i^{j+1} are two branch points then we require the image of S_i^{j+1} to be within the innermost component, that is, the disk component of $F(S_i^j)$.

Let F' be the restriction of F to the closure in M of the complement of the disks enclosed by these circles, and let f' be the restriction of F' to the boundary of this

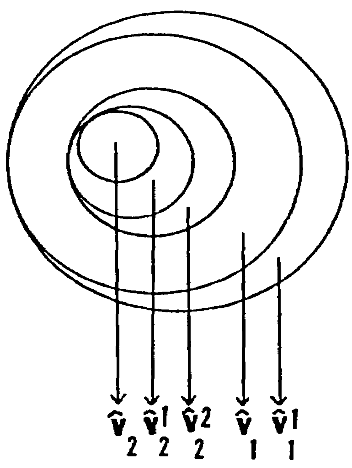

Figure 7

subsurface of M. Note that this subsurface still has genus k. Since F' has no branch points it is an immersion extending f'. We will show how to relate a word for f', $w(f')$, to $w(f)$ and a k-grouping of $w(f')$ to a k-grouping of an appended $w(f)$.

First choose rays $\hat{v}_i^1, \hat{v}_i^2, \ldots, \hat{v}_i^{t(i)}$, where: $t(i) = \sum_{j=1}^{g(i)} m(i,j) - 1$ which are paralell to $\hat{v}_i$ and lie within a tubular neighborhood of $\hat{v}_i$ containing rays from the system for f; each $\hat{v}_i^n$ is to the right of $\hat{v}_i^{n+1}$, where 'right' is with respect to the forward direction (towards ∞) of the rays; and such that each $\hat{v}_i^{n+1}$ intersects the image of f' in one fewer place than $\hat{v}_i^n$ for each $n \geq 0$, where $\hat{v}_i^0 = \hat{v}_i$ (see Figure 7).

Construct a word for f' using the ray system for f. One obtains a word for f' by performing the following operations on $w(f)$:

Type I. Replace each instance of a branch value v_i in $w(f)$ (resp. v_i^{-1}) with the string $v_i v_i^1 v_i^2 \cdots v_i^{t(i)}$ (resp. $[v_i^{t(i)}]^{-1} \cdots [v_i^2]^{-1}[v_i^1]^{-1}[v_i]^{-1}$).

Type II. Construct L additional words, where L is the total number of branch points and where each new word is of the form

$$[v_i]^{-1} \cdot ([v_i^1]^{-1}[v_i]^{-1}) \cdots ([v_i^{m(j+1)}]^{-1} \cdots [v_i^1]^{-1}[v_i]^{-1}),$$

where $m(j) = \sum_{k=1}^{j} m(i,k) - 1$. That is, introduce a word for the circle S_i^J, $J \geq j$.

An appendage to $w(f)$ is obtained by deleting from each type II word all the letters $[v_i^j]^{-1}$ where $j > 0$.

Since F' is an immersion extending f', the word $w(f')$ has a natural k-grouping that is given by the pairings determined by the inverse image under F' of the rays in the system for f'. We now show that these pairings also yield a k-grouping for the appended word $w(f)^*$. Recall that a given pairing can be regarded as a joining pair, part of a linked pair, or as a grouping pair. Observe that if a string of negative letters from either of the two operation types contains a letter which is part of a join or a linked pair, then the remaining letters, after rearrangement in the latter case, are from grouped pairs. In fact, if one performs reduction after joining or performing an assemblage, these letters are eliminated. Thus the pairings in $w(f)^*$ obtained by restricting the pairings in $w(f')$ have the same number of joins and linked pairs as $w(f')$, and hence is a k-grouping.

Furthermore, the total rotation numbers of f and f' satisfy

$$\tau(f') = \tau(f) - \sum_{ij} m(i,j),$$

since each boundary component S_i^j has rotation number $-m(i,j)$. Since $X(M) - L$ is the Euler characteristic of the domain of F' and $\sum_{ij} m(i,j) = \text{val}$, we have $\tau(f) = X(M) + \text{val} - L$. Thus the condition of Theorem 2 is necessary.

Conversely, assume there is an appended word for $w(f)$ with a k-grouping. Choose L circles in M which bound L disjoint disks. For each appendage word $[v_i^{-1}]^k$ select an immersion from one of these circles into the component containing the initial point of the ray $\hat{v}_i$. We select these mappings so that the images are in standard form, as described earlier. Denote the resulting extension of f by f'. Construct additional rays as before. The pairings in $w(f)^*$ extend to pairings in $w(f')$ so that adjacent negative letters in a type I or type II string are paired with adjacent positive letters in a type I string of positive letters. Furthermore, since each such negative string contains at most one join letter or one linked pair letter and the remaining letters are in groupings which reduce, the pairings in $w(f')$ do comprise a k-grouping.

Since the total rotation number of f' equals the Euler characteristic of the subsurface of M which is the closure of the complement in M of the disks, by Theorem 1 there is an immersion F' from this subsurface extending f'.

Finally, extend F' to the interior of each disk using a polymersion with one branch point. Thus one obtains a polymersion from M extending f. Note that the number of branch points, their multiplicities, and the components of $\mathbf{R}^2 - \mathrm{Im}(f)$ where their values lie can be discerned from the appendage of $w(f)$. Conversely, this data determines an appendage. □

Equivalence classes of extensions. Since any polymersion is equivalent to one with no branch value on the image of f and with no component of $\mathbf{R}^2 - \mathrm{Im}(f)$ having more than one branch value, we consider only these polymersions. Below is the image in $\mathbf{R}^2$ of an f with word *abcabb* which admits two extensions to D^2.

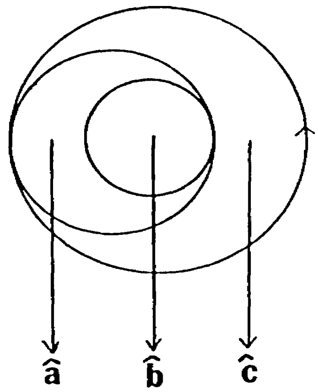

Figure 8

One extension is determined by the appendage $b^{-1}b^{-1}b^{-1}$ to *bcabba*, and the pairings

$$b^{-1}b^{-1}b^{-1}bcabba.$$

Another extension is determined by appending words $a^{-1}a^{-1}$ and $b^{-1}b^{-1}$ to *bbabca*, and by the pairings

$$a^{-1}a^{-1}b^{-1}b^{-1}bbabca.$$

We do not distinguish between two pairings which differ by a cyclic permutation of the letters in an appendage.

THEOREM 3. *Two polymersion extensions of f are equivalent if and only if they induce the same appendage for $w(f)$ and the same pairings on the appended word.*

Proof. If F_1 and F_2 are equivalent extensions, choose circles S_i^j for F_1 as in the proof of Theorem 2 and let $h(S_i^j)$ be the corresponding circles for F_2. The diffeomorphism h restricts to a diffeomorphism from the surface obtained by deleting the interior of the disks enclosed by the S_i^j to the corresponding subsurface obtained by deleting the interiors of the $h(S_i^j)$. By reparameterizing these latter circles, if necessary, we can assume that $F_1=F_2$ on the boundary of the subsurface. Let F_1' and F_2' denote the restrictions of F_1 and F_2. By Theorem 1, F_1' and F_2' are equivalent immersions extending the map f' on the boundary. Choose ray systems for f and f' as in the proof of Theorem 2. Note that the appendages determined by F_1 and F_2 are the same since they are equivalent extensions. By Theorem 1, the pairings given by F_1' and F_2' on this appended word are the same.

Conversely, assume F_1 and F_2 are polymersions extending f which determine the same appendage for f and the same pairings. One can choose L circles S_i^j around the branch points of F_1 and L circles $h(S_i^j)$, diffeomorphic images of S_i^j, around the branch points of F_2. Recall that the appendage of $w(f)$ determines the number of branch points, their multiplicities, and the component of $\mathbf{R}^2-\operatorname{Im}(f)$ where each branch point maps. Furthermore, we may choose these circles so that $F_1=F_2\circ h$ on S_i^j.

As before, we consider the subsurfaces of M obtained by deleting the interiors of the circles chosen. By Theorem 1, the restriction of F_1 and F_2 are equivalent by a diffeomorphism of the subsurfaces extending h. Since on each S_i^j, $F_1=F_2\circ h$ and is in standard form, we furthermore can extend h, uniquely up to equivalence, to the interior of the circles. Hence the polymersions are equivalent. □

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