NONCOMPACT RIEMANNIAN MANIFOLDS WITH PURELY CONTINUOUS SPECTRUM

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1. Introduction. Let (M^n, ds^2) be an n-dimensional Riemannian manifold with Laplacian Δ, where

$$\Delta u = \frac{1}{\sqrt{g}} \Sigma_{ij} \left(\frac{\partial}{\partial x^i} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} u \right)$$

in local coordinates. Here $ds^2 = \Sigma_{i,j} g_{ij} dx^i dx^j$, $g = \det(g_{ij})$, and $(g^{ij}) = (g_{ij})^{-1}$. It is well-known that if M^n is compact then the spectrum of Δ is discrete ([5]). On the other hand, if M^n is noncompact the spectrum may be purely continuous, as it is for Euclidean space (and hyperbolic space, see [9]), purely discrete (see [4], [11], [16]), or a mixture of the two types, possibly with eigenvalues embedded in the continuous spectrum (cf. [10]).

It was first shown by Pinsky [20] (for rotation invariant metrics) and then by Donnelly [10] (for general metrics) that if M is simply-connected and has curvature $K(r, \theta) \leq -k^2$ with $K \to -k^2 < 0$ and angular derivatives K_θ and $K_{\theta\theta} \to 0$ (all with sufficient speed) as $r \to \infty$, then the Laplacian has no eigenvalues (i.e., the spectrum is purely continuous as it is for the constant curvature case $K \equiv -k^2 < 0$). In this paper we prove an analogous result for manifolds with $K \to 0$, involving only conditions on K and K_θ. Our method is related to Rellich's original work on this problem for Euclidean space [23], while Donnelly uses a modification of Kato's extension [17] of Rellich's work to operators with variable coefficients. The general Rellich-Kato procedure, which involves estimating the growth of integrals of solutions of the eigenvalue equation (in this case $\Delta f + \lambda f = 0$) has already proved itself useful in mathematical physics (cf. [1], [25]).

It is a pleasure to thank M. Pinsky and H. Donnelly for sending us preprints of their work. As in [10] and [20], we write out the proofs only for two-dimensional surfaces for simplicity of exposition. There is a known technique for translating these results to manifolds of dimension ≥ 3 which has already been sketched in [10].

In discussing eigenvalues (as opposed to the continuous spectrum) it is important to fix a specific self-adjoint extension of a given symmetric operator. For this reason we give, in Section 2, a quick proof of Gaffney's result [12] concerning the essential self-adjointness of Δ.

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2. Preliminaries: essential self-adjointness and Gaffney's theorem. We take [22] as a basic reference for results on self-adjoint operators.

Let T be a densely defined symmetric operator on a Hilbert space H, with adjoint operator T^*: $T \subseteq T^*$. Recall that T is essentially self-adjoint if and only if its closure $\bar{T}$ is self-adjoint, and this occurs if and only if T has a unique self-adjoint extension A. In fact, $A = \bar{T}$.

Now the existence of isolated eigenvalues of finite multiplicity in the spectrum of a self-adjoint extension of T depends, in general, on the extension chosen. Thus a discussion of the non-existence of eigenvalues in the spectrum of (a self-adjoint extension of) $T = \Delta$, where say $\Delta : C_0^\infty \to L^2$ initially, takes on a well-defined (i.e., only *metric* dependent) meaning only if Δ is essentially self-adjoint. This motivates our discussion of

GAFFNEY'S THEOREM [12]. *If (M^n, ds^2) is complete then Δ, defined on C_0^∞ initially, is essentially self-adjoint.*

Gaffney's original proof of this result used some functional analysis together with a special representation of Δ. A second proof of this result was given by P. Chernoff [7] as a corollary of some results concerning hyperbolic equations. We give a third proof which uses elliptic theory. We use the following standard fact (cf. [22]): A densely defined symmetric operator T is essentially self-adjoint if and only if $\operatorname{Ker}(T^* \pm i) = 0$. Here $T : D(T) \subseteq H \to H$ and H is a complex Hilbert space. Using this fact, the proof of Gaffney's theorem is very simple. We take $\Delta = T$ on $C_0^\infty(M^n) \subseteq$ complex $L^2(M^n, \text{dvol})$. Then:

(i) It is easy to check that $f \in D(T^*)$ if and only if $f \in L^2$ and Δf (defined as a distribution) $\in L^2$, and then $T^* f = \Delta f$. Thus if $f \in \operatorname{Ker}(T^* \pm i)$ then $f \in L^2 \cap C^\infty$ (by "elliptic regularity", cf. [19]), $T^* f = \Delta f$ in the classical sense, and $\Delta f \pm i f = 0$.

(ii) LEMMA. *If (M^n, ds^2) is complete, $f \in C^\infty \cap L^2$ and $\Delta f \in L^2$, then $\nabla f \in L^2$ and*

$$-\int_M \bar{f} \Delta f \,\text{dvol} = \int_M |\nabla f|^2 \,\text{dvol}.$$

It follows immediately from the lemma and (i) that $\operatorname{Ker}(T^* \pm i) = 0$. To prove the lemma we choose an exhaustion $\{K_j\}$ of M^n (i.e., $K_j \subset M$, $K_j \subset \operatorname{int}(K_{j+1})$ and $M = \cup K_j$) and functions $\psi_j \in C_0^\infty(M)$ with $\psi_j \equiv 1$ on K_j, $\psi_j \equiv 0$ on $M - K_{j+1}$, $0 \le \psi_j \le 1$, and $\sup_{x,j} |\nabla \psi_j(x)| < \infty$ (cf. [24] p. 187). Then, with $\psi = \psi_j$,

$$\int_M \operatorname{div}(\psi^2 \bar{f} \nabla f) \,\text{dvol} = 0$$

and so

$$\int_M \psi^2 |\nabla f|^2 \,\text{dvol} + \int_M \psi^2 \bar{f} \Delta f \,\text{dvol} = -2 \int_M \psi \bar{f} \langle \nabla \psi, \nabla f \rangle \,\text{dvol}. \tag{2.1}$$

Now

$$\left| 2 \int_M \psi \bar{f} \langle \nabla \psi_j, \nabla f \rangle \,\text{dvol} \right| \le \frac{1}{2} \int_M \psi_j^2 |\nabla f|^2 \,\text{dvol} + 2 \int_M |\nabla \psi_j|^2 |f|^2 \,\text{dvol},$$

since $2ab \le \frac{1}{2}a^2 + 2b^2$. Putting this into (2.1) and noting that $f\Delta f \in L^1$, $\psi_j \to 1$, and $\sup|\nabla\psi_j| < \infty$ we find that $|\nabla f| \in L^2$. But then

$$\left|2\int_M \psi_j \bar{f}\langle\nabla\psi_j, \Delta f\rangle\right| < \int_{A_j} |\nabla\psi_j|^2 |f|^2 \,\mathrm{dvol} + \int_{A_j} \psi_j^2 |\nabla f|^2 \,\mathrm{dvol},$$

where $A_j = K_{j+1} - K_j$ (since $\nabla\psi_j \equiv 0$ off A_j), and so the right side of (2.1) tends to zero as $j \to \infty$ (since f and ∇f are in L^2). This completes the proof of the lemma, and thus also of Gaffney's theorem. □

REMARKS. (a) This proof extends immediately to the Hodge Laplacian on forms and certain other geometric differential operators.

(b) The proof of Gaffney's theorem given above was described in the course of lectures on elliptic equations given by the author during the Special Year in Differential Geometry at the University of Maryland, College Park, 1981–1982. A similar proof has recently been given by Strichartz [26] who uses results of Yau [27] which are closely related to the lemma above.

We conclude this introductory section with the following consequence of essential self-adjointness of Δ. For convenience, we write Δ for the unique self-adjoint extension of Δ (which is initially defined on C_0^∞).

PROPOSITION. *Let (M^n, ds^2) be complete. Then the self-adjoint operator Δ has domain $D(\Delta) = \{u \in L^2 : \Delta u$, defined in the sense of distributions, is in $L^2\}$.*

Proof. The operator Δ coincides with the Friedrich's extension of Δ on C_0^∞ (since Δ on C_0^∞ has a unique self-adjoint extension). The characterization of $D(\Delta)$ now follows from a general result for Friedrich's extensions (cf. [28, p. 318]).

3. The main theorem. Let M^n be a simply-connected manifold of nonpositive curvature. Then given $p \in M^n$, $\exp_p$ is a diffeomorphism and we can introduce geodesic polar coordinates $(r, \theta^1, \ldots, \theta^n) \in \mathbf{R}^+ \times S^{n-1}$ so that $ds^2 = dr^2 + \sum \gamma_{ij}(r, \theta)\, d\theta^i\, d\theta^j$, (cf. [14]). If $G = \sqrt{\det \gamma_{ij}}$ then the Riemannian measure is given by $\mathrm{dvol} = G\, dr \wedge d\theta$ where $d\theta = d\theta^1 \wedge \cdots \wedge d\theta^n$ is the standard volume form on S^{n-1}. Let $S(r) = \{x : \mathrm{dist}(x, p) = r\}$ and $B(r) = \{x : \mathrm{dist}(x, p) < r\}$. Then

$$\int_{B(R)} f = \int_0^R dr \int_{S(r)} f\, d\sigma(r)$$

where $\sigma(r) = G\,d\theta =$ the induced measure on $S(r)$. Note that when $n = 2$, $ds^2 = dr^2 + G^2(r, \theta)\, d\theta^2$, and the Jacobi equation

$$G_{rr} + KG = 0, \tag{3.1}$$

where $K(r, \theta)$ is the Gaussian curvature, relates G and K. Regularity of ds^2 at p leads to the conditions

$$\begin{aligned} G(0+, \theta) &= 0, \\ G_r(0+, \theta) &= 1. \end{aligned} \tag{3.2}$$

If $n>2$ then G is related to the sectional curvature via the Jacobi system of equations [6].

It is known (cf., [6, p. 353] and [13, Chapter 2]) that if M^n has nonpositive sectional curvature then

$$(G^{1/(n-1)})_r \geq 1 \quad \text{and} \quad G \geq r^{n-1}. \tag{3.3}$$

If $n=2$ this follows easily from (3.1) and (3.2) by comparison techniques. For $n>2$ the method is similar.

The Laplacian Δ can be written in terms of (r,θ). We mention explicitly only the formula for $n=2$:

$$\Delta u = \frac{1}{G}(Gu_r)_r + \frac{1}{G}\left(\frac{u_\theta}{G}\right)_\theta. \tag{3.4}$$

For the general case see, for example, [14, p. 445].

Using the coordinates (r,θ) we can state our main result. As in [10] and [20], we restrict ourselves to a complete discussion in the case $n=2$. This considerably simplifies the exposition. The translation to dimension greater than two is essentially standard and has already been described briefly in [10].

THEOREM A. *If (M^2, ds^2) is a complete simply-connected surface of nonpositive curvature K that satisfies*

$$\text{(i)} \qquad K \geq -\frac{1}{(r+e)^2 \log(r+e)} \cdot \left(1+\frac{2e}{r}\right)$$

and

$$\text{(ii)} \qquad \sup_\theta \int_0^\infty |K_\theta| r \log^2(r+e)\, dr \leq 2, \tag{3.5}$$

then Δ has no eigenvalues (i.e., the spectrum of Δ is purely continuous).

COROLLARY. *If (M, ds^2) is a simply-connected surface of nonpositive curvature K with ds^2 radially symmetric about some point $p \in M$ and $K \geq -1/(r^2 \log r)$ for all $r =$ distance to $p \geq$ some fixed r_0, then the spectrum of Δ is purely continuous.*

REMARKS. (a) It has been shown by Pinsky [20] that even if $K \leq 0$, $\pi_1 M = 0$, and ds^2 is radially symmetric there still may exist eigenvalues in the spectrum of Δ. See also the remarks in [9]. Moreover, there are known examples of radially symmetric metrics on $\mathbf{R}^2$ with $K \leq 0$ for which the spectrum of Δ is discrete (i.e., consists entirely of isolated eigenvalues of finite multiplicity). For example, if $ds^2 = dr^2 + G(r)^2 d\theta^2$ (where G satisfies $G'' + KG = 0$, $G(0)=0$, and $G'(0)=1$) and $K=K(r)$ is any function with $K(r) \to -\infty$ as $r \to \infty$ (cf. [11] and generalizations in [16]), then the spectrum is discrete.

(b) It may be remarked that the condition $K \geq -1/(r^2 \log r)$ implies that (M^2, ds^2) is conformally equivalent to the plane $\mathbf{C}$ (cf. [2] and the discussions in [13] and [18]). This result is generalized in [15].

(c) Kato's procedure [17], which is used by Donnelly [10], may also be used to prove the nonexistence of eigenvalues when $K \to 0$ as $r \to \infty$, if K, K_θ, and $K_{\theta\theta}$ all tend to zero rapidly enough. We leave this result to the reader and emphasize that the theorem above only imposes conditions on K and K_θ.

Theorem A follows from the more general Theorem B.

THEOREM B. *If the metric of a complete simply-connected surface of nonpositive curvature has the representation* $ds^2 = dr^2 + G(r,\theta)^2\, d\theta^2$,

$$\int_1^\infty \left[\max_\theta G(r,\theta)\right]^{-1} dr = +\infty \tag{3.6}$$

and

$$\sup_M \frac{|G_\theta|}{G} \le 2,$$

then the spectrum of Δ *is purely continuous.*

COROLLARY. *If the simply-connected surface of nonpositive curvature has a complete radially symmetric metric about some point,* $ds^2 = dr^2 + G(r)^2\, d\theta^2$, *and* $\int_1^\infty dr/G(r) = +\infty$, *then the spectrum of* Δ *is purely continuous.*

REMARKS. (a) Theorem B and its corollary are rather sharp. The example in [20, p. 615] with $G \sim e^{kr}$ shows that even when $K \le 0$ and $G = G(r)$ the condition $\int_1^\infty 1/G = +\infty$ is necessary. The example in [10] with $G(r) = e^{-r}$ shows that even when exp is a diffeomorphism about some point $p \in M$, and $G = G(r)$ with $\int_1^\infty dr/G(r) = +\infty$, the hypothesis $K \le 0$ is still necessary. Similarly, the examples of (even rotation invariant) metrics with discrete spectrum (cf. [4], [11], [16]) either violate (on a compact set) the conditions $K \le 0$ or violate the condition $\int_1^\infty ds/G = \infty$.

(b) A condition similar to $\sup_M(|G_\theta|/G) \le 2$ appears in [21].

(c) The condition $\int_1^\infty ds/G = \infty$ is important also in [8].

We first prove Theorem B and then show how Theorem A follows from Theorem B.

Proof of Theorem B. Let $f \in L^2$ be real and satisfy $\Delta f + \lambda f = 0$, where Δf is defined as in the proposition at the end of Section 2. It follows from standard elliptic theory (see [19], for example) that we can take $f \in C^\infty$ and Δf has its classical meaning. If $\lambda = 0$ then f is an L^2 harmonic function and hence a constant. (See [26] or use the lemma in Section 2.) Since vol $M = +\infty$, as a consequence of the fact that $G \ge r$, it follows that $f = 0$ if $\lambda = 0$. Assume then that $\lambda > 0$ (again, by results of [26] or from the lemma, it follows, as just above, that we cannot have $\lambda < 0$ unless $f = 0$). Set

$$F(t) = \int_{S^1} G(t,\theta)^2 f(t,\theta)^2\, d\theta.$$

We have

$$F'(t) = 2\int_{S^1} d\theta\, G^2 f_t f + 2\int_{S^1} d\theta\, f^2 G_t G.$$

Integrating and using the equation $\Delta f + \lambda f = 0$ we obtain

$$F(r) = \frac{-2}{\lambda}\int_0^r dt \int_{S^1} d\theta\, G^2 f_t\, \Delta f + 2\int_0^r dt \int_{S^1} d\theta\, f^2 G_t G. \tag{3.7}$$

Using (3.4) and integration by parts we have

$$\begin{aligned}\int_0^r dt \int_{S^1} d\theta\, G^2 f_t\, \Delta f &= \int_0^r dt \int_{S^1} d\theta\, G\cdot f_t\left[(Gf_t)_t + \left(\frac{f_\theta}{G}\right)_\theta\right] \\ &= \frac{1}{2}\int_{S^1} d\theta (Gf_t)^2 - \int_0^r dt \int_{S^1} d\theta\left[\frac{1}{2}(f_\theta^2)_t + \frac{G_\theta}{G} f_t f_\theta\right],\end{aligned} \tag{3.8}$$

and, since $|ab| \le 1/2(a^2+b^2)$,

$$\begin{aligned}\int_0^r dt \int_{S^1} d\theta \left|\frac{G_\theta}{G} f_t \cdot f_\theta\right| &\le \frac{1}{2}\sup_{B(r)}\left|\frac{G_\theta}{G}\right| \int_0^r \int_{S^1}\left[f_t^2 + f_\theta^2 \frac{1}{G^2}\right] G\, dt\, d\theta \\ &= \frac{1}{2}\sup_{B(r)}\left|\frac{G_\theta}{G}\right| \cdot \int_{B(r)} |\nabla f|^2 \,\text{dvol}.\end{aligned} \tag{3.9}$$

Combining (3.7)-(3.9) we find

$$\lambda \int_{B(r)} f^2 G_t \,\text{dvol} \le E(r) + \frac{1}{2}\sup_{B(r)} \frac{|G_\theta|}{G} \cdot \int_{B(r)} |\nabla f|^2 \text{dvol}, \tag{3.10}$$

where

$$E(r) = \frac{\lambda}{2} F(r) + \frac{1}{2}\int_{S(r)} G f_t^2\, d\sigma - \frac{1}{2}\int_{S(r)} f_\theta^2 \frac{1}{G}\, d\sigma, \quad \text{and} \quad d\sigma = G\, d\theta.$$

From the lemma of Section 2 we see that $\nabla f \in L^2$ and $\int_M |\nabla f|^2 \text{dvol} = \lambda \int_M f^2 \text{dvol}$. We claim that $\liminf_{r\to\infty} E(r) \le 0$ Assuming this for a moment, we can choose an appropriate sequence $r = r_k \to \infty$ in (3.10) and obtain

$$\lambda \int \left(G_t - \frac{1}{2}\sup\frac{|G_\theta|}{G}\right) f^2 \text{dvol} \le 0,$$

thus completing the proof. In fact $G_t \ge 1$ [(3.3)] and if $G_t \equiv 1$ then $K \equiv 0$ and the theorem is trivial. If $G_t > 1$ somewhere then f vanishes on the open set where $G_t > \frac{1}{2}\sup(|G_\theta|/G)$ and, consequently, everywhere [1]. To prove the claim we note that

$$E(r) \le \frac{1+\lambda}{2}\left(\max_\theta G\right)\left[\int_{S(r)} |\nabla f|^2\, d\sigma + \int_{S(r)} f^2\, d\sigma\right] \equiv H(r)$$

and $\liminf_{r\to\infty} H(r) = 0$. In fact, if $H(r) \ge c > 0$ for some constant $c > 0$ and all r greater than some r_0, then

$$\frac{1+\lambda}{2c}\left(\int_M |\nabla f|^2 + \int_M f^2\right) \ge \int_{r_0}^\infty \left[\max_\theta G(t,\theta)\right]^{-1} dt = +\infty,$$

contradicting $\nabla f \in L^2$. This completes the proof. $\square$

In order to prove Theorem A we will require some preliminary results.

PROPOSITION. *Let* (M, ds^2) *be a complete Riemannian* 2*-manifold.*
(a) *If the Gaussian curvature* K *satisfies*

$$K \geq \frac{-1}{r^2(x)\log r(x)}$$

for all $r(x) >$ *some* r_0 *and* $r(x) = \operatorname{dist}(x, p)$, p *fixed, then in geodesic polar coordinates* (r, θ) *centered at* p, $G(r, \theta) \leq Ar\log r$ *for some* $A > 0$ *and all* $r \geq r_0$.
(b) *If the Gaussian curvature* K *satisfies*

$$K \geq \frac{-1}{(r+e)^2\log(r+e)}\left(1+\frac{2e}{r}\right), \quad r = \operatorname{dist}(x, p),$$

then for all $r \geq 0$, $G(r, \theta) \leq r\log(r+e)$.

Proof. Let $G_0(r,\theta) = r\log(r+e)$. Then

$$\frac{-G''}{G} = K \geq \frac{-G_0''}{G_0}, \quad G(0, \theta) = G_0(0, \theta) = 0, \quad \text{and} \quad G_r(0, \theta) = G_{0r}(0, \theta) = 1.$$

It follows that $G \leq G_0$ (cf. [18]). This proves (b). The proof of (a) is similar. □

As an immediate consequence we have the following.

COROLLARY. *If* (M^2, ds^2) *is complete and the Gaussian curvature satisfies the conditions in* (a) *or* (b) *of the Proposition, then*

$$\int_1^\infty \left[\max_\theta G(r, \theta)\right]^{-1} dr = +\infty.$$

REMARK. Since, for $\epsilon > 0$, the function $G_\epsilon(r)_{\text{def}} = r(\log r)^{1+\epsilon}$ for $r \geq 2$ (and smooth on $\mathbf{R}^+$ with $G_\epsilon(0) = 0$, $G_\epsilon'(0) = 1$) satisfies

$$\frac{-G_\epsilon''}{G_\epsilon} = K_\epsilon \geq \frac{-(1+2\epsilon)}{r^2(\log r)^{1+\epsilon}} \quad \text{for } r \geq 2,$$

the conditions of the proposition are the weakest possible that still yield $\int_1^\infty 1/G = +\infty$ (cf. [18]). □

We can now give the

Proof of Theorem A. On the basis of the Corollary to the Proposition, Theorem A will follow from Theorem B if it is shown that $\sup_M(|G_\theta|/G) \leq 2$ follows from hypotheses (i) and (ii) of Theorem A. Now $G'' + KG = 0$ so that $G_\theta'' + KG_\theta = -K_\theta G$. Since $G_\theta = (G_\theta)_r = 0$ at $r = 0$, $G_\theta(r, \theta) = \int_0^r R(r, t)(-K_\theta G)\,dt$ where $R(r, t)$ is the Green's function (for the initial value problem) for the operator $L = d^2/dr^2 + K$. Since G and $H =_{\text{def}} -G\int_1^r ds/G^2$ are linearly independent solutions of $Lu = 0$ on $(0, \infty)$ and the Wronskian $HG' - GH' \equiv 1$, we have

$$R(r, t) = G(r, \theta)H(t, \theta) - H(r, \theta)G(t, \theta) = G(r, \theta)G(t, \theta)\int_t^r \frac{ds}{G^2}$$

for $0 < t < r$. Thus

$$\frac{G_\theta}{G}(r,\theta) = \int_0^r -K_\theta(t,\theta)G(t,\theta)^2 \int_t^r \frac{ds}{G^2}\,dt$$

and

$$\sup_M \left|\frac{G_\theta}{G}\right| \le \sup_\theta \int_0^\infty \left(|K_\theta|\,G(t,\theta)^2 \int_t^\infty \frac{ds}{G^2}\right) dt$$

$$\le \sup_\theta \int_0^\infty |K_\theta(t,\theta)|\,t\log^2(t+e)\,dt,$$

since $\int_t^\infty ds/G^2 \le 1/t$ (recall that $G(s,\theta) \ge s$ if $K \le 0$) and $G(t,\theta) \le t\log(t+e)$ (as in the Proposition). This completes the proof of the Theorem. The corollary follows in the same way (from Theorem B and the Proposition) even though it only assumes that $K = K(r) \ge -1/(r^2 \log r)$ outside a compact set.

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