# STABILITY PROPERTIES OF THE YANG-MILLS FUNCTIONAL NEAR THE CANONICAL CONNECTION

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**Introduction.** Connections and curvature were introduced in the early part of this century by Cartan and Weyl, and by the 1950's these ideas were a well-established part of differential geometry ([9], [14]). In the past few years, this area of mathematics has again received widespread attention by both mathematicians and physicists in the form of Yang-Mills theory.

The Yang-Mills functional gives a measure of the total curvature of a connection in a principal bundle. The critical points of this functional, the so-called Yang-Mills connections, appear to play a fundamental role in physics. In §1, we define the functional and derive the corresponding variational equations for its critical points.

Because of the physical applications, much of the work in Yang-Mills theory has dealt with the case of principal bundles over four-dimensional manifolds ([2], [4], [7], [17]). The functional for bundles over Riemann surfaces has also been studied in some detail ([3]). In this paper, we are primarily concerned with the Yang-Mills functional for bundles of the form $P: G \to G/H$ and for associated principal bundles, $P_\lambda = G \times_H U \to G/H$, where $\lambda: H \to U$ is a Lie group homomorphism. These bundles have a canonical $G$-invariant connection, $\omega_0$, and we are especially interested in the behavior of the functional near $\omega_0$.

The key to doing explicit calculations on these homogeneous spaces is that geometric objects on $G/H$ are given by sections of bundles which are associated to $P$ by representations of $H$. The space of such sections becomes a $G$-module, called the induced representation, whose structure is given by Frobenius reciprocity (Theorem 2.1). In §2, we also develop the notation of equivariant functions which is used for subsequent calculations.

In §3, we study the Yang-Mills functional near the canonical connection. In particular, we show that $\omega_0$ is Yang-Mills (Theorem 3.1) and we derive methods for computing the index and nullity at $\omega_0$ (Theorems 3.3 and 3.4). These formulas involve Laplacians and Casimir operators and, in §4, these operators are related to representation theory. The index of a representation, originally introduced by Dynkin ([8]), plays an important role.

Finally, in §5 we consider some examples using Theorems 3.3 and 3.4. In particular, we determine the index and nullity of the Yang-Mills functional at the canonical connection for the bundles $G \to G/H$ when $G/H$ is a compact irreducible Riemannian symmetric space. These results are given by Theorem 5.1 and Table II. The canonical connection gives a stable critical point in all cases except for spheres $S^n$ when $n \geq 5$, compact simple Lie groups, quaternionic projective spaces $\mathrm{SP}(p+1)/\mathrm{SP}(p) \times \mathrm{SP}(1)$, and the exceptional symmetric spaces $E_6/F_4$

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and $F_4/\mathrm{Spin}(9)$. The index is nonzero in all these cases except for the quaternionic projective spaces (which include the sphere $S^4$) where the nullity is nonzero. This positive nullity is an indication that the Yang-Mills functional is invariant under the action of a larger group than the gauge group. It seems analogous to the case of $S^4$ where the conformal invariance of the functional in four dimensions contributes to the nullity.

**1. Yang-Mills theory.** Suppose $P \xrightarrow{\pi} M$ is a principal $H$-bundle, where $M$ is a compact oriented Riemannian manifold and $H$ is a compact Lie group. A connection for the principal bundle $P$ is an $H$-invariant splitting, $\omega_A$, of the exact sequence

$$0 \to T_F P \underset{\omega_A}{\rightleftarrows} TP \to \pi^* TM \to 0 \tag{1.1}$$

of vector bundles over $P$. Here $T_F P$ is the tangent space to the fibers of $P$ and the kernel of $\omega_A$ gives the horizontal subspaces of the connection. Alternatively, the connection can be viewed as a splitting of an exact sequence of bundles over $M$. Proceeding as in [3], we let $E(P)$ be the bundle over $M$ with fibers

$$E(P)_m = \Gamma\{TP \mid \pi^{-1} m\}^H \tag{1.2}$$

and we let $\mathrm{ad}(P)$ be the bundle associated to $P$ by the adjoint action of $H$ on its Lie algebra $\mathfrak{h}$. Then

$$\Gamma E(P) \cong H\text{-invariant vector fields on } P \tag{1.3}$$

and

$$\Gamma\, \mathrm{ad}(P) \cong H\text{-invariant vertical vector fields on } P. \tag{1.4}$$

A connection for $P$ can now be viewed as a splitting of the exact sequence

$$0 \to \mathrm{ad}(P) \underset{\omega_A}{\rightleftarrows} E(P) \to TM \to 0. \tag{1.5}$$

This sequence goes into (1.1) under pullback to $P$.

The spaces of sections of $E(P)$ and $\mathrm{ad}(P)$ have natural Lie algebra structures induced by the isomorphisms (1.3) and (1.4). The curvature measures to what extent $\omega_A$ fails to be a Lie algebra homomorphism. More precisely, if $X$ and $Y$ are vector fields on $M$, then the connection gives horizontal lifts

$$\tilde{X}^A, \tilde{Y}^A \in \Gamma E(P) \tag{1.6}$$

and the curvature $F_A \in \Omega^2(M; \mathrm{ad}(P))$ is defined by

$$F_A(X, Y) = \omega_A[\tilde{X}^A, \tilde{Y}^A] \in \Gamma\, \mathrm{ad}(P). \tag{1.7}$$

The Lie algebra structure of $\Gamma\, \mathrm{ad}(P)$ can be extended to all of $\Omega^*(M; \mathrm{ad}(P))$ by using exterior multiplication of forms together with the bracket of $\Gamma\, \mathrm{ad}(P)$. In particular, if $\eta \in \Omega^1(M; \mathrm{ad}(P))$ then

$$[\eta, \eta](X, Y) = 2[\eta(X), \eta(Y)] \in \Gamma\, \mathrm{ad}(P). \tag{1.8}$$

We clearly have

$$[\omega^p, \omega^q] = (-1)^{pq+1}[\omega^q, \omega^p] \tag{1.9}$$

and the corresponding Jacobi identity

$$[\omega^p, [\omega^q, \omega^r]] = [[\omega^p, \omega^q], \omega^r] + (-1)^{pq}[\omega^q, [\omega^p, \omega^r]]. \tag{1.10}$$

The Riemannian metric on $M$ together with an Ad-invariant metric on $\mathfrak{h}$ can be used to define a natural Riemannian structure on $\Omega^*(M; \mathrm{ad}(P))$. The metric on $\mathfrak{h}$ along with exterior multiplication of forms gives a pairing

$$\Omega^p(M; \mathrm{ad}(P)) \otimes \Omega^q(M; \mathrm{ad}(P)) \xrightarrow{\wedge} \Omega^{p+q}(M; \mathbf{R}). \tag{1.11}$$

The identity

$$\omega^p \wedge [\omega^q, \omega^r] = [\omega^p, \omega^q] \wedge \omega^r \tag{1.12}$$

comes from the infinitesimal version of Ad-invariance of the metric on $\mathfrak{h}$. We let $\mathrm{vol}(M) \in \Omega^*(M; \mathbf{R})$ be the unique form of length one in the orientation of $M$. There is a natural $*$-operator

$$\Omega^p(M; \mathrm{ad}(P)) \xrightarrow{*} \Omega^{\dim M - p}(M; \mathrm{ad}(P)) \tag{1.13}$$

which is characterized by

$$\omega \wedge {}^*\omega = \langle \omega, \omega \rangle \, \mathrm{vol}(M). \tag{1.14}$$

We obtain a natural inner product in $\Omega^*(M; \mathrm{ad}(P))$ by letting

$$(\theta, \phi) = \int_M \theta \wedge {}^*\phi. \tag{1.15}$$

The Yang-Mills functional on the space $\mathfrak{C}(P)$ of connections is now defined by

$$\mathrm{YM}(A) = \|F_A\|^2 = (F_A, F_A). \tag{1.16}$$

Thus, the Yang-Mills functional gives a measure of the total curvature of the connection. Critical points of the functional, the so-called Yang-Mills connections, appear to play an important role in mathematics and physics. For example, when $H = U(1)$ the curvature of a critical connection corresponds to a solution of Maxwell's equations in vacuo ([2]).

The choice of a connection in $P$ induces a covariant derivative of sections of all associated vector bundles. A section $\phi$ of an associated vector bundle, $P(V) = P \times_H V$, corresponds to an $H$-equivariant function, $\hat{\phi}: P \to V$, by letting $\phi(\pi(p)) = (p, \hat{\phi}(p))$. The covariant derivative is given by

$$(\nabla_X^A \phi)\hat{} = \tilde{X}^A \hat{\phi}. \tag{1.17}$$

In particular, if $P(V) = \mathrm{ad}(P)$ then

$$\nabla_X^A \phi = \omega_A[\tilde{X}^A, \phi] \in \Gamma\, \mathrm{ad}(P). \tag{1.18}$$

Note that $[\tilde{X}^A, \phi]$, viewed as a vector field on $P$, is automatically vertical. The covariant derivative gives a natural differential operator

$$\Omega^0(M; P(V)) \xrightarrow{d_A} \Omega^1(M; P(V)) \tag{1.19}$$

by letting

$$(d_A \phi)(X) = \nabla_X^A \phi. \tag{1.20}$$

This operator extends uniquely to all of $\Omega^*(M; P(V))$ if we require compatibility with the natural pairing

$$\Omega^p(M; \mathbf{R}) \otimes \Omega^q(M; P(V)) \xrightarrow{\wedge} \Omega^{p+q}(M; P(V)) \tag{1.21}$$

given by multiplication of forms. In particular, if $\eta \in \Omega^1(M; P(V))$ then

$$(d_A \eta)(X, Y) = \nabla_X^A \eta(Y) - \nabla_Y^A \eta(X) - \eta[X, Y]. \tag{1.22}$$

In the case of $\mathrm{ad}(P)$-valued forms on $M$, the operator $d_A$ behaves like a derivation with respect to both the bracket and the wedge operations.

The exact sequence (1.5) shows that the space of connections is an affine space and the difference of two connections is given by a 1-form on $M$ with values in $\mathrm{ad}(P)$. If $\omega_B - \omega_A = \eta \in \Omega^1(M; \mathrm{ad}(P))$, then the horizontal lifts of the two connections are related by

$$\tilde{X}^B = \tilde{X}^A - \eta(X) \in \Gamma E(P) \tag{1.23}$$

and the corresponding operators satisfy

$$d_B \xi = d_A \xi - [\eta, \xi] \quad \forall \xi \in \Omega^*(M; \mathrm{ad}(P)). \tag{1.24}$$

It follows that the curvatures of the two connections are related by

$$F_B = F_A - d_A \eta + \tfrac{1}{2}[\eta, \eta]. \tag{1.25}$$

Since the space of connections is an affine space, we can study the critical points of the Yang-Mills functional by considering the variation of the functional along lines

$$\omega_t = \omega_A + t\eta, \tag{1.26}$$

where $\eta \in \Omega^1(M; \mathrm{ad}(P))$. By (1.25) we have

$$\begin{aligned} \|F_t\|^2 = \|F_A\|^2 - 2t(F_A, d_A \eta) + t^2\{\|d_A \eta\|^2 + (F_A, [\eta, \eta])\} \\ - t^3(d_A \eta, [\eta, \eta]) + \frac{t^4}{4} \|[\eta, \eta]\|^2. \end{aligned} \tag{1.27}$$

So a connection $A$ gives a critical point if and only if $(F_A, d_A \eta) = 0$ for all $\eta \in \Omega^1(M; \mathrm{ad}(P))$, or equivalently, if and only if

$$d_A^* F_A = 0, \tag{1.28}$$

where $d_A^*$ is the adjoint to the operator $d_A$ relative to the inner product. Equation (1.28) along with Bianchi's identity,

$$d_A F_A = 0, \tag{1.29}$$

which is true for any connection, are called the Yang-Mills equations.

The variational equation (1.27) also gives the second variation of the Yang-Mills functional at a critical connection $A$. The Hessian at the critical connection is the quadratic form

$$(1.30)\qquad \begin{aligned} Q(\eta,\eta) &= \|d_A\eta\|^2 + (F_A,[\eta,\eta]) \\ &= (d_A^* d_A\eta + {}^*[{}^*F_A,\eta],\eta). \end{aligned}$$

Now, the Yang-Mills functional is invariant under the action of a large group ([3]). The gauge group $\mathfrak{G}(P)$ for the bundle $P \to M$ consists of all bundle automorphisms of $P$ which cover the identity map of $M$. The group acts on the space of connections by pullback of horizontal spaces. The tangent space to the orbit of $\mathfrak{G}(P)$ at a critical connection $A$ is given by the image of $d_A$ mapping $\Omega^0(M;\mathrm{ad}(P))$ into $\Omega^1(M;\mathrm{ad}(P))$; so if $\eta = d_A\phi$ then $Q(\eta,\eta)=0$. Thus, when we consider the index and nullity of $Q$, we really want to study $Q$ on the orthogonal complement to the image of $d_A$ in $\Omega^1(M;\mathrm{ad}(P))$. This is precisely the kernel of $d_A^*$. We let the operators $\Delta_A$ and $\mathfrak{F}_A$ be defined by

$$(1.31)\qquad \Delta_A = d_A^* d_A + d_A d_A^*$$

and

$$(1.32)\qquad \mathfrak{F}_A = {}^*[{}^*F_A,\cdot\,].$$

Then the quadratic form

$$(1.33)\qquad \underline{Q}(\eta,\eta) = (\Delta_A\eta + \mathfrak{F}_A\eta,\eta)$$

agrees with $Q$ on the kernel of $d_A^*$ and is strictly positive definite on the image of $d_A$. So to determine the index and nullity of $Q$, it suffices to consider the index and nullity of $\underline{Q}$ on all of $\Omega^1(M;\mathrm{ad}(P))$.

**2. Homogeneous spaces.** In this paper, we are primarily concerned with the Yang-Mills functional for bundles of the form $P: G \to G/H$ and for associated principal bundles

$$(2.1)\qquad P_\lambda = G\times_H U = G\times U/(gh,u) \sim (g,\lambda(h)u)$$

where $\lambda: H \to U$ is a Lie group homomorphism. We assume $G$, $H$, and $U$ are compact Lie groups with $G$ semisimple so that the Killing form, $B_\mathfrak{g}$, is negative definite. This Killing form gives a splitting of the Lie algebra $\mathfrak{g}$ into $\mathfrak{h}\oplus\mathfrak{m}$ where $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. The Ad-invariance of $B_\mathfrak{g}$ implies $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$. The Killing form is also used to define the Riemannian metric on $G/H$, i.e.,

$$(2.2)\qquad \langle dL_g\,d\pi x, dL_g\,d\pi y\rangle_{gH} = -B_\mathfrak{g}(x,y)\quad \forall x,y\in\mathfrak{m}.$$

Given a representation $\rho: H \to \mathrm{Aut}\,V$, we can form the associated vector bundle

$$(2.3)\qquad G(V) = G\times_H V = G\times V/(gh,v) \sim (g,\rho(h)v).$$

Sections of this bundle are given by $H$-equivariant functions on $G$ with values in $V$, i.e., (2.4) $$\sigma(gH) = (g, \hat{\sigma}(g)) \in G \times_H V,$$

where the $H$-equivariance condition is

(2.5) $$\hat{\sigma}(gh) = \rho(h^{-1})\hat{\sigma}(g) \quad \forall h \in H.$$

The space of such functions becomes a $G$-module—called the induced representation, $i_* V$—by letting

(2.6) $$g \cdot \hat{\sigma} = \hat{\sigma} \circ L_{g-1}.$$

Conversely, given a $G$-module $W$, we define an $H$-module $i^* W$ by restricting the action from $G$ to $H$. The structure of the induced representation is given by the following Frobenius Reciprocity Theorem ([5]).

THEOREM 2.1. *If $W$ is a $G$-module and $V$ is an $H$-module then*

$$\mathrm{Hom}_G(W, i_* V) = \mathrm{Hom}_H(i^* W, V).$$

This theorem is of important computational value because it connects finite dimensional representation theory with the geometry of homogeneous spaces. In particular, we have

(2.7) $$\Gamma T_*(G/H) \cong i_*(\mathfrak{m}),$$

(2.8) $$\Omega^p(G/H; \mathrm{ad}(P)) \cong i_*(\wedge^p \mathfrak{m}^* \otimes \mathfrak{h}),$$

and

(2.9) $$\Omega^p(G/H; \mathrm{ad}(P_\lambda)) \cong i_*(\wedge^p \mathfrak{m}^* \otimes \mathfrak{u}).$$

Note that in (2.9) the Lie algebra $\mathfrak{u}$ is the $H$-representation $\mathrm{Ad} \circ \lambda$ and that, in general, the bundles associated to $P_\lambda$ are also associated to the bundle $G \to G/H$.

For computational purposes, we fix an orthonormal basis $\{u_p\}$ for $\mathfrak{g}$, relative to the metric $-B_{\mathfrak{g}}$, so that the initial $u$'s span $\mathfrak{h}$ and the remaining $u$'s span $\mathfrak{m}$. We will use subscripts $i, j, k, \ldots$ for $u$'s in $\mathfrak{h}$, subscripts $\alpha, \beta, \gamma, \ldots$ for $u$'s in $\mathfrak{m}$ and subscripts $p, q, r, \ldots$ for $u$'s in $\mathfrak{g}$. In particular, we use this for the summation convention. We let $\{\mu^p\}$ be the dual basis of $\mathfrak{g}^*$ and we let $\{U_p\}$ be the corresponding left invariant vector fields on $G$. The structure constants for $\mathfrak{g}$ are defined by

(2.10) $$[u_p, u_q] = B^r_{pq} u_r.$$

Similarly, when we are dealing with the bundle $P_\lambda$, we let $\{v_p\}$ be an orthonormal basis for $\mathfrak{u}$ relative to some positive definite Ad-invariant metric, and we let the corresponding structure constants be defined by

(2.11) $$[v_p, v_q] = C^r_{pq} v_r.$$

The Ad-invariance of the metrics on $\mathfrak{g}$ and $\mathfrak{u}$ gives a cyclic symmetry to the structure constants $B$ and $C$ along with the usual skew symmetries. We define constants $\lambda^p_i$ by

(2.12) $$\lambda(u_i) = \lambda^p_i v_p.$$

In terms of these bases, a vector field on $G/H$ is given by

$$\hat{X}(g) = x^\alpha(g) u_\alpha \tag{2.13}$$

and a form $\xi \in \Omega^p(G/H; \mathrm{ad}(P_\lambda))$ is given by

$$\hat{\xi}(g) = \frac{1}{p!} \xi^q_{\alpha_1,\ldots,\alpha_p}(g) \mu^{\alpha_1} \wedge \cdots \wedge \mu^{\alpha_p} \otimes v_q. \tag{2.14}$$

The metric and *-operator on $\Omega^*(G/H; \mathrm{ad}(P_\lambda))$ are determined by letting the elements

$$\mu^{\alpha_1} \wedge \cdots \wedge \mu^{\alpha_p} \otimes v_q \quad \alpha_1 < \cdots < \alpha_p \tag{2.15}$$

form an orthonormal basis of $\wedge^p \mathfrak{m}^* \otimes \mathfrak{u}$. The $H$-equivariance of the functions $\hat{\xi} \in i_*(\wedge^p \mathfrak{m}^* \otimes \mathfrak{u})$ shows that this gives a well defined Riemannian metric. The integral of a highest dimensional real-valued form $\omega$ on $G/H$ can also be defined by the integral over $G$ of the corresponding function $\hat{\omega}: G \to \wedge^m \mathfrak{m}^* \cong \mathbf{R}$. This differs from the integral of the form on $G/H$ by a constant factor (the volume of $H$), and thus defines an equivalent inner product on $\Omega^*(G/H; \mathrm{ad}(P_\lambda))$. We will use the inner product defined by the integral of $\hat{\omega}$ whenever our base manifold is $G/H$. The Lie algebra structure on $\Gamma\,\mathrm{ad}(P_\lambda) = i_*(\mathfrak{u})$ is given by

$$[\sigma_1, \sigma_2]\hat{}(g) = -[\hat{\sigma}_1(g), \hat{\sigma}_2(g)] \tag{2.16}$$

where we get the minus sign because the bracket in $\mathfrak{u}$ is the bracket of left invariant vector fields, whereas the sections of $\mathrm{ad}(P_\lambda)$ give vector fields which are invariant under right translations by $U$.

The bundle $G \to G/H$ has a canonical connection, $\omega_0$, whose horizontal subspaces are given by the left translates of $\mathfrak{m}$ by $G$. Since connections move forward to associated bundles ([14]), we also obtain a canonical $G$-invariant connection in $P_\lambda$. Connections in $P$ or in $P_\lambda$ which are invariant under the left action of $G$ are called homogeneous connections. The following theorem describing the space of homogeneous connections is due to Wang ([19]).

THEOREM 2.2. *The homogeneous connections in $P_\lambda$ are in one-to-one correspondence with linear maps $\phi: \mathfrak{g} \to \mathfrak{u}$ such that*

$$\phi(x) = \lambda(x) \quad \forall x \in \mathfrak{h}$$

*and*

$$\phi(\mathrm{Ad}(h)x) = \mathrm{Ad}(\lambda(h))\phi(x) \quad \forall h \in H,\ x \in \mathfrak{g}.$$

The map $\phi$ is completely determined by its restriction to $\mathfrak{m}$ and, in the case of the canonical connection, $\phi$ is trivial on $\mathfrak{m}$. This theorem shows that the space of homogeneous connections corresponds to the set of trivial $H$-representations in $\mathfrak{m}^* \otimes \mathfrak{u}$. By Frobenius reciprocity, this is equivalent to the set of constant $H$-equivariant functions with values in $\mathfrak{m}^* \otimes \mathfrak{u}$. The correspondence between $\phi$ and $\omega_0 + \eta$ is given by

$$\hat{\eta} = \mu^\alpha \otimes \phi(u_\alpha) \in i_*(\mathfrak{m}^* \otimes \mathfrak{u}). \tag{2.17}$$

The symmetric spaces are an especially important class of homogeneous spaces. In this case, the Lie group $G$ has an involutive automorphism $\sigma$ and $H$ is the fixed point set of $\sigma$. At the level of Lie algebras we have $\mathfrak{h} = \{x \mid \sigma(x) = x\}$ and $\mathfrak{m} = \{x \mid \sigma(x) = -x\}$. Infinitesimally, the condition that $G/H$ is a symmetric space is equivalent to the condition $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$. Any compact symmetric space splits into a product of compact irreducible symmetric spaces of types I and II ([12]).

The type II symmetric spaces are precisely the simple Lie groups viewed as $G \times G/\Delta$ where $\Delta$ is the diagonal subgroup. In this case $\mathfrak{h} = \{(x, x) \in \mathfrak{g} \oplus \mathfrak{g}\}$ and $\mathfrak{m} = \{(x, -x) \in \mathfrak{g} \oplus \mathfrak{g}\}$, so both $\mathfrak{h}$ and $\mathfrak{m}$ are isomorphic to $\mathfrak{g}$ as $\Delta$-representations. Since $G$ is simple, $\mathfrak{h}$ and $\mathfrak{m}$ are irreducible and $\mathfrak{m}^* \otimes \mathfrak{h}$ includes exactly one trivial representation. This one-dimensional family of homogeneous connections is described by the maps $\phi_t : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{h}$, where

$$\phi_t(x, y) = \left(\left(\frac{1+t}{2}\right)x + \left(\frac{1-t}{2}\right)y,\ \left(\frac{1+t}{2}\right)x + \left(\frac{1-t}{2}\right)y\right). \tag{2.18}$$

The canonical connection is given by $t = 0$ while the connections with $t = 1$ and $t = -1$ are called the $(+)$ and the $(-)$ connections. The $(+)$ and $(-)$ connections are flat and they are gauge equivalent. The map $\theta : G \times G \to G \times G$ which is defined by

$$\theta(g_1, g_2) = (g_1 g_2^{-1} g_1, g_1) \tag{2.19}$$

is an element of the gauge group, and we have

$$\omega_- \cdot \theta = \omega_+. \tag{2.20}$$

LEMMA 2.3. *If $G/H$ is a symmetric space with $G$ simple, then $\mathfrak{m}$ is an irreducible $H$-module.*

*Proof.* Let $\mathfrak{m}_1$ be an $H$-invariant subspace of $\mathfrak{m}$ and let $\mathfrak{m}_2 = \mathfrak{m}_1^\perp$ relative to the restriction of $B_\mathfrak{g}$ to $\mathfrak{m}$. Then $\mathfrak{m}_2$ is also an $H$-module and $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$. This is clear because if $x \in \mathfrak{m}_1$ and $y \in \mathfrak{m}_2$, then $[y, [x, y]] \in \mathfrak{m}_2$ and we have

$$B_\mathfrak{g}([x, y], [x, y]) = B_\mathfrak{g}(x, [y, [x, y]]) = 0. \tag{2.21}$$

It follows that $\mathfrak{m}_1 \oplus [\mathfrak{m}_1, \mathfrak{m}_1]$ is an ideal in $\mathfrak{g}$ and the assumption that $G$ is simple gives the desired result. □

If $G/H$ is a type I symmetric space, then by the classification of the irreducible symmetric spaces ([1]) it follows that $G$ is simple. By the lemma, the decomposition of $\mathfrak{g}$ into irreducible $H$-modules is given by

$$\mathfrak{g} = (\bigoplus \mathfrak{h}_i) \oplus \mathfrak{m}. \tag{2.22}$$

Here the $\mathfrak{h}_i$ are the ideals in $\mathfrak{h}$ and we have

$$[\mathfrak{h}_i, \mathfrak{h}_j] = \{0\} \quad \forall i \neq j. \tag{2.23}$$

Now for any $i$ we have

$$[\mathfrak{h}_i, \mathfrak{m}] \neq \{0\}, \tag{2.24}$$

or else $\mathfrak{h}_i$ would be an ideal in $\mathfrak{g}$. A homogeneous connection in $G \to G/H$ is given by an $H$-module map $\phi: \mathfrak{m} \to \mathfrak{h}$. By irreducibility, $\phi = 0$ or $\phi$ is an isomorphism of $\mathfrak{m}$ onto one of the $\mathfrak{h}_i$'s. But this clearly contradicts (2.23) and (2.24) if there are at least two $\mathfrak{h}_i$'s. The possibility that $\mathfrak{m} = \mathfrak{h}$ is also eliminated by the classification of the type I symmetric spaces. Thus we have the following theorem.

THEOREM 2.4. *The bundle $G \to G/H$ has a unique homogeneous connection if $G/H$ is a type I symmetric space. It has a one-dimensional family of homogeneous connections if $G/H$ is a type II symmetric space.*

**3. The canonical connection.** In this section, we are interested in the behavior of the Yang-Mills functional near the canonical connection, $\omega_0$. The canonical connection is a critical point of the Yang-Mills functional and, by using the bases and equivariant functions described in the previous section, we derive formulas for the second variation at $\omega_0$.

If $X$ is a vector field on $G/H$ with corresponding equivariant function $\hat{X} = x^\alpha u_\alpha : G \to \mathfrak{m}$, then the horizontal lift of $X$ for the canonical connection in $G \to G/H$ is given by

(3.1) $$\tilde{X}^0 = x^\alpha U_\alpha.$$

Now the curvature $F_0 \in \Omega^2(G/H; \mathrm{ad}(P))$ satisfies

(3.2) $$\begin{aligned} F_0(X, Y) &= \omega_0[\tilde{X}^0, \tilde{Y}^0] \\ &= x^\alpha y^\beta B^k_{\alpha\beta} u_k, \end{aligned}$$

so

(3.3) $$\hat{F}_0 = (1/2) B^k_{\alpha\beta} \mu^\alpha \wedge \mu^\beta \otimes u_k.$$

Since the canonical connection in $P_\lambda$ is induced by the canonical connection in $G \to G/H$, it follows that the curvature of the canonical connection in $P_\lambda$ is given by

(3.4) $$\begin{aligned} \hat{F}_0 &= (1/2) B^k_{\alpha\beta} \mu^\alpha \wedge \mu^\beta \otimes \lambda(u_k) \\ &= (1/2) B^k_{\alpha\beta} \lambda^p_k \mu^\alpha \wedge \mu^\beta \otimes v_p. \end{aligned}$$

The $d_0$ and $d_0^*$ operators can be expressed in terms of equivariant functions. In particular, if

(3.5) $$\hat{\phi} = \phi^p v_p \in i_*(\mathfrak{u}) = \Omega^0(G/H; \mathrm{ad}(P_\lambda)),$$

then

(3.6) $$(d_0 \phi)\hat{} = U_\alpha(\phi^p) \mu^\alpha \otimes v_p \in i_*(\mathfrak{m}^* \otimes \mathfrak{u})$$

and if

(3.7) $$\hat{\eta} = \eta^p_\alpha \mu^\alpha \otimes v_p,$$

then by (1.22) we have

(3.8) $$(d_0 \eta)\hat{} = (1/2)(U_\alpha \eta^p_\beta - U_\beta \eta^p_\alpha - B^\gamma_{\alpha\beta} \eta^p_\gamma) \mu^\alpha \wedge \mu^\beta \otimes v_p.$$

The adjoint operator now satisfies

$$d_0^*(\eta_\alpha^p \mu^\alpha \otimes v_p) = -U_\alpha(\eta_\alpha^p) v_p \tag{3.9}$$

and

$$d_0^*(\tfrac{1}{2}\xi_{\alpha\beta}^p \mu^\alpha \wedge \mu^\beta \otimes v_p) = -(U_\beta \xi_{\beta\alpha}^p + \tfrac{1}{2}\xi_{\beta\gamma}^p B_{\beta\gamma}^\alpha)\mu^\alpha \otimes v_p. \tag{3.10}$$

It is easily verified that both these expressions are $H$-equivariant. So to prove these equations, it suffices to show

$$(\eta, d_0 \phi) = (d_0^* \eta, \phi) \tag{3.11}$$

and

$$(\xi, d_0 \eta) = (d_0^* \xi, \eta) \tag{3.12}$$

whenever $\phi \in \Omega^0(G/H; \mathrm{ad}(P_\lambda))$, $\eta \in \Omega^1$, and $\xi \in \Omega^2$. This follows directly from (3.6), (3.8), and integration by parts.

THEOREM 3.1. *If $G$ is a compact semisimple Lie group and if the metric on $\mathfrak{g}$ which is used to define the Riemannian structure on $G/H$ is the negative of the Killing form, then the canonical connections in $G \to G/H$ and in $P_\lambda$ are Yang–Mills connections.*

*Proof.* We need to show $d_0^* F_0 = 0$. By (3.4) and (3.10) it follows that

$$d_0^* F_0 = -(1/2) B_{\beta\gamma}^\alpha B_{\beta\gamma}^k \mu^\alpha \otimes \lambda(u_k). \tag{3.13}$$

Now for $u_\alpha \in \mathfrak{m}$ and $u_k \in \mathfrak{h}$ we have

$$\begin{aligned} B_{\mathfrak{g}}(u_\alpha, u_k) &= \operatorname{trace} \mathrm{ad}(u_\alpha)\,\mathrm{ad}(u_k) \\ &= \langle [u_\alpha, [u_k, u_p]], u_p \rangle \\ &= B_{k\beta}^\gamma B_{\alpha\gamma}^\beta. \end{aligned} \tag{3.14}$$

Thus

$$d_0^* F_0 = (1/2) B_{\mathfrak{g}}(u_\alpha, u_k) \mu^\alpha \otimes \lambda(u_k) \tag{3.15}$$

which is zero by the orthogonality of $\mathfrak{h}$ and $\mathfrak{m}$. □

In order to compute the index and nullity of the Yang–Mills functional at the canonical connection, it is necessary to express the operator $\Delta_0 + \mathfrak{F}_0$ in terms of our bases. Given an $H$-representation $\rho: H \to \operatorname{Aut} V$, we let $\Gamma_V$ be the Casimir operator of the representation relative to the basis $\{u_i\}$ of $\mathfrak{h}$, i.e.,

$$\Gamma_V = \sum_i (\rho(u_i))^2. \tag{3.16}$$

If $V$ is irreducible, then $\Gamma_V$ is a negative or zero scalar. When dealing with a space $W$ of functions on $G$ with values in some vector space, we let $\Delta$ be the Laplacian with positive eigenvalues computed relative to the basis $\{u_p\}$ of $\mathfrak{g}$, i.e.,

$$\Delta f = -\sum U_p^2 f = -\sum X_p^2 f \tag{3.17}$$

where the $U_p$ (respectively $X_p$) are the left (respectively right) invariant vector fields on $G$ which agree with the $u_p$ at the identity. On an irreducible $G$-representation in $W$, $\Delta$ is a nonnegative scalar and is equal to the Casimir operator of the representation relative to the orthonormal basis $\{u_p\}$ of $\mathfrak{g}$.

The space of $\mathrm{ad}(P_\lambda)$-valued-forms splits up into a direct sum of induced representations. We have

$$\Omega^1(G/H; \mathrm{ad}(P_\lambda)) = \bigoplus_{a,b} i_*(\mathfrak{m}_a^* \otimes \mathfrak{u}_b), \tag{3.18}$$

where $\mathfrak{m} = \bigoplus \mathfrak{m}_a$ and $\mathfrak{u} = \bigoplus \mathfrak{u}_b$ are decompositions of $\mathfrak{m}$ and $\mathfrak{u}$ into irreducible $H$-representations. The proof of the following proposition is given in the appendix.

PROPOSITION 3.2. *If* $\hat{\eta} \in i_*(\mathfrak{m}_a^* \otimes \mathfrak{u}_b)$, *then*

$$(\Delta_0 + \mathfrak{F}_0)(\eta)\hat{} = (\Delta + \Gamma_{\mathfrak{u}_b} + \tfrac{1}{2} + \Gamma_{\mathfrak{m}_a})(\hat{\eta}) + B^\gamma_{\alpha\beta} U_\gamma(\eta^p_\beta)\mu^\alpha \otimes v_p.$$

If $G/H$ is a symmetric space, then all structure constants of the form $B^\gamma_{\alpha\beta}$ are zero and by equation (A.15) in the appendix, we have $\Gamma_{\mathfrak{m}_a} = -\frac{1}{2}$. Thus for $\eta \in i_*(\mathfrak{m}_a^* \otimes \mathfrak{u}_b)$ we have

$$(\Delta_0 + \mathfrak{F}_0)(\eta)\hat{} = (\Delta + \Gamma_{\mathfrak{u}_b})(\hat{\eta}). \tag{3.19}$$

Similarly, if $\eta$ gives a variation in the space of homogeneous connections, then $\mathfrak{m}_a = \mathfrak{u}_b$ and $\hat{\eta}$ is a constant equivariant function, so

$$(\Delta_0 + \mathfrak{F}_0)(\eta)\hat{} = (\tfrac{1}{2} + 2\Gamma_{\mathfrak{m}_a})(\hat{\eta}). \tag{3.20}$$

Finally, if $G/H$ is a symmetric space and if $\eta$ gives a variation in the space of homogeneous connections, then

$$(\Delta_0 + \mathfrak{F}_0)(\eta)\hat{} = -\tfrac{1}{2}\hat{\eta}. \tag{3.21}$$

Thus we have the following two theorems.

THEOREM 3.3. *Suppose $G/H$ is a symmetric space with $G$ compact and semisimple; $P_\lambda = G \times_H U$ is an associated principal bundle; and $\mathfrak{u} = \bigoplus \mathfrak{u}_k$ is a decomposition of $\mathfrak{u}$ into irreducible $H$-modules. Let $\Delta$ be the positive Laplacian relative to the metric $-B_{\mathfrak{g}}$ on $\mathfrak{g}$ and let $\Gamma_{\mathfrak{u}_k}$ be the negative Casimir operator of $\mathfrak{u}_k$ relative to the restriction of this metric to $\mathfrak{h}$. Then the index (respectively nullity) of the Yang-Mills functional at the canonical connection in $P_\lambda$ is given by the sum over $k$ of the number of negative (respectively null) eigendirections of the operator $\Delta + \Gamma_{\mathfrak{u}_k}$ on $i_*(\mathfrak{m}^* \otimes \mathfrak{u}_k)$. In particular, the canonical connection is a local maximum on the space of homogeneous connections in $P_\lambda$.*

THEOREM 3.4. *If $G/H$ is a homogeneous space with $G$ compact and semisimple and if $P_\lambda = G \times_H U$, then the behavior of the Yang-Mills functional near the canonical connection on the space of homogeneous connections in $P_\lambda$ is determined as follows. We decompose $\mathfrak{m}$ and $\mathfrak{u}$ into irreducible $H$-representations, $\mathfrak{m} = \bigoplus \mathfrak{m}_a$ and $\mathfrak{u} = \bigoplus \mathfrak{u}_b$. Each pair $(a, b)$ with $\mathfrak{m}_a = \mathfrak{u}_b$ gives a homogeneous direction. If $\Gamma_{\mathfrak{m}_a} < -\frac{1}{4}$, then the direction decreases the Yang-Mills action. If $\Gamma_{\mathfrak{m}_a} = -\frac{1}{4}$ then, infinitesimally, the direction is a null direction for the functional.*

*Otherwise the direction increases the Yang-Mills action. Here,* $\Gamma_{\mathfrak{m}_a}$ *is the Casimir operator of* $\mathfrak{m}_a$ *relative to the restriction to* $\mathfrak{h}$ *of the metric* $-B_{\mathfrak{g}}$.

**4. Indices and Casimir operators.** In order to use Theorems 3.3 and 3.4 to compute the index and nullity of the Yang-Mills functional at the canonical connection, it is necessary to compute Casimir operators of representations relative to various metrics on $\mathfrak{g}$ and $\mathfrak{h}$. In particular, the Laplacian $\Delta$ on an irreducible $G$-representation in $i_*(\mathfrak{m}^* \otimes \mathfrak{u})$ is, up to sign, the Casimir operator of that representation relative to the Killing metric. In what follows, because of the equivalence between the real and complex representation theories of semisimple Lie groups ([1]), we can complexify the Lie algebras and representations whenever it is necessary. Also, when we describe an irreducible representation of a simple Lie group by means of its maximal weight, we will use the ordering of simple roots which appears in [6], [12], and [13].

If $\rho: G \to \operatorname{Aut} V$ is a representation of the Lie group $G$, then we let $C_V$ be the positive Casimir operator of the representation relative to the Killing metric. By [13], if $\rho$ is an irreducible representation with maximal weight $\wedge$, then

$$C_V = B_{\mathfrak{g}}^*(\wedge + 2\delta, \wedge). \tag{4.1}$$

Here, $\delta$ is one-half the sum of the positive roots and $B_{\mathfrak{g}}^*$ is the metric on $\mathfrak{g}^*$ which is dual to the Killing metric.

If $G$ is a simple Lie group, then an Ad-invariant metric on $\mathfrak{g}$ is unique up to a scalar multiple. We let $(\ ,\ )_{\mathfrak{g}}$ be the metric on $\mathfrak{g}$ which is normalized so that the longest root has length squared equal to 2. If $L(G)$ is the length squared of the longest root relative to the Killing form, then

$$(x, y)_{\mathfrak{g}} = (1/2)L(G)B_{\mathfrak{g}}(x, y) \quad \forall x, y \in \mathfrak{g}. \tag{4.2}$$

Note that we obtain the factor $L(G)/2$ rather than $2/L(G)$ because the roots lie in the dual to the Cartan subalgebra of $\mathfrak{g}$. In Table I, we list the $L(G)$ constants ([12]) along with the least nonzero Casimir operator $C_V$ for the simple Lie groups.

Dynkin ([8]) defines the index $k_V$ of a possibly reducible representation $\rho: G \to \operatorname{Aut} V$ of a simple Lie group by

$$\operatorname{tr}(\rho(x)\rho(y)) = k_V \cdot (x, y)_{\mathfrak{g}} \quad \forall x, y \in \mathfrak{g}. \tag{4.3}$$

If $x$ lies in the Cartan subalgebra of $\mathfrak{g}$ and if $\Pi(V)$ is the set of weights of $\rho$, then

$$\operatorname{tr}(\rho(x)\rho(x)) = \sum_{\lambda \in \Pi(V)} (\lambda(x))^2. \tag{4.4}$$

The index of a representation is an integer ([8]) and in the case of the adjoint representation we have

$$B_{\mathfrak{g}}(x, y) = \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y) = k_{\mathfrak{g}}(x, y)_{\mathfrak{g}} \tag{4.5}$$

so

$$k_{\mathfrak{g}} = 2/L(G). \tag{4.6}$$

| Lie Group | $L(G)$ | $C(\Lambda)_{\mathrm{MIN}}$ |
|---|---|---|
| $A_n$ | $\frac{1}{n+1}$ | $\frac{n(n+2)}{2(n+1)^2}$ |
| $B_n$ | $\frac{1}{2n-1}$ $(n \geq 2)$ | $\frac{5}{12}$ $(n=2)$ |
| | | $\frac{21}{40}$ $(n=3)$ |
| | | $\frac{n}{2n-1}$ $(n \geq 4)$ |
| $C_n$ | $\frac{1}{n+1}$ | $\frac{2n+1}{4(n+1)}$ |
| $D_n$ | $\frac{1}{2n-2}$ $(n \geq 3)$ | $\frac{2n-1}{4(n-1)}$ $(n \geq 4)$ |
| $E_6$ | $\frac{1}{12}$ | $\frac{13}{18}$ |
| $E_7$ | $\frac{1}{18}$ | $\frac{19}{24}$ |
| $E_8$ | $\frac{1}{30}$ | 1 |
| $F_4$ | $\frac{1}{9}$ | $\frac{2}{3}$ |
| $G_2$ | $\frac{1}{4}$ | $\frac{1}{2}$ |

**Table I.** Length squared of the longest root and least positive eigenvalue of the Laplacian for the simple Lie Groups. (Metrics given by the Killing form.)

The index of an irreducible representation with maximal weight $\wedge$ is closely related to the Casimir operator of the representation. If the trace form

$$t_\rho(x,y) = \operatorname{tr}(\rho(x)\rho(y)) = (1/2)k_V L(G) B_{\mathfrak{g}}(x,y) \tag{4.7}$$

is used to define orthonormal bases, then the Casimir operator is the scalar $\dim \mathfrak{g}/\dim V$ ([13]). So, relative to the Killing metric, we have

$$C_V = \frac{\dim \mathfrak{g}}{\dim V} \cdot \frac{k_V L(G)}{2} = \frac{\dim \mathfrak{g}}{\dim V} \cdot \frac{k_V}{k_{\mathfrak{g}}}. \tag{4.8}$$

Suppose $G$ and $H$ are both simple Lie groups. Dynkin ([8]) defines the index $j$ of an embedding $i: H \to G$ by

(4.9) $$(ix, iy)_{\mathfrak{g}} = j(x, y)_{\mathfrak{h}} \quad \forall x, y \in \mathfrak{h}.$$

This index, which is an integer, has the following cohomological interpretation ([4]). Since $G$ is simple, the 3-form $\omega$, defined by

(4.10) $$\omega_g(dL_g x, dL_g y, dL_g z) = ([x, y], z)_{\mathfrak{g}},$$

is a generator of $H^3(G; \mathbf{R})$. Because of the normalization, this form represents an integral class and the index of the embedding is precisely the degree of the map

(4.11) $$\mathbf{Z} \cong H^3(G; \mathbf{Z}) \xrightarrow{i^*} H^3(H; \mathbf{Z}) \cong \mathbf{Z}.$$

Tables of indices of embeddings along with indices of representations appear in [8] and [15]. By (4.4) and consideration of weights, we have the following theorem.

THEOREM 4.1.
(i) $k_V = 0$ *if* $V$ *is a trivial representation*;
(ii) $k_{V^*} = k_V$;
(iii) $k_{V \oplus W} = k_V + k_W$;
(iv) $k_{V \otimes W} = d_V k_W + d_W k_V$, *where* $d_V = \dim V$;
(v) $k_{\wedge^2 V} = (d_V - 2) k_V$;
(vi) $k_{S^2 V} = (d_V + 2) k_V$;
(vii) $k_{i^* W} = j \cdot k_W$, *where* $i: H \to G$ *has index* $j$ *and* $W$ *is a* $G$*-representation.*

By (4.8) and (4.9), the Casimir operator of a representation of $H$, computed relative to the restriction to $\mathfrak{h}$ of $-B_{\mathfrak{g}}$, is given by

(4.12) $$\Gamma_V = \frac{-\dim \mathfrak{h}}{\dim V} \cdot \frac{k_V L(G)}{2j} = \frac{-\dim \mathfrak{h}}{\dim V} \cdot \frac{k_V}{k_{i^*\mathfrak{g}}}.$$

In particular, if $\rho$ is the adjoint representation then

(4.13) $$\Gamma_{\mathfrak{h}} = \frac{-L(G)}{jL(H)} = \frac{-k_{\mathfrak{h}}}{k_{i^*\mathfrak{g}}}.$$

The embeddings of a compact simple Lie group $H$ in one of the classical groups $A_n = SU(n+1)$, $B_n = SO(2n+1)$, $C_n = SP(n)$, or $D_n = SO(2n)$ correspond to faithful, possibly reducible, representations of $H$ of degrees $n+1$, $2n+1$, $2n$, or $2n$ respectively. The representation can be arbitrary for embeddings in $A_*$, it must be orthogonal for embeddings in $B_*$ or $D_*$, and it must be symplectic for embeddings in $C_*$. We let $\rho: H \to G$ be an embedding of $H$ in one of the classical groups, viewed as a representation of $H$. The Lie algebra $\mathfrak{g}$ is the $H$-representation $\mathrm{Ad}_G \circ \rho$ with weights

(4.14) $$\Pi(\mathrm{Ad}_G \circ \rho) = \{\rho^* \alpha \mid \alpha \text{ is a root of } G\}.$$

The following theorem shows how this representation can be expressed in terms of the representation $\rho$, and also how the index of the embedding depends on $k_\rho$.

THEOREM 4.2. *Let* $\rho: H \to G$ *be an embedding of* $H$ *in one of the classical groups, viewed as a representation of* $H$*. Then the following table expresses the* $H$*-representation* $\mathrm{Ad}_G \circ \rho$ *in terms of* $\rho$.

| $G$ | $\dim \rho$ | $\dim \mathfrak{g}$ | $\mathrm{Ad}_G \circ \rho$ | $j$ |
|---|---|---|---|---|
| $A_n$ | $n+1$ | $n^2+2n$ | $\rho \otimes \rho^* - (0)$ | $k_\rho$ |
| $B_n$ | $2n+1$ | $2n^2+n$ | $\wedge^2 \rho$ | $\frac{1}{2} k_\rho$ |
| $C_n$ | $2n$ | $2n^2+n$ | $S^2 \rho$ | $k_\rho$ |
| $D_n$ | $2n$ | $2n^2-n$ | $\wedge^2 \rho$ | $\frac{1}{2} k_\rho$ |

*Proof.* Theorems of Dynkin ([8]), Gantmacher ([11]), or Navon and Patera ([16]) show that we can choose a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{h}$ so that $\rho(\mathfrak{h}_0) \subseteq \mathfrak{g}_0$, where $\mathfrak{g}_0$ is a given Cartan subalgebra of $\mathfrak{g}$. For example, if $G = A_n$ then we can assume $\rho$ maps $\mathfrak{h}_0$ into the set of diagonal matrices. For $x \in \mathfrak{g}_0$ let $e_i(x)$ be the $i$th element of the diagonal of $x$. Then the weights of $\rho$ are

$$\Pi(\rho) = \{\lambda_i = \rho^* e_i \mid 1 \leq i \leq n+1\}. \tag{4.15}$$

Since the roots of $G$ are

$$\{e_i - e_j \mid 1 \leq i, j \leq n+1\} - \{0\} \tag{4.16}$$

we have

$$\Pi(\mathrm{Ad}_G \circ \rho) = \{\lambda_i - \lambda_j \mid 1 \leq i, j \leq n+1\} - \{0\}, \tag{4.17}$$

and thus $\mathrm{Ad}_G \circ \rho = \rho \otimes \rho^* - (0)$. The cases $G = B_n$, $C_n$, or $D_n$ are similar. Now

$$\begin{aligned} k_{\mathrm{Ad}_G \circ \rho}(x, y)_{\mathfrak{h}} &= \mathrm{tr}(\mathrm{ad}\, \rho(x)\, \mathrm{ad}\, \rho(y)) = B_{\mathfrak{g}}(\rho(x), \rho(y)) \\ &= \frac{2j}{L(G)} (x, y)_{\mathfrak{h}}. \end{aligned} \tag{4.18}$$

So the index of the embedding $H \to G$ satisfies

$$j = (1/2) L(G) \cdot k_{\mathrm{Ad}_G \circ \rho}, \tag{4.19}$$

and by Theorem 4.1 and Table I it follows that the index $j$ is given by the above table. □

**5. Index and nullity computations.** In this section, we consider some examples using Theorems 3.3 and 3.4. In particular, we determine the index and nullity of the Yang-Mills functional at the canonical connection on the space of all connections in $G \to G/H$ whenever $G/H$ is a compact irreducible symmetric space.

THEOREM 5.1. *Let $G$ be a simple Lie group. Then the bundle $G \times G \to G \times G/\Delta$ has a one-dimensional family of homogeneous connections. The index of the Yang-Mills functional on the space of all connections is* 1 *at the canonical connection and the nullity is* 0.

*Proof.* By Theorems 2.4 and 3.3 we know that the canonical connection is a local maximum on the one-dimensional space of homogeneous connections. Since both $\mathfrak{m}$ and $\mathfrak{h}$ are isomorphic to the adjoint representation of $G$, we have $\Gamma_{\mathfrak{m}} = \Gamma_{\mathfrak{h}} = -\frac{1}{2}$. So the second variation at the canonical connection is given by

$$(\Delta_0 + \mathfrak{F}_0)(\eta) = (\Delta - \tfrac{1}{2})(\eta) \quad \eta \in i_*(\mathfrak{m}^* \otimes \mathfrak{h}), \tag{5.1}$$

and we need to determine which irreducible representations of $G\times G$ with Casimir operator less than or equal to $\frac{1}{2}$ occur in $i_*(\mathfrak{m}^*\otimes\mathfrak{h})$. The irreducible representations of $G\times G$ are given by tensor products of irreducible representations of $G$. The corresponding Casimir operators satisfy

$$C(\rho_1\otimes\rho_2)=C(\rho_1)+C(\rho_2), \tag{5.2}$$

where all these operators are computed relative to the respective Killing forms. Now the only Lie groups which have $C(\rho)\leq\frac{1}{2}$ for any non-trivial $\rho$ are $A_n$, $C_n$, and $G_2$. In these cases, it is necessary to determine how often the representations

$$(\lambda_1)\otimes(0),\ (\lambda_n)\otimes(0),\ (0)\otimes(\lambda_1),\ (0)\otimes(\lambda_n) \tag{5.3}$$

of $A_n\times A_n$, or the representations

$$(\lambda_1)\otimes(0),\ (0)\otimes(\lambda_1) \tag{5.4}$$

of $C_n\times C_n$ or $G_2\times G_2$, appear in the induced representation $i_*(\mathfrak{m}^*\otimes\mathfrak{h})$. By Frobenius reciprocity and Steinberg's formula ([13]), none of these actually occur in $i_*(\mathfrak{m}^*\otimes\mathfrak{h})$. Thus, the only direction which decreases the Yang–Mills action is a variation in the space of homogeneous connections. □

In Table II, we list the index and nullity of the Yang–Mills functional at the canonical connection in the case of the bundle $G\to G/H$ over a type I symmetric space. To perform these calculations, we first decompose $\mathfrak{g}$ into a direct sum of $H$-modules

$$\mathfrak{g}=\left(\mathfrak{z}\bigoplus_k\mathfrak{h}_k\right)\oplus\mathfrak{m} \tag{5.5}$$

where $\mathfrak{z}$ is the center of $\mathfrak{h}$ and the $\mathfrak{h}_k$ are the simple ideals in $\mathfrak{h}$. By Theorem 2.4, no trivial $G$-modules occur in the induced representation

$$\Omega^1(G/H;\mathrm{ad}(P))=i_*(\mathfrak{m}^*\otimes\mathfrak{z})\bigoplus_k i_*(\mathfrak{m}^*\otimes\mathfrak{h}_k). \tag{5.6}$$

Since the action of $H$ on $\mathfrak{z}$ is trivial, directions in $i_*(\mathfrak{m}^*\otimes\mathfrak{z})$ do not contribute to the index or nullity at the canonical connection. An irreducible $G$-representation $V$ in $i_*(\mathfrak{m}^*\otimes\mathfrak{h}_k)$ contributes to the index (respectively nullity) if and only if $C_V+\Gamma_{\mathfrak{h}_k}<0$ (respectively $=0$). Since the $\mathfrak{h}_k$ are ideals in $\mathfrak{h}$, the Casimir operator of $\mathfrak{h}_k$ as an $H$-representation is the same as the Casimir operator of $\mathfrak{h}_k$ as an $H_k$-representation, where in both cases we use the restriction of the metric on $\mathfrak{g}$ to define orthonormal bases. By (4.13) we have

$$\Gamma_{\mathfrak{h}_k}=\frac{-L(G)}{j_k L(H_k)}, \tag{5.7}$$

where $j_k$ is the index of the embedding $H_k\to G$. These indices, which are also listed in Table II, can be determined from the tables of Dynkin ([8]) or McKay and Patera ([15]) and by the methods of §4. A $G$-representation $V$ in $i_*(\mathfrak{m}^*\otimes\mathfrak{h}_k)$ with $C_V\leq-\Gamma_{\mathfrak{h}_k}$ will contribute $\dim(V)$ directions to the index or nullity at the canonical connection. Frobenius reciprocity can be used to determine if such $G$-representations occur. Branching rules ([15], [16]) are used to decompose $i^*V$

| Symmetric Space | | | $j(H_i)$ | Index | Nullity |
|---|---|---|---|---|---|
| AI | $SU(n)/SO(n)$ | $n=2$ | abelian $H$ | 0 | 0 |
| | | $n=3$ | 4 | 0 | 0 |
| | | $n=4$ | 2, 2 | 0 | 0 |
| | | $n\geq 5$ | 2 | 0 | 0 |
| AII | $SU(2n)/SP(n)$ | $n=2$ | 1 | 6 | 0 |
| | | $n\geq 3$ | 1 | 0 | 0 |
| AIII | $SU(p+q)/S(U_p\times U_q)$ | $p=q=1$ | abelian $H$ | 0 | 0 |
| | | $p>q=1$ | 1 | 0 | 0 |
| | | $p\geq q\geq 2$ | 1, 1 | 0 | 0 |
| BDI | $SO(p+q)/SO(p)\times SO(q)$ | $p=3,\ q=2$ | 1 | 0 | 0 |
| | | $p=4,\ q=2$ | 1, 1 | 0 | 0 |
| | | $p\geq 5,\ q=2$ | 1 | 0 | 0 |
| | | $p=3,\ q=3$ | 2, 2 | 0 | 0 |
| | | $p=4,\ q=3$ | 1, 1 ,2 | 0 | 0 |
| | | $p\geq 5,\ q=3$ | 1, 2 | 0 | 0 |
| | | $p=4,\ q=4$ | 1, 1 ,1 ,1 | 0 | 0 |
| | | $p\geq 5,\ q=4$ | 1, 1, 1 | 0 | 0 |
| | | $p\geq q\geq 5$ | 1, 1 | 0 | 0 |
| BDII | $SO(p+1)/SO(p)$ | $p=2$ | abelian $H$ | 0 | 0 |
| | | $p=3$ | type II | 1 | 0 |
| | | $p=4$ | 1, 1 | 0 | 10 |
| | | $p\geq 5$ | 1 | $p+1$ | 0 |
| CI | $SP(n)/U(n)$ | $n=1$ | abelian $H$ | 0 | 0 |
| | | $n\geq 2$ | 2 | 0 | 0 |
| CII | $SP(p+q)/SP(p)\times SP(q)$ | $p=q=1$ | 1, 1 | 0 | 10 |
| | | $p>q=1$ | 1, 1 | 0 | $p(2p+3)$ |
| | | $p\geq q\geq 2$ | 1, 1 | 0 | 0 |
| DIII | $SO(2n)/U(n)$ | $n\geq 3$ | 1 | 0 | 0 |
| EI | $E_6/SP(4)$ | | 1 | 0 | 0 |
| EII | $E_6/SU(2)\cdot SU(6)$ | | $\geq 1, 1$ | 0 | 0 |
| EIII | $E_6/SO(2)\cdot \mathrm{Spin}(10)$ | | 1 | 0 | 0 |
| EIV | $E_6/F_4$ | | 1 | 54 | 0 |
| EV | $E_7/SU(8)$ | | 1 | 0 | 0 |
| EVI | $E_7/SU(2)\cdot SO(12)$ | | $\geq 1, 1$ | 0 | 0 |
| EVII | $E_7/SO(2)\cdot E_6$ | | 1 | 0 | 0 |
| EVIII | $E_8/SO(16)$ | | 1 | 0 | 0 |
| EIX | $E_8/SU(2)\cdot E_7$ | | $\geq 1, 1$ | 0 | 0 |
| FI | $F_4/SU(2)\cdot SP(3)$ | | $\geq 1, 1$ | 0 | 0 |
| FII | $F_4/\mathrm{Spin}(9)$ | | 1 | 26 | 0 |
| GI | $G_2/SU(2)\times SU(2)$ | | 1, 3 | 0 | 0 |

**Table II.** Index and nullity of the Yang-Mills functional at the canonical connection for the type I symmetric spaces. The $j(H_i)$ are the indices of the embeddings in $G$ of the simple subgroups of $H$.

into $H$-representations and the methods in [10] and [13] are useful for decomposing the tensor products, $\mathfrak{m}^* \otimes \mathfrak{h}_k$. The results of these computations are given in Table II.

The Cayley plane, $F_4/\mathrm{Spin}(9)$, is an example of an exceptional symmetric space which exhibits interesting behavior at the canonical connection. As $B_4$-representations, we have $\mathfrak{h} = (\lambda_2)$ (the adjoint representation) and $\mathfrak{m} = (\lambda_4)$. By the tables in [8], the index of the embedding $B_4 \to F_4$ is one and thus the Casimir operator $\Gamma_{\mathfrak{h}} = -\frac{7}{9}$. So we need to determine which representations of $F_4$ with $C_V \leq \frac{7}{9}$ occur in the induced representation $i_*(\mathfrak{m}^* \otimes \mathfrak{h})$. Now the only nontrivial representation of $F_4$ which satisfies this is the degree 26 representation $(\lambda_4)$ for which $C_V = \frac{2}{3}$. Frobenius reciprocity shows that this representation actually occurs in $i_*(\mathfrak{m}^* \otimes \mathfrak{h})$. The tables in [8] and [15] give $i^*(\lambda_4) = (\lambda_4) \oplus (\lambda_1) \oplus (0)$. By [10], we have $\mathfrak{m}^* \otimes \mathfrak{h} = (\lambda_2 + \lambda_4) \oplus (\lambda_1 + \lambda_4) \oplus (\lambda_4)$. Since $(\lambda_4)$ occurs once in both $i^*(\lambda_4)$ and $\mathfrak{m}^* \otimes \mathfrak{h}$, we conclude that the $F_4$-representation $(\lambda_4)$ occurs once in $i_*(\mathfrak{m}^* \otimes \mathfrak{h})$. Thus the index of the Yang-Mills functional at the canonical connection is 26 in this case.

The Yang-Mills functional for bundles over the $p$-sphere has also been studied by Bourguignon, Lawson and Simons ([7]). A direction in $\mathbf{R}^{p+1}$ gives a natural vector field on $S^p$. By contracting the curvature, $F_A$, with one of these vector fields, they produce an $\mathrm{ad}(P)$ valued 1-form on $S^p$. This infinitesimal variation decreases the Yang-Mills functional if the connection $A$ is Yang-Mills and $p \geq 5$. If $p = 4$ then the functional is invariant under this sort of conformal transformation. In this case, we get 10 null directions instead of just 5 because of the splitting of $SO(4)$ into two simple subgroups. Since $S^3$ is a Lie group, the case $p = 3$ is an example of Theorem 5.1. From the point of view of Theorem 3.3, the second variation at the canonical connection is given by

$$(\Delta_0 + \mathfrak{F}_0)(\eta) = \left(\Delta - \frac{p-2}{p-1}\right)(\eta). \tag{5.8}$$

When $p \geq 5$, the ordinary representation is the only representation of $G = SO(p+1)$ which occurs in $i_*(\mathfrak{m}^* \otimes \mathfrak{h})$ and which has Casimir operator less than or equal to $(p-2)/(p-1)$. It occurs with multiplicity one and has dimension $p+1$. So the index of the Yang-Mills functional at the canonical connection for the bundle $SO(p+1) \to S^p$ is equal to $p+1$ whenever $p \geq 5$.

As Theorem 2.4 indicates, examples of bundles over symmetric spaces which have nontrivial homogeneous connections are rare. In the case of the $p$-sphere, $S^p = \mathrm{Spin}(p+1)/\mathrm{Spin}(p)$, let $\rho: H \to SU(n+1) = U$ be an embedding viewed as a representation of $H$. Then $\mathfrak{u} = \rho \otimes \rho^* - (0)$ as an $H$-representation and the associated bundle $G \times_H U$ will have nontrivial homogeneous connections if and only if the ordinary representation of $H$ appears in $\mathfrak{u}$. By Steinberg's formula ([13]), the only representations $(\lambda)$ that can occur in $(\lambda') \otimes (\lambda'')$ have maximal weight of the form $\lambda = \mu + \lambda''$ where $\mu$ is a weight of $(\lambda')$. Using this, it is clear that if the embedding $\rho$ comes from a representation of $SO(p)$, then the ordinary representation of $H$ does not appear in $\mathfrak{u}$. If $\rho$ is a representation of $\mathrm{Spin}(p)$ which does not come from $SO(p)$ then it is possible for $G \times_H U$ to have nontrivial homogeneous connections. For example, if $\rho$ is the total spin representation,

$$\rho = \begin{cases} \Delta & p \text{ odd} \\ \Delta^+ \oplus \Delta^- & p \text{ even,} \end{cases} \tag{5.9}$$

then $\rho \otimes \rho^*$ includes one copy of $(\lambda_1)$ when $p$ is odd and two copies of $(\lambda_1)$ when $p$ is even ([18]). So the associated total spin bundle has a one- or two-dimensional family of homogeneous connections, and the canonical connection is a local maximum for the Yang-Mills functional on this space of homogeneous connections.

**Appendix: Proof of Proposition 3.2.** Let $\hat{\eta} = \eta_\alpha^p \mu^\alpha \otimes v_p \in i_*(\mathfrak{m}^* \otimes \mathfrak{u})$. By (1.8) and (2.16) we have

$$[\eta, \eta]\hat{} = -\eta_\alpha^p \eta_\beta^q C_{pq}^r \mu^\alpha \wedge \mu^\beta \otimes v_r. \tag{A.1}$$

It now follows that

$$(\mathfrak{F}_0 \eta)\hat{} = -B_{\alpha\beta}^k \lambda_k^r C_{pq}^r \eta_\beta^q \mu^\alpha \otimes v_p. \tag{A.2}$$

This is clear because this function is $H$-equivariant and

$$(F_0, [\eta, \eta]) = \int_G -B_{\alpha\beta}^k \lambda_k^r C_{pq}^r \eta_\beta^q \eta_\alpha^p. \tag{A.3}$$

LEMMA A.1. *The infinitesimal version of H-equivariance for* 1*-forms* $\hat{\eta} \in i_*(\mathfrak{m}^* \otimes \mathfrak{u})$ *is*

$$U_k \eta_\alpha^p = B_{k\alpha}^\beta \eta_\beta^p - \lambda_k^r C_{rq}^p \eta_\alpha^q.$$

*Proof.* We have

$$\begin{aligned} U_k(\hat{\eta})\,|_g &= \frac{d}{dt}\bigg|_{t=0} \eta_\alpha^p(g \cdot \exp(tu_k))\mu^\alpha \otimes v_p \\ &= \frac{d}{dt}\bigg|_{t=0} \eta_\alpha^p(g)(\mathrm{Ad}^* \otimes \mathrm{Ad} \circ \lambda)(\exp(-tu_k))\mu^\alpha \otimes v_p \\ &= \eta_\alpha^p(g) B_{k\beta}^\alpha \mu^\beta \otimes v_p - \eta_\alpha^p(g) \lambda_k^r C_{rp}^q \mu^\alpha \otimes v_q. \end{aligned} \tag{A.4}$$

Equating coefficients of $\mu^\alpha \otimes v_p$ gives the desired result. □

Now by (3.6), (3.8), (3.9), and (3.10) we have

$$\begin{aligned} \Delta_0 \eta &= d_0 d_0^* \eta + d_0^* d_0 \eta \\ &= \begin{Bmatrix} -U_\alpha U_\beta \eta_\beta^p - U_\beta(U_\beta \eta_\alpha^p - U_\alpha \eta_\beta^p - B_{\beta\alpha}^\gamma \eta_\gamma^p) \\ -(1/2) B_{\beta\gamma}^\alpha (U_\beta \eta_\gamma^p - U_\gamma \eta_\beta^p - B_{\beta\gamma}^\delta \eta_\delta^p) \end{Bmatrix} \mu^\alpha \otimes v_p \\ &= \begin{Bmatrix} -U_\beta U_\beta \eta_\alpha^p + [U_\beta, U_\alpha]\eta_\beta^p \\ +2B_{\beta\alpha}^\gamma U_\beta \eta_\gamma^p + (1/2) B_{\beta\gamma}^\alpha B_{\beta\gamma}^\delta \eta_\delta^p \end{Bmatrix} \mu^\alpha \otimes v_p. \end{aligned} \tag{A.5}$$

By using Lemma A.1 twice, it follows that

$$\begin{aligned} U_k U_k \eta_\alpha^p &= B_{k\alpha}^\beta U_k \eta_\beta^p - \lambda_k^r C_{rq}^p U_k \eta_\alpha^q \\ &= B_{k\alpha}^\beta U_k \eta_\beta^p - \lambda_k^r C_{rq}^p B_{k\alpha}^\beta \eta_\beta^q + \lambda_k^r \lambda_k^s C_{rq}^p C_{st}^q \eta_\alpha^t. \end{aligned} \tag{A.6}$$

Combining (A.2), (A.5), and (A.6) gives

$$(\mathrm{A}.7)\qquad (\Delta_0+\mathfrak{F}_0)(\eta)=\left\{\begin{array}{l}-\Sigma_t\, U_t^2\eta_\alpha^p+\lambda_k^r\lambda_k^s C_{rq}^p C_{st}^q\eta_\alpha^t\\ +B_{\alpha\beta}^\gamma U_\gamma\eta_\beta^p+(1/2)B_{\delta\gamma}^\alpha B_{\delta\gamma}^\beta\eta_\beta^p\end{array}\right\}\mu^\alpha\otimes v_p.$$

Since the metrics on $\mathfrak{m}$ and $\mathfrak{u}$ are Ad-invariant, the distinct $\mathfrak{m}_a$'s and $\mathfrak{u}_b$'s in the decompositions $\mathfrak{m}=\bigoplus\mathfrak{m}_a$ and $\mathfrak{u}=\bigoplus\mathfrak{u}_b$ can be assumed to be orthogonal. In particular, when the bases for $\mathfrak{m}$ and $\mathfrak{u}$ are chosen, it can be assumed that each $u_\alpha$ or $v_p$ lies entirely in one of these irreducible submodules. We define index subsets by

$$(\mathrm{A}.8)\qquad I(\mathfrak{m}_a)=\{\alpha\mid u_\alpha\in\mathfrak{m}_a\}$$

and

$$(\mathrm{A}.9)\qquad I(\mathfrak{u}_b)=\{p\mid v_p\in\mathfrak{u}_b\}.$$

We need to compute $(\Delta_0+\mathfrak{F}_0)(\eta)$ when $\eta\in i_*(\mathfrak{m}_a^*\otimes\mathfrak{u}_b)$. If $v_t\in\mathfrak{u}_b$ then

$$(\mathrm{A}.10)\qquad \begin{aligned}\Gamma_{\mathfrak{u}_b}v_t&=\sum((\mathrm{ad}\circ\lambda)(u_k))^2v_t\\ &=\lambda_k^r\lambda_k^s C_{st}^q C_{rq}^p v_p.\end{aligned}$$

Since $\Gamma_{\mathfrak{u}_b}$ is a scalar, we have

$$(\mathrm{A}.11)\qquad \lambda_k^r\lambda_k^s C_{st}^q C_{rq}^p=\delta_{pt}\Gamma_{\mathfrak{u}_b}\quad t\in I(\mathfrak{u}_b).$$

Similarly, if $u_\alpha\in\mathfrak{m}_a$ then

$$(\mathrm{A}.12)\qquad \Gamma_{\mathfrak{m}_a}u_\alpha=B_{k\alpha}^\gamma B_{k\gamma}^\beta u_\beta,$$

so

$$(\mathrm{A}.13)\qquad B_{k\alpha}^\gamma B_{k\gamma}^\beta=\delta_{\alpha\beta}\cdot\Gamma_{\mathfrak{m}_a}\quad \alpha\in I(\mathfrak{m}_a).$$

Since the metric on $\mathfrak{g}$ is $-B_\mathfrak{g}$, it follows that

$$(\mathrm{A}.14)\qquad -\delta_{\alpha\beta}=B_\mathfrak{g}(u_\alpha,u_\beta)=2B_{\beta k}^\gamma B_{\alpha\gamma}^k+B_{\beta\gamma}^\delta B_{\alpha\delta}^\gamma$$

and thus for $\beta\in I(\mathfrak{m}_a)$

$$(\mathrm{A}.15)\qquad B_{\delta\gamma}^\alpha B_{\delta\gamma}^\beta=\delta_{\alpha\beta}+2\delta_{\alpha\beta}\Gamma_{\mathfrak{m}_a}.$$

Now for $\eta\in i_*(\mathfrak{m}_a^*\otimes\mathfrak{u}_b)$ we have $\eta_\alpha^p=0$ unless $\alpha\in I(\mathfrak{m}_a)$ and $p\in I(\mathfrak{u}_b)$. So by (A.7), (A.11), and (A.15) it follows that

$$(\mathrm{A}.16)\qquad (\Delta_0+\mathfrak{F}_0)(\eta)=(\Delta+\Gamma_{\mathfrak{u}_b}+\tfrac{1}{2}+\Gamma_{\mathfrak{m}_a})(\eta_\alpha^p\mu^\alpha\otimes v_p)+B_{\alpha\beta}^\gamma U_\gamma(\eta_\beta^p)\mu^\alpha\otimes v_p.$$

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