ON CERTAIN TRANSCENDENTAL NUMBERS

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In this note we deduce some consequences of the Lindemann–Weierstrass theorem and the Gelfond–Schneider theorem. In a simple fashion we prove certain pairs of numbers are algebraically independent. We also show certain number classes contain only transcendental numbers. Examples include the algebraic independence of $e^2 \cos 3$ and $e^2 \sin 3$ and the transcendence of

$$\log \sin 2, \quad \sin \log 2, \quad \cos^{-1}(\exp(2)), \quad \exp(\cos^{-1} 2), \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(\sqrt{2}n)}{n}.$$

One form of the Lindemann–Weierstrass theorem [1, p. 20] states that if $\alpha_1, \dots, \alpha_n$ are algebraic, then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent if and only if $\alpha_1, \dots, \alpha_n$ are linearly independent over the rationals. It follows at once that e^α is transcendental for all non-zero algebraic α. But what about the arithmetic nature of the real and imaginary parts of e^α? The transcendence of e^α insures that at least one of $\mathrm{Re}(e^\alpha)$ and $\mathrm{Im}(e^\alpha)$ is transcendental. But much more is true if the real and imaginary parts of α are both non-zero. In this case, a corollary of the following theorem shows that $\mathrm{Re}(e^\alpha)$ and $\mathrm{Im}(e^\alpha)$ are in fact algebraically independent.

THEOREM 1. *Suppose α and β are algebraic numbers. Then $e^\alpha \cos \beta$ and $e^\alpha \sin \beta$ are algebraically independent if and only if α and βi are linearly independent over the rationals.*

Proof. Let K denote the field of algebraic numbers. Then the following implications hold:

$e^\alpha \cos \beta$ and $e^\alpha \sin \beta$ are algebraically dependent.

$\Leftrightarrow$ $\mathrm{Tr.\,deg.}_K\, K(e^\alpha \cos \beta, e^\alpha \sin \beta) \leqslant 1$.

$\Leftrightarrow$ $\mathrm{Tr.\,deg.}_K\, K(e^\alpha \cos \beta + ie^\alpha \sin \beta, e^\alpha \cos \beta - ie^\alpha \sin \beta) \leqslant 1$.

$\Leftrightarrow$ $\mathrm{Tr.\,deg.}_K\, K(e^{\alpha+\beta i}, e^{\alpha-\beta i}) \leqslant 1$.

$\Leftrightarrow$ $e^{\alpha+\beta i}$ and $e^{\alpha-\beta i}$ are algebraically dependent.

$\Leftrightarrow$ $\alpha + i\beta$ and $\alpha - i\beta$ are linearly dependent over the rationals (by the Lindemann–Weierstrass theorem).

$\Leftrightarrow$ α and βi are linearly dependent over the rationals.

The proof is complete. □

COROLLARY. *Suppose α is an algebraic number whose real and imaginary parts are both non-zero. Then the real and imaginary parts of e^α are algebraically independent.*

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Proof. Let $\alpha = a + bi$ where a and b are real numbers. Then a and b are both algebraic. Since $ab \neq 0$, a and bi are linearly independent over the rationals. Noting that $\operatorname{Re}(e^{\alpha}) = e^{a} \cos b$ and $\operatorname{Im}(e^{\alpha}) = e^{a} \sin b$, the corollary follows from Theorem 1. □

If α and β are algebraic with $\alpha, \beta i$ linearly independent over the rationals, we see from Theorem 1 that $e^{\alpha} \cos \beta$ and $e^{\alpha} \sin \beta$ are both transcendental. The next theorem shows that we can relax the condition on α and β and achieve the same result. We shall need the following version of the Lindemann-Weierstrass theorem [1, p. 23]: For any distinct algebraic numbers $\alpha_1, \ldots, \alpha_n$ and any non-zero algebraic numbers $\beta_1, \ldots, \beta_n$ we have

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0.$$

THEOREM 2. *Suppose α and β are algebraic numbers. Then the following are transcendental:*

(a) $e^{\alpha} \cos \beta$ *for* $(\alpha, \beta) \neq (0,0)$, $e^{\alpha} \sin \beta$ *and* $e^{\alpha} \tan \beta$ *for* $\beta \neq 0$;

(b) $\log(\alpha \cos \beta)$ *for* $\alpha \neq 0$ *and* $(\alpha, \beta) \neq (1,0)$, *and* $\log(\alpha \sin \beta)$ *and* $\log(\alpha \tan \beta)$ *for* $\alpha\beta \neq 0$ (*regardless of the branch of the natural logarithm*);

(c) $\cos^{-1}(\alpha e^{\beta})$ *for* $(\alpha, \beta) \neq (1,0)$, $\sin^{-1}(\alpha e^{\beta})$ *for* $\alpha \neq 0$, *and* $\tan^{-1}(\alpha e^{\beta})$ *for* $\alpha \neq 0$ *and* $(\alpha, \beta) \neq (\pm i, 0)$ (*regardless of the branches of the inverse trigonometric functions*;

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\alpha}{n}$ *for* $-\pi < \alpha < \pi$, $\sum_{n=1}^{\infty} \frac{\cos n\alpha}{n}$ *for* $0 < \alpha < 2\pi$, *and* $\sum_{n=1}^{\infty} \frac{\cos[(2n-1)\alpha]}{(2n-1)}$ *for* $0 < \alpha < \pi$.

Proof. We will prove only the first result in each of (a)-(d). Similar arguments apply to the other results. For (a) suppose that $\gamma = e^{\alpha} \cos \beta$ is algebraic. It suffices to show $\alpha = \beta = 0$. Now

$$e^{\alpha + \beta i} + e^{\alpha - \beta i} - 2\gamma e^{0} = 0. \tag{1}$$

By the second version of the Lindemann-Weierstrass theorem, at least one of the following inequalities holds: $\alpha + \beta i = \alpha - \beta i$, $\alpha + \beta i = 0$, $\alpha - \beta i = 0$. In the first case $\beta = 0$. Then from (1) we have e^{α} is algebraic so that $\alpha = 0$. In the other cases $\alpha = \pm \beta i$. From (1) we see that $e^{\pm 2\beta i}$ is algebraic, and again $\alpha = \beta = 0$.

As for (b), suppose that $\gamma = \log(\alpha \cos \beta)$ is algebraic. Since $\alpha \neq 0$, it suffices to show $(\alpha, \beta) = (1, 0)$. Now $e^{-\gamma} \cos \beta = \alpha^{-1}$. Using (a) we see that $\beta = \gamma = 0$, and thus $\alpha = 1$.

For (c) we suppose that $\gamma = \cos^{-1}(\alpha e^{\beta})$ is algebraic. It suffices to show $(\alpha, \beta) = (1, 0)$. Now $e^{-\beta} \cos \gamma = \alpha$. From (a) we deduce that $\beta = \gamma = 0$ so that $\alpha = 1$.

Next, we consider (d). Using the theory of Fourier series we know

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\alpha}{n} = -\log\left|2\cos\frac{\alpha}{2}\right|, \quad -\pi<\alpha<\pi,$$

$$\sum_{n=1}^{\infty} \frac{\cos n\alpha}{n} = -\log\left|2\sin\frac{\alpha}{2}\right|, \quad 0<\alpha<2\pi,$$

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)\alpha}{2n-1} = -\frac{1}{2}\log\left|\tan\frac{\alpha}{2}\right|, \quad 0<\alpha<\pi.$$

Hence (d) follows from (b), and Theorem 2 is proved. □

We remark that results can also be proved when the cosines occurring in (d) are "replaced" by sines. For example, the transcendence of π implies the transcendence of $\sum_{n=1}^{\infty} \sin n\alpha/n = (\pi-\alpha)/2$ for all algebraic α with $0<\alpha<2\pi$.

Finally, we shall make an application of the Gelfond-Schneider theorem [1, p. 76]: If α and β are algebraic with $\alpha \neq 0$, 1 and β irrational, then $\alpha^{\beta} = \exp(\beta \log \alpha)$ is transcendental (regardless of the branch of the natural logarithm). The next theorem provides a complement to parts (b) and (c) of Theorem 2.

THEOREM 3. *Suppose α, β are algebraic with αi irrational. Then the following numbers are transcendental:*

(a) $\cos(\alpha \log \beta)$, $\sin(\alpha \log \beta)$, *and* $\tan(\alpha \log \beta)$ *for* $\beta \neq 0, 1$ (*regardless of the branch of the natural logarithm*);

(b) $\exp(\alpha \cos^{-1} \beta)$ *for* $\beta \neq 1$, $\exp(\alpha \sin^{-1} \beta)$ *for* $\beta \neq 0$, *and* $\exp(\alpha \tan^{-1} \beta)$ *for* $\beta \neq 0$, $\pm i$ (*regardless of the branches of the inverse trigonometric functions*).

Proof. We prove only the first result in both (a) and (b). The other results are proved in a similar fashion. For (a) assume that $\gamma = \cos(\alpha \log \beta)$ is algebraic with $\beta \neq 0, 1$. Then $e^{i\alpha \log \beta} + e^{-i\alpha \log \beta} = 2\gamma$. Hence $\exp(i\alpha \log \beta)$ is algebraic. Since αi is irrational and algebraic, the Gelfond-Schneider theorem implies $\beta = 0$ or $\beta = 1$, a contradiction.

As for (b), recall that

$$\cos^{-1}(z) = -i[\log(z + i(1-z^2)^{1/2})], \quad \sin^{-1} z = -i[\log(iz + (1-z^2)^{1/2})],$$

and $\tan^{-1} z = (i/2)\log((i+z)/(i-z))$ for specific branches of the square root and natural logarithm. Suppose $\exp(\alpha \cos^{-1} \beta)$ is algebraic with $\beta \neq 1$. Then $\exp(-\alpha i[\log(\beta + i(1-\beta^2)^{1/2})])$ is also algebraic. But $-\alpha i$ is algebraic and irrational while $\beta + i(1-\beta^2)^{1/2}$ is algebraic. The Gelfond-Schneider theorem implies $\beta + i(1-\beta^2)^{1/2} = 0$ or 1. The first case is impossible and the second leads to $\beta = 1$, a contradiction. Theorem 3 is proved. □

We conclude by noting that it was essential in Theorem 3 to assume the irrationality of αi.

REFERENCES

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