ON TWO NOTIONS OF THE LOCAL SPECTRUM FOR SEVERAL COMMUTING OPERATORS

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S. Frunză and E. Albrecht initiated the study of spectral decompositions for finite systems $a=(a_1,\dots,a_N)$ of commuting operators on a Banach space X. S. Frunză used the results of J. L. Taylor [8; 9] concerning the spectrum and the analytic functional calculus to develop his concept of decomposable N-tuples. He proved in [6] that a spectral capacity E of an N-tuple a is necessarily of the form

(1) $$E(F)=\{x\in X;\ \sigma_a(x)\subset F\},$$

where $\sigma_a(x)$ is the so-called local spectrum of a in x, i.e. the complement of the union of all open sets $U\subset \mathbf{C}^N$ on which there is a solution of

$$xs_1\wedge\cdots\wedge s_N=(\bar\partial\oplus\alpha)\psi.$$

Earlier E. Albrecht had suggested another definition of the local spectrum, now called the local analytic spectrum $\gamma_a(x)$ of a in x. In [1] he defined $\gamma_a(x)$ to be the complement of the union of all open sets $U\subset \mathbf{C}^N$ on which there are analytic functions $f_i\colon U\to X$ satisfying

$$x=\sum_{i=1}^{N}(z_i-a_i)f_i(z),\quad z\in U.$$

S. Frunză has shown in [6] that $\sigma_a(x)\subset\gamma_a(x)$, that equality holds for decomposable N-tuples and in [7] that $\gamma_a(x)$ is contained in the spectral hull of $\sigma_a(x)$.

The aim of this paper is to prove that $\sigma_a(x)=\gamma_a(x)$ holds for each $x\in X$. This enables us to give a very simple proof of Equation (1) for decomposable N-tuples.

Equality of the local spectra. Let X be a complex Banach space and $a=(a_1,\dots,a_N)$ a system of commuting continuous linear operators on X. For an open set $U\subset\mathbf{C}^N$ we denote by $\mathcal{O}(U,X)$ the space of all analytic functions and by $C^\infty(U,X)$ the space of all C^∞-functions on U with values in X. If B is a K-module let $A_p(\mathbf{C}^N,B)$ be the K-module of all forms with degree p over $\mathbf{C}^N$ and coefficients in B. Special 1-forms needed are

$$\alpha(z)=(z_1-a_1)s_1+\cdots+(z_N-a_N)s_N,\qquad \alpha=(z_1-a_1)s_1+\cdots+(z_N-a_N)s_N,$$

$$\bar\partial=(\partial/\partial\bar z_1)d\bar z_1+\cdots+(\partial/\partial\bar z_N)d\bar z_N\quad\text{and}$$

$$\bar\partial\oplus\alpha=(\partial/\partial\bar z_1)d\bar z_1+\cdots+(\partial/\partial\bar z_N)d\bar z_N+(z_1-a_1)s_1+\cdots+(z_N-a_N)s_N.$$

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The first three operate by left exterior multiplication on $A_p(\mathbf{C}^N, B)$, where B is one of the spaces X, $\mathcal{O}(U,X)$, $C^\infty(U,X)$ and the last one operates on $A(\mathbf{C}^{2N}, C^\infty(U,X))$. The Taylor spectrum of a is by definition

$$\sigma(a,X)=\{z\in\mathbf{C}^N; 0\to X\xrightarrow{\alpha(z)}\cdots\xrightarrow{\alpha(z)}A_N(\mathbf{C}^N,X)\to 0 \text{ is non exact}\},$$

and the local spectrum of a in $x\in X$ in the sense of S. Frunză $\sigma_a(x)$ is the complement of the union of all open sets $U\subset\mathbf{C}^N$ on which there is a form $\psi\in A_{N-1}(\mathbf{C}^{2N}, C^\infty(U,X))$ satisfying

$$xs_1\wedge\cdots\wedge s_N=(\bar\partial\oplus\alpha)\psi \tag{2}$$

(see [6]). Let the local analytic spectrum $\gamma_a(x)$ of a in $x\in X$ in the sense of E. Albrecht ([1]) be defined as above. In [6] S. Frunză observed that $\sigma_a(x)\subset\gamma_a(x)\subset\sigma(a,X)$: If $z^0\notin\gamma_a(x)$ there are analytic functions $f_1,\ldots,f_N\in\mathcal{O}(U,X)$ on an open neighbourhood U of z^0 satisfying $x=\sum_{i=1}^N(z_i-a_i)f_i(z)$, $z\in U$. But then clearly $\psi=\sum_{i=1}^N(-1)^{i-1}f_is_1\wedge\cdots\wedge\hat{s}_i\wedge\cdots\wedge s_N$ is a C^∞-form satisfying (2) on U. If $z^0\notin\sigma(a,X)$ there is an open polydisc D contained in $\mathbf{C}^N-\sigma(a,X)$ with center z^0. On D

$$0\to X\xrightarrow{\alpha(z)}A_1(\mathbf{C}^N,X)\xrightarrow{\alpha(z)}\cdots\xrightarrow{\alpha(z)}A_N(\mathbf{C}^N,X)\to 0$$

is an analytical parameterized chain complex in the sense of J. L. Taylor. Since it is exact in each $z\in D$, the cohomology groups

$$H^p(\mathcal{O}(D,X),\alpha)=\frac{\operatorname{Ker}\{\alpha: A_p(\mathbf{C}^N,\mathcal{O}(D,X))\to A_{p+1}(\mathbf{C}^N,\mathcal{O}(D,X))\}}{\operatorname{Im}\{\alpha: A_{p-1}(\mathbf{C}^N,\mathcal{O}(D,X))\to A_p(\mathbf{C}^N,\mathcal{O}(D,X))\}}$$

vanish for $p=0,\ldots,N$ (see [8: Theorem 2.2]). In particular each $f\in\mathcal{O}(D,X)$ is of the form $f(z)=(z_1-a_1)f_1(z)+\cdots+(z_N-a_N)f_N(z)$, $z\in D$, with appropriate $f_1,\ldots,f_N\in\mathcal{O}(D,X)$.

Now we are able to prove our main result:

THEOREM 1. *For each finite system $a=(a_1,\ldots,a_N)$ of commuting continuous linear operators on a Banach space X, and for each $x\in X$, the local spectrum of a in x and the local analytic spectrum of a in x coincide; i.e., $\sigma_a(x)=\gamma_a(x)$.*

Proof. If $z^0\notin\sigma_a(x)$ there is an open polydisc D centered at z^0 and a form $\psi\in A_{N-1}(\mathbf{C}^{2N}, C^\infty(D,X))$ such that $xs_1\wedge\cdots\wedge s_N=(\bar\partial\oplus\alpha)\psi$. It is well-known (cf. [8: proof of Lemma 2.3]) that the X-valued $\bar\partial$-sequence

$$0\to\mathcal{O}(D,X)\xrightarrow{i}C^\infty(D,X)\xrightarrow{\bar\partial}A_1(\mathbf{C}^N,C^\infty(D,X))\xrightarrow{\bar\partial}\cdots \\ \cdots\xrightarrow{\bar\partial}A_N(\mathbf{C}^N,C^\infty(D,X))\to 0 \tag{3}$$

is exact. ψ can be written as the sum $\psi=\psi_{0,N-1}+\psi_{1,N-2}+\cdots+\psi_{N-1,0}$ of forms $\psi_{p,q}$ of degree p in $d\bar z_1,\ldots,d\bar z_N$ and degree q in $s_1,\ldots,s_N$, and $xs_1\wedge\cdots\wedge s_N=(\bar\partial\oplus\alpha)\psi$ is equivalent to

$$(4)\qquad \begin{aligned} xs_1\wedge\cdots\wedge s_N&=\alpha\psi_{0,N-1},\\ 0&=\bar\partial\psi_{0,N-1}+\alpha\psi_{1,N-2},\ldots,\\ 0&=\bar\partial\psi_{N-2,1}+\alpha\psi_{N-1,0},\\ 0&=\bar\partial\psi_{N-1,0}. \end{aligned}$$

Due to the exactness of (3) we can find a form $\varphi_{N-2,0}$ such that $\psi_{N-1,0}=\bar\partial\varphi_{N-2,0}$. Replacing this in the next to the last equation of (4) we obtain $\bar\partial(\psi_{N-2,1}-\alpha\varphi_{N-2,0})=0$. Successively we can choose $\varphi_{N-3,1},\ldots,\varphi_{0,N-2}$ such that

$$\psi_{N-l,l-1}-\alpha\varphi_{N-l,l-2}=\bar\partial\varphi_{N-l-1,l-1}\quad(2\leqslant l\leqslant N-1).$$

Applying the second equation of (4) to the case $l=N-1$ we obtain $\bar\partial(\psi_{0,N-1}-\alpha\varphi_{0,N-2})=0$. Hence $\chi=\psi_{0,N-1}-\alpha\varphi_{0,N-2}\in A_{N-1}(\mathbf{C}^N,\mathcal{O}(D,X))$ is a form with analytic coefficients which satisfies $xs_1\wedge\cdots\wedge s_N=\alpha\psi_{0,N-1}=\alpha\chi$. □

S. Frunză has proved in [6] that the condition $H^{N-1}(C^\infty(U,X),\bar\partial\oplus\alpha)=0$ for all open sets $U\subset\mathbf{C}^N$ is sufficient to guarantee for each $x\in X$ the existence of a global solution ψ of (2) on the largest possible domain $\rho_a(x)=\mathbf{C}^N-\sigma_a(x)$. Our proof shows in this case that there are analytic solutions $f_1,\ldots,f_N\in\mathcal{O}(U,X)$ of $x=\sum_{i=1}^N(z_i-a_i)f_i(z)$, $z\in U$, on each open set $U\subset\rho_a(x)$ on which the X-valued $\bar\partial$-sequence

$$0\to\mathcal{O}(U,X)\xrightarrow{i}C^\infty(U,X)\xrightarrow{\bar\partial}A_1(\mathbf{C}^N,C^\infty(U,X))\xrightarrow{\bar\partial}\cdots$$
$$\cdots\xrightarrow{\bar\partial}A_N(\mathbf{C}^N,C^\infty(U,X))\to 0$$

is exact. Using tensor product methods it is not hard to show that this is true for every domain of holomorphy $U\subset\mathbf{C}^N$. For this and other related questions see [5].

An application to decomposable *N*-tuples. Theorem 1 can be used to give a much simpler proof of Equation (1) for decomposable N-tuples. Namely, it is sufficient to prove

$$E(F)=\{x\in X;\gamma_a(x)\subset F\}.$$

For the definition of a decomposable N-tuple we refer the reader to Definition 3.1 in [6].

THEOREM 2. *If E is a spectral capacity for $a=(a_1,\ldots,a_N)$, then for each $x\in X$ we have $\gamma_a(x)=\bigcap\,(F;\ F\subset\mathbf{C}^N$ closed such that $x\in E(F))$.*

Proof. We shall only sketch the proof. Details can be found in [4]. Let us denote the intersection on the right side by $\operatorname{supp}(E,x)$. The inclusion $\gamma_a(x)\subset\operatorname{supp}(E,x)$ is obvious. Conversely, if D is any open polydisc such that $x=\sum_{i=1}^N(z_i-a_i)f_i(z)$ has analytic solutions f_i on an open neighbourhood U of $\bar D$, let us choose another open polydisc D' larger than D and such that $\bar D'\subset U$. Then $X=Y+Z$ for $Y=E(\mathbf{C}^N-D)$, $Z=E(\bar D')$, $\sigma(a,X/Y)=\sigma(a,Z/Y\cap Z)\subset\sigma(a,Z)\cup\sigma(a,Y\cap Z)\subset\bar D'$ and

$$(2\pi i)^N x/Y = \int_{\Gamma_1} \cdots \int_{\Gamma_N} \prod_{i=1}^{N} R(z_i, a_i/Y) \sum_{j=1}^{N} (z_j - a_j/Y)(f_j(z)/Y)\, dz_N \ldots dz_1 = 0$$

for suitably chosen Γ_i and $R(z_i, a_i/Y) = (z_i - a_i/Y)^{-1}$. Hence $D \subset \mathbf{C}^N - \operatorname{supp}(E, x)$. Now the quoted result of S. Frunză is an easy consequence (cf. [6: Theorem 3.3]). □

COROLLARY. *If $a = (a_1, \ldots, a_N)$ is decomposable, its spectral capacity is given by $E(F) = \{x \in X;\ \sigma_a(x) \subset F\}$. In particular, E is uniquely determined and all spaces $E(F)$ are spectral maximal* (generalization of [2: Definition 3.1]).

Proof. If $x \in E(F)$ we have $\sigma_a(x) \subset \sigma(a, E(F)) \subset F$. On the other hand, for $\sigma_a(x) \subset F$ it follows by the intersection property of the spectral capacity and the above theorems that $x \in E(\sigma_a(x)) \subset E(F)$.

If Y is any closed a-invariant subspace with $\sigma(a, Y) \subset \sigma(a, E(F))$ it follows that $\sigma_a(y) \subset \sigma(a, Y) \subset F$ for all $y \in Y$, hence $Y \subset E(F)$. □

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