(BCP)-OPERATORS ARE REFLEXIVE

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Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If T is a contraction in $\mathcal{L}(\mathcal{H})$ (i.e., $\|T\| \leqslant 1$) that has no nontrivial reducing subspace on which it acts as a unitary operator, then T is called a *completely nonunitary contraction*. A subset S of the open unit disc $D = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$ is said to be *dominating for* ∂D if almost every point of ∂D is a nontangential limit point of S. (An alternative characterization of sets dominating for ∂D was given in [8].) The set of all completely nonunitary contractions T in $\mathcal{L}(\mathcal{H})$ with the property that the intersection $\sigma_e(T) \cap D$ of the essential (i.e., Calkin) spectrum of T with D is dominating for ∂D will be denoted by (BCP); we permit ourselves the indulgence of referring to operators in this set as (BCP)-operators.

Such operators were first studied in [7], where a certain structure theorem ([7, Lemma 4.9]) was obtained that had as a consequence the existence of nontrivial invariant subspaces for all (BCP)-operators. (Proposition 1 below is a strengthening of this structure theorem.) The study of the class (BCP) continued in [2], [3], [4], [13], [14], and [16], and, as a result, we now know considerably more about the structure of (BCP)-operators. (In particular, we owe to [16] the clarification of the correct definition of the class (BCP). Before [16] a somewhat more restrictive definition of (BCP)-operator was in use.)

The purpose of this paper is to make an additional contribution to the theory of (BCP)-operators by proving the theorem of the title and related results. Recall that if $T \in \mathcal{L}(\mathcal{H})$, then $\operatorname{Lat}(T)$ is by definition the lattice of all invariant subspaces of T, and $\operatorname{Alg}\operatorname{Lat}(T)$ is the algebra of all operators A in $\mathcal{L}(\mathcal{H})$ such that $\operatorname{Lat}(T) \subset \operatorname{Lat}(A)$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *reflexive* if $\operatorname{Alg}\operatorname{Lat}(T) = \mathcal{W}(T)$, where $\mathcal{W}(T)$ is the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the weak operator topology (WOT).

The first examples of reflexive operators were given by Sarason in [17], where he showed that normal operators and analytic Toeplitz operators in $\mathcal{L}(\mathcal{H})$ are reflexive. This line of research was continued by Deddens [10], who showed that every isometry in $\mathcal{L}(\mathcal{H})$ is reflexive, and these results of Sarason and Deddens are particular cases of the recent beautiful theorem of Olin and Thomson [15] which says that all subnormal operators in $\mathcal{L}(\mathcal{H})$ are reflexive. In another direction, Deddens and Fillmore [11] characterized those operators acting on a finite dimensional space that are reflexive. This result was extended to C_0-operators (in the sense of [18]) in [5].

Our central result is as follows.

THEOREM 1. *Every* (BCP)*-operator in* $\mathcal{L}(\mathcal{H})$ *is reflexive.*

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The third author, in [14], obtained some interesting special cases of this result. We now introduce some notation and terminology from [7] that will be needed in the proof of Theorem 1.

Suppose $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}(T)$ be the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology (cf. [12, Chapter I]), and let $\mathcal{Q}(T)$ denote the quotient space $(\tau c)/{}^a\mathcal{A}(T)$, where (τc) denotes the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}^a\mathcal{A}(T)$ denotes the preannihilator of $\mathcal{A}(T)$ in (τc). Then $\mathcal{A}(T)$ is the dual space of $\mathcal{Q}(T)$ and the duality is given by

$$\langle A, [L] \rangle = \operatorname{tr}(AL), \qquad A \in \mathcal{A}(T), \qquad [L] \in \mathcal{Q}(T),$$

where $[L]$ is the image in $\mathcal{Q}(T)$ of the operator L in (τc). Furthermore, the weak* topology that accrues to $\mathcal{A}(T)$ by virtue of this duality coincides with the ultraweak operator topology on $\mathcal{A}(T)$ (see [7] for details). Moreover if x and y are vectors in $\mathcal{H}$ and we write, as usual, $x \otimes y$ for the rank-one operator in (τc) defined by

$$(x \otimes y)(u) = (u, y)x, \qquad u \in \mathcal{H},$$

then an easy computation shows that for any A in $\mathcal{A}(T)$, we have $\langle A, [x \otimes y] \rangle = \operatorname{tr}(A(x \otimes y)) = (Ax, y)$. Furthermore the content of [7, Lemma 4.9], mentioned above, is that if $T \in$ (BCP), then every element of $\mathcal{Q}(T)$ has the form $[x \otimes y]$ for some vectors x, y in $\mathcal{H}$. The following proposition from [3] is an improvement of this result.

PROPOSITION 1. *Suppose* $T \in$ (BCP), $[L] \in \mathcal{Q}(T)$, ϵ *is any positive number, and* x, y *are vectors in* $\mathcal{H}$. *Then there exist* x', y' *in* $\mathcal{H}$ *such that*

$$[x' \otimes y'] = [L],$$
$$\|x - x'\| < \|[L] - [x \otimes y]\|^{1/2} + \epsilon, \quad \textit{and}$$
$$\|y - y'\| < \|[L] - [x \otimes y]\|^{1/2} + \epsilon.$$

An easy consequence of Proposition 1 is the analog for (BCP)-operators of the fact from [15] that if T is a subnormal operator, then $\mathcal{W}(T) = \mathcal{A}(T)$.

COROLLARY 1. *If* $T \in$ (BCP), *then* $\mathcal{W}(T) = \mathcal{A}(T)$ *and the weak operator and ultraweak operator topologies coincide on* $\mathcal{A}(T)$.

Proof. One knows that $\mathcal{A}(T) \subset \mathcal{W}(T)$, so to prove $\mathcal{W}(T) = \mathcal{A}(T)$ it suffices to prove the reverse inclusion. Thus, let A be any operator in $\mathcal{W}(T)$, and let $\{A_\alpha\}$ be a net in $\mathcal{A}(T)$ that is WOT-convergent to A. We assert that the formula

$$\phi([L]) = \lim_\alpha \langle A_\alpha, [L] \rangle, \qquad [L] \in \mathcal{Q}(T), \tag{1}$$

defines a bounded linear functional on $\mathcal{Q}(T)$. Indeed, if $[L] \in \mathcal{Q}(T)$ it follows from Proposition 1 that we can write $[L] = [h \otimes k]$ for certain vectors h and k in $\mathcal{H}$ satisfying $\|h\|, \|k\| \leqslant \sqrt{2}\|[L]\|^{1/2}$ (take $x = 0$, $y = 0$, $\epsilon = (\sqrt{2} - 1)\|[L]\|^{1/2}$, $h = x'$, and $k = y'$). Therefore

$$\lim_\alpha \langle A_\alpha, [L] \rangle = \lim_\alpha \langle A_\alpha, [h \otimes k] \rangle = \lim_\alpha (A_\alpha h, k) = (Ah, k), \tag{2}$$

so that the limit in (1) exists and equals (Ah,k). Moreover, ϕ is obviously linear and

$$|\phi([L])| = |(Ah,k)| \leqslant \|A\| \|h\| \|k\| \leqslant 2\|A\| \|[L]\|,$$

so ϕ is bounded. Since $\mathcal{A}(T) = (\mathcal{Q}(T))^*$, there exists an operator B in $\mathcal{Q}(T)$ such that

$$\phi([L]) = \langle B, [L]\rangle, \qquad [L] \in \mathcal{Q}(T).$$

Since $(Bx,y) = \langle B, [x \otimes y]\rangle = \phi([x\otimes y]) = \lim_\alpha \langle A_\alpha, [x\otimes y]\rangle = (Ax,y)$ for all vectors x and y in $\mathcal{H}$, we conclude that $B = A$ and hence $A \in \mathcal{A}(T)$. To conclude the proof we note that (2) can now be rewritten as

$$\lim_\alpha \langle A_\alpha, [L]\rangle = \langle A, [L]\rangle, \qquad [L] \in \mathcal{Q}(T),$$

from which it follows that the net $\{A_\alpha\}$ converges to A in the weak* or ultraweak operator topology. Since the ultraweak operator topology is, by definition, stronger than the weak operator topology, the proof is complete. □

We now proceed to the proof of Theorem 1, which depends upon the following sequence of lemmas.

LEMMA 1. *Suppose A and T belong to $\mathcal{L}(\mathcal{H})$. If for every positive integer n and every pair of sequences $\{h_1, \ldots, h_n\}$ and $\{k_1, \ldots, k_n\}$ of vectors from $\mathcal{H}$, the equation $\sum_{i=1}^n [h_i \otimes k_i] = 0$ in $\mathcal{Q}(T)$ implies $\sum_{i=1}^n (Ah_i, k_i) = 0$, then $A \in \mathcal{W}(T)$.*

Proof. Suppose, to the contrary, that $A \notin \mathcal{W}(T)$. Then the Hahn–Banach theorem (cf. [6, Proposition 15.9]) implies the existence of a weakly continuous linear functional ϕ on $\mathcal{L}(\mathcal{H})$ such that $\phi(\mathcal{W}(T)) = 0$ but $\phi(A) \neq 0$. By [12, Theorem 1, Section I.3], there exist sequences $\{h_1, \ldots, h_n\}$ and $\{k_1, \ldots, k_n\}$ in $\mathcal{H}$ such that ϕ has the form

$$\phi(X) = \sum_{i=1}^n (Xh_i, k_i), \qquad X \in \mathcal{L}(\mathcal{H}).$$

Thus, in particular, if $X \in \mathcal{A}(T) \subset \mathcal{W}(T)$, we have

$$\langle X, \sum_{i=1}^n [h_i \otimes k_i]\rangle = \sum_{i=1}^n (Xh_i, k_i) = \phi(X) = 0,$$

so $\sum_{i=1}^n [h_i \otimes k_i] = 0$ in $\mathcal{Q}(T)$. On the other hand, $\phi(A) = \sum_{i=1}^n (Ah_i, k_i) \neq 0$, which contradicts the hypothesis, so the lemma is proved. □

LEMMA 2. *If B is an operator acting on any complex Hilbert space and there exists a nonzero polynomial p such that p has only simple zeros and $p(B) = 0$, then B is reflexive and $\mathcal{W}(B)$ coincides with the algebra of all polynomials in B.*

Proof. One can give various proofs of this elementary lemma. For example, it is easy to see that B is similar to a normal operator with finite spectrum, and the reflexivity of B then follows from [17] and the fact that the reflexivity is invariant under similarity transformations. This reflexivity result is also an easy corollary of [5, Theorem A]. That $\mathcal{W}(B)$ coincides with the algebra of all polynomials in B follows from the fact that this algebra is finite dimensional and therefore weakly closed. □

LEMMA 3. *Suppose* $T \in \mathcal{L}(\mathcal{H})$, *and* $\mathfrak{M}, \mathfrak{N} \in \operatorname{Lat}(T)$ *with* $\mathfrak{N} \subset \mathfrak{M}$. *Suppose also that* $A \in \operatorname{Alg}\operatorname{Lat}(T)$, *and write* $\tilde{T} = P_{\mathfrak{M}\ominus\mathfrak{N}} T \mid \mathfrak{M} \ominus \mathfrak{N}$ *and* $\tilde{A} = P_{\mathfrak{M}\ominus\mathfrak{N}} A \mid \mathfrak{M} \ominus \mathfrak{N}$, *where* $P_{\mathfrak{M}\ominus\mathfrak{N}}$ *is the projection onto* $\mathfrak{M} \ominus \mathfrak{N}$. *Then* $\tilde{A} \in \operatorname{Alg}\operatorname{Lat}(\tilde{T})$. *Moreover, if* $\mathfrak{M}_* \in \operatorname{Lat}(T^*)$ *and* $\mathfrak{N} = \mathfrak{M} \cap \mathfrak{M}_*^{\perp}$, *then a necessary and sufficient condition that* $\tilde{A} = 0$ *is that* $(Ah, k) = 0$ *for all* h *in* $\mathfrak{M}$ *and* k *in* $\mathfrak{M}_*$.

Proof. We have $\operatorname{Lat}(T \mid \mathfrak{M}) \subset \operatorname{Lat}(T)$ so that obviously $A \mid \mathfrak{M} \in \operatorname{Alg}\operatorname{Lat}(T \mid \mathfrak{M})$, which in turn implies that $(A \mid \mathfrak{M})^* \in \operatorname{Alg}\operatorname{Lat}((T \mid \mathfrak{M})^*)$. Since $\mathfrak{M} \ominus \mathfrak{N}$ is invariant for $(T \mid \mathfrak{M})^*$, a similar argument shows that

$$\tilde{A}^* = (A \mid \mathfrak{M})^* \mid \mathfrak{M} \ominus \mathfrak{N} \in \operatorname{Alg}\operatorname{Lat}\{(T \mid \mathfrak{M})^* \mid \mathfrak{M} \ominus \mathfrak{N}\} = \operatorname{Alg}\operatorname{Lat}(\tilde{T}^*),$$

or, equivalently, that $\tilde{A} \in \operatorname{Alg}\operatorname{Lat}(\tilde{T})$.

To prove the second statement, note that $\tilde{A} = 0$ if and only if $A\mathfrak{M} \subset \mathfrak{N}$. Indeed, the fact that $A\mathfrak{M} \subset \mathfrak{N}$ implies $\tilde{A} = 0$ is trivial, and since

$$\begin{aligned} A\mathfrak{M} &= A(\mathfrak{M} \ominus \mathfrak{N}) + A\mathfrak{N} \\ &\subset \tilde{A}(\mathfrak{M} \ominus \mathfrak{N}) + P_{\mathfrak{N}} A(\mathfrak{M} \ominus \mathfrak{N}) + A\mathfrak{N} \subset \tilde{A}(\mathfrak{M} \ominus \mathfrak{N}) + \mathfrak{N}, \end{aligned}$$

it is clear that $\tilde{A} = 0$ implies $A\mathfrak{M} \subset \mathfrak{N}$. Now since $\mathfrak{N} = \mathfrak{M} \cap \mathfrak{M}_*^{\perp}$, a vector x in $\mathfrak{M}$ belongs to $\mathfrak{N}$ if and only if $(x, k) = 0$ for all k in $\mathfrak{M}_*$. Thus, for any h in $\mathfrak{M}$, $Ah \in \mathfrak{N}$ if and only if $(Ah, k) = 0$ for all k in $\mathfrak{M}_*$, and the lemma follows from these remarks. □

The following proposition is a special case of the basic matricial factorization theorem of [3]. Note that for $n = 1$, its content is exactly Proposition 1. For a proof of Proposition 2 that does not rely upon the Sz.-Nagy–Foiaş functional model of a contraction, see [16].

PROPOSITION 2. *Suppose* $T \in$ (BCP), n *is a positive integer, and* $\{[L_{ij}]\}_{i,j=1}^{n}$ *is a doubly indexed family of elements of* $\mathcal{Q}(T)$. *Suppose also that* δ *and* ϵ *are any positive numbers, and that there exist sequences* $\{h_1, \dots, h_n\}$ *and* $\{k_1, \dots, k_n\}$ *of vectors in* $\mathcal{H}$ *such that*

$$\|[L_{ij}] - [h_i \otimes k_j]\| \leqslant \delta, \qquad 1 \leqslant i, j \leqslant n.$$

Then there exist sequences $\{h_1', \dots, h_n'\}$ *and* $\{k_1', \dots, k_n'\}$ *in* $\mathcal{H}$ *such that*

$$[L_{ij}] = [h_i' \otimes k_j'], \qquad 1 \leqslant i, j \leqslant n,$$

and

$$\|h_i' - h_i\|_{\mathcal{H}} < n\delta^{1/2} + \epsilon, \qquad \|k_i' - k_i\|_{\mathcal{H}} < n\delta^{1/2} + \epsilon, \qquad 1 \leqslant i \leqslant n.$$

PROOF OF THEOREM 1. Let T be a (BCP)-operator. Since it is obvious that $\mathcal{W}(T) \subset \operatorname{Alg}\operatorname{Lat}(T)$, it suffices to prove that every A in $\operatorname{Alg}\operatorname{Lat}(T)$ belongs to $\mathcal{W}(T)$. We will accomplish this via Lemma 1, so let n be a positive integer and let $\{h_1, \dots, h_n\}$ and $\{k_1, \dots, k_n\}$ be sequences from $\mathcal{H}$ such that $\sum_{i=1}^{n} [h_i \otimes k_i] = 0$ in $\mathcal{Q}(T)$. We must show that $\sum_{i=1}^{n} (Ah_i, k_i) = 0$. If $n = 1$, and p is any polynomial, then $(p(T)h_1, k_1) = \langle p(T), [h_1 \otimes k_1] \rangle = 0$, so the cyclic subspace $\mathfrak{M} = \{p(T)h_1\}^-$ is orthogonal to k_1. Thus the equality $(Ah_1, k_1) = 0$ follows from the fact that

$\mathfrak{M} \in \operatorname{Lat}(A)$. We may therefore suppose that $n > 1$. To show that $\sum_{i=1}^{n}(Ah_i, k_i) = 0$, we fix an arbitrary positive number $\gamma < 1$ and show that $|\sum_{i=1}^{n}(Ah_i, k_i)| < \gamma$. To accomplish this, we first set $M = 2n^3(\|A\|+1)(K+1)$, where

$$K = \max\{\|h_1\|, \ldots, \|h_n\|, \|k_1\|, \ldots, \|k_n\|\},$$

and we observe that a simple calculation shows that it suffices to exhibit sequences $\{h_1', \ldots, h_n'\}$ and $\{k_1', \ldots, h_n'\}$ from $\mathcal{H}$ such that $\sum_{i=1}^{n}(Ah_i', k_i') = 0$ and such that

$$\|h_i' - h_i\| < \frac{n^2\gamma}{M}, \qquad \|k_i' - k_i\| < \frac{n^2\gamma}{M}, \qquad i = 1, \ldots, n. \tag{3}$$

Indeed, given such sequences $\{h_1', \ldots, h_n'\}$ and $\{k_1', \ldots, k_n'\}$, we have

$$\begin{aligned} |(Ah_i, k_i) - (Ah_i', k_i')| &\leqslant |(Ah_i, k_i - k_i')| + |(A(h_i - h_i'), k_i')| \\ &\leqslant \|A\|\,\|h_i\|\,\|k_i - k_i'\| + \|A\|\,\|h_i - h_i'\|\,\|k_i'\| < \gamma/n, \end{aligned}$$

and therefore

$$\left|\sum_{i=1}^{n}(Ah_i, k_i)\right| \leqslant \sum_{i=1}^{n}|(Ah_i, k_i) - (Ah_i', k_i')| < \gamma.$$

To construct the desired sequences $\{h_i'\}$ and $\{k_i'\}$, we choose, using [7, Lemma 4.7], for each fixed pair (i,j) of integers satisfying $1 \leqslant i,j \leqslant n$ and $(i,j) \neq (n,n)$, finite sequences $\{\alpha_k^{i,j}\}_{k=1}^{N(i,j)} \subset \mathbf{C}$ and $\{\lambda_k^{i,j}\}_{k=1}^{N(i,j)} \subset D$ satisfying

$$\left\|[h_i \otimes k_j] - \sum_{i=1}^{N(i,j)} \alpha_k^{i,j}[C_{\lambda_k^{i,j}}]\right\| < \gamma^2/M^2, \tag{4}$$

where the elements $[C_\lambda]$ in $\mathcal{Q}(T)$ satisfy $\langle p(T), [C_\lambda]\rangle = p(\lambda)$ for all polynomials p. For all pairs (i,j) except the pair (n,n), we now set

$$[L_{i,j}] = \sum_{k=1}^{N(i,j)} \alpha_k^{i,j}[C_{\lambda_k^{i,j}}], \tag{5}$$

and we also set

$$[L_{n,n}] = -\sum_{i=1}^{n-1}[L_{i,i}] = \sum_{k=1}^{N(n,n)} \alpha_k^{n,n}[C_{\lambda_k^{n,n}}] \tag{6}$$

where the $\alpha_k^{n,n}$ and $\lambda_k^{n,n}$ are defined in the obvious way in terms of the $\alpha_k^{i,i}$ and $\lambda_k^{i,i}$, $1 \leqslant i \leqslant n-1$. By virtue of (4) and (5), it follows that

$$\|[h_i \otimes k_j] - [L_{i,j}]\| < \gamma^2/M^2, \qquad 1 \leqslant i,j \leqslant n, \qquad (i,j) \neq (n,n). \tag{7}$$

For $(i,j) = (n,n)$, we have from (6) and (7),

$$\|[h_n \otimes k_n] - [L_{n,n}]\| = \left\|-\sum_{i=1}^{n-1}[h_i \otimes k_i] + \sum_{i=1}^{n-1}[L_{i,i}]\right\| \leqslant (n-1)\gamma^2/M^2.$$

Thus, by Proposition 2 (with $\delta = n\gamma^2/M^2$ and ϵ sufficiently small), there exist sequences $\{h_1', \ldots, h_n'\}$ and $\{k_1', \ldots, k_n'\}$ in $\mathcal{H}$ such that

$$[h_i' \otimes k_j'] = [L_{i,j}], \qquad 1 \leq i, j \leq n, \tag{8}$$

and such that (3) is valid. Hence, to complete the argument, it suffices to show that $\sum_{i=1}^n (Ah_i', k_i') = 0$. But, by virtue of (6) and (8), we know that $\sum_{i=1}^n [h_i' \otimes k_i'] = 0$ in $\mathcal{Q}(T)$, and thus for every polynomial p, we have

$$\sum_{i=1}^n (p(T)h_i', k_i') = 0.$$

Thus it suffices to show that there is some polynomial q such that

$$(Ah_i', k_i') = (q(T)h_i', k_i'), \qquad 1 \leq i \leq n. \tag{9}$$

To accomplish this, let

$$\begin{aligned} \mathfrak{M} &= \{p_1(T)h_1' + \cdots + p_n(T)h_n' : p_1, \ldots, p_n \text{ any polynomials}\}^-, \qquad \text{and} \\ \mathfrak{M}_* &= \{r_1(T^*)k_1' + \cdots + r_n(T^*)k_n' : r_1, \ldots, r_n \text{ any polynomials}\}^-. \end{aligned} \tag{10}$$

It is obvious that $\mathfrak{M} \in \operatorname{Lat}(T)$ and $\mathfrak{M}_* \in \operatorname{Lat}(T^*)$, so we set $\mathfrak{N} = \mathfrak{M} \cap \mathfrak{M}_*^\perp$, and note that $\mathfrak{N} \in \operatorname{Lat}(T)$. Let $\tilde{T}$ and $\tilde{A}$ be as in Lemma 3. To show that there is a polynomial q such that (9) is valid, it suffices, by virtue of Lemma 3, to show that $\tilde{A} = q(\tilde{T})$ for some polynomial q. Furthermore, since $\tilde{A} \in \operatorname{Alg}\operatorname{Lat}(\tilde{T})$ by that same lemma, we conclude from Lemma 2 that it is enough to show that $\tilde{T}$ satisfies a polynomial equation $s(\tilde{T}) = 0$, where s has only simple zeros.

Let F denote the finite set of all those (distinct) $\lambda_k^{i,j}$ that appear in (4) (and (6)), and let s be the monic polynomial with simple zeros at exactly the points of F. The proof of the theorem can be completed by showing that $s(\tilde{T}) = 0$, or, what comes to the same thing (Lemma 3), by showing that $(s(T)h, k) = 0$ where h runs over a dense set of vectors in $\mathfrak{M}$ and k runs over a dense set of vectors in $\mathfrak{M}_*$. But if h and k have the form as in (10):

$$h = p_1(T)h_1' + \cdots + p_n(T)h_n',$$
$$k = r_1(T^*)k_1' + \cdots + r_n(T^*)k_n'$$

then we have (using (5), (6), and (8))

$$\begin{aligned} (s(T)h, k) &= \sum_{i,j=1}^n (s(T)p_i(T)h_i', r_j(T^*)k_j') \\ &= \sum_{i,j=1}^n (r_j^\sim(T)s(T)p_i(T)h_i', k_j') \\ &= \sum_{i,j=1}^n \langle r_j^\sim(T)s(T)p_i(T), [L_{ij}]\rangle \\ &= \sum_{i,j=1}^n \sum_{k=1}^{N(i,j)} \alpha_k^{i,j} r_j^\sim(\lambda_k^{i,j}) s(\lambda_k^{i,j}) p_i(\lambda_k^{i,j}) = 0 \end{aligned}$$

since s was defined so that $s(\lambda_k^{i,j}) = 0$ for all i, j, k. Thus the theorem is proved. □

A careful perusal of the above argument shows that in any "(BCP)-like" setting in which one has a version of Proposition 2, one will be able to prove reflexivity. One such setting occurs in [9] and [16], so we are able to conclude immediately the following result.

THEOREM 2. *If T is any polynomially bounded operator in $\mathcal{L}(\mathcal{H})$ such that $\sigma_{le}(T) \cap D$ is dominating for ∂D and such that $\{T^n\}$ converges to zero in the strong operator topology, then T is reflexive.*

It is clear that Theorem 2 is not a generalization of Theorem 1 (because, for example, of the hypothesis that $\{T^n\}$ converges strongly to zero). But [3] contains another setting in which Proposition 2 is valid and which does, indeed, yield a generalization of Theorem 1. If T is any contraction in $\mathcal{L}(\mathcal{H})$ and $\lambda \in D$, then the operator

$$T_\lambda = (T - \lambda I)(I - \bar{\lambda} T)^{-1}$$

is also a contraction, and we may define the sets

$$A_\theta = \{\lambda \in D : \min\{\inf \sigma_e(T_\lambda^* T_\lambda), \inf \sigma_e(T_\lambda T_\lambda^*)\} \leqslant \theta^2\}, \qquad 0 \leqslant \theta < 1.$$

THEOREM 3. *Every completely nonunitary contraction T in $\mathcal{L}(\mathcal{H})$ with the property that A_θ is dominating for ∂D whenever $0 < \theta < 1$ is reflexive.*

It is easy to see from the definitions (cf. [3]) that if T is a (BCP)-operator, then A_θ is a dominating set for $\theta = 0$, so Theorem 3 does generalize Theorem 1.

In fact, an example given below (Example 2) shows that there exist contractions T in $\mathcal{L}(\mathcal{H})$ satisfying the hypotheses of Theorem 3 such that $\sigma(T) \cap D = \varnothing$, so the generalization is certainly not vacuous. Furthermore, as was noted in [3], the operators studied by Apostol in [1, Theorem 2.3] also satisfy the hypotheses of Theorem 3. Thus we obtain the following.

THEOREM 4. *If T is a completely nonunitary contraction, with the property that, for each $0 < \theta < 1$, the set*

$$\{\lambda \in D : \lambda \in \sigma_e(T) \ \text{or} \ \theta \|(\pi(T) - \lambda)^{-1}\| \geqslant (1 - |\lambda|)^{-1}\}$$

is dominating for ∂D, where $\pi(T)$ is the image of T in the Calkin algebra, then T is reflexive.

It was noted in [14] that the hypothesis that T is completely nonunitary plays no essential role in the preceding results. For example, one has the following theorem.

THEOREM 5. *Every contraction T in $\mathcal{L}(\mathcal{H})$ such that $\sigma_e(T) \cap D$ is dominating for ∂D is reflexive.*

Proof. We may suppose that such a T has the form $T_0 \oplus U$ where $T_0 \in$ (BCP) and U is a unitary operator acting on a nontrivial Hilbert space. Write $U = U_a \oplus U_s$, where U_a and U_s are, respectively, absolutely continuous and singular unitary operators. It is known that every absolutely continuous unitary operator is similar to a completely nonunitary contraction. (In fact, the bilateral shift W of multiplicity one is obviously similar to a bilateral weighted shift all of whose weights except one are equal to 1. Furthermore, every absolutely continuous unitary operator is unitarily equivalent to the restriction to some reducing subspace of $W \otimes 1_{\mathcal{K}}$.) Thus U_a is similar to some completely nonunitary contraction T_1, and it clearly suffices to show that $T' = T_0 \oplus T_1 \oplus U_s$ is reflexive.

Write $T_2 = T_0 \oplus T_1$ and observe that $T_2 \in$ (BCP). Now let $A \in \operatorname{Alg}\operatorname{Lat}(T')$. Then, of course, $A = A_2 \oplus B$ where $A_2 \in \operatorname{Alg}\operatorname{Lat}(T_2)$ and $B \in \operatorname{Alg}\operatorname{Lat}(U_s)$. Since T_2 is reflexive by Theorem 1 and $\mathcal{W}(T_2) = \mathcal{A}(T_2) = \{u(T_2) : u \in H^\infty\}$ (cf. [7]), we have $A_2 = v(T_2)$ for some v in H^∞. Consider next $v(W) \oplus B$, which belongs to $\operatorname{Alg}\operatorname{Lat}(W \oplus U_s)$ because $\operatorname{Lat}(W \oplus U_s) = \{\mathfrak{M} \oplus \mathfrak{N} : \mathfrak{M} \in \operatorname{Lat}(W),\ \mathfrak{N} \in \operatorname{Lat}(U_s)\}$ (cf., for example, [10]). Since $W \oplus U_s$ is normal (and hence reflexive), there exists a net $\{p_\alpha\}$ of polynomials such that $\{p_\alpha(W \oplus U_s)\}$ is WOT convergent to $v(W) \oplus B$. Thus, in particular, the net

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} p_\alpha(e^{it})h(e^{it})\overline{g(e^{it})}\,dt\right\}$$

converges to

$$\frac{1}{2\pi}\int_0^{2\pi} v(e^{it})h(e^{it})\overline{g(e^{it})}\,dt$$

for all functions h and g in $L_2[0,2\pi]$, and since every function in $L_1[0,2\pi]$ is a product $h\bar{g}$ where $h, g \in L_2[0,2\pi]$, we conclude that the net $\{p_\alpha\}$ converges to v in the weak* topology on H^∞. Therefore (cf. [7]), the net $\{p_\alpha(T_2)\}$ converges ultraweakly to $v(T_2) = A_2$. Consequently the net $\{p_\alpha(T') = p_\alpha(T_2) \oplus p_\alpha(U_s)\}$ is WOT-convergent to $A = A_2 \oplus B$, so $A \in \mathcal{W}(T')$ and the theorem is proved. □

The following examples show that the above theorems have several interesting consequences.

EXAMPLE 1. It was shown in [13] that there exists an invertible (BCP)-operator T with the property that $\operatorname{Lat}(T^{-1})$ is linearly ordered. Thus T is reflexive while T^{-1} is not reflexive. The existence of such operators T is a striking phenomenon, which certainly cannot happen on finite dimensional spaces.

EXAMPLE 2. As promised above, we will now exhibit an operator T satisfying the hypotheses of Theorem 3 such that $\sigma(T) \cap D = \varnothing$. For every λ in ∂D and every positive number a we denote by $\theta_{\lambda,a}$ the inner function defined by

$$\theta_{\lambda,a}(z) = \exp\left(a\frac{z+\lambda}{z-\lambda}\right), \quad z \in D.$$

Note that $\lim_{z\to 1}\theta_{1,a}(z) = 0$ where $z \to 1$ nontangentially. Therefore the relation

$$\theta_{\lambda,a}(z) = \theta_{1,a}(\bar{\lambda}z), \qquad z \in D,$$

implies that $\lim_{z\to\lambda}\theta_{\lambda,a}(z) = 0$ uniformly in λ as $z \to \lambda$ nontangentially. It follows from the Sz.-Nagy–Foiaş model theory (cf. [3]) that the operator $T = \bigoplus_{n=1}^\infty S(\theta_{\lambda_n,1})$, where the $S(\theta_{\lambda_n,1})$ are as defined in [5] and $\{\lambda_n\}$ is a dense sequence in ∂D, satisfies the hypotheses of Theorem 3 and also satisfies $\sigma(T) \cap D = \varnothing$. A more careful analysis shows that the operator $T' = \bigoplus_{n=1}^\infty S(\theta_{\lambda_n,a_n})$ also satisfies the hypotheses of

Theorem 3 for some sequences $\{a_n\}$ with $\lim_{n\to\infty} a_n = 0$; thus such operators T' are reflexive. Interestingly enough, if the sequence $\{a_n\}$ satisfies the stronger condition $\sum_{n=1}^{\infty} a_n < \infty$, the operator T' is an operator of class C_0 in the sense of [18] and T' is not reflexive (cf. [5]).

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