IDEALS OF INJECTIVE DIMENSION 1

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Introduction. Throughout this paper R is an integral domain with quotient field $Q \neq R$, and $K = Q/R$. The completion of R in the R-topology is denoted by H. Let I be a non-zero ideal of R, $S = \{1 - a \mid a \in I\}$, and $E(R/I)$ the injective envelope of R/I. If $\mathfrak{J}(R)$ denotes the Jacobson radical of R, then $I \subset \mathfrak{J}(R)$ if and only if $R = R_S$. It has been proved elsewhere that $H/IH \simeq R/I$.

The main purpose of this paper is to examine the relationship between the injective envelope of R/I and the torsion-free cover of R/I in order to shed light on the condition: $\text{inj.dim}_R I = 1$. This suggests consideration of the successively weaker conditions: (a) $E(R/I) = Q/I$, (b) $\text{inj.dim}_R I = 1$ (i.e. Q/I is injective); and (c) $E(R/I) \subset Q/I$.

Condition (a) naturally leads to the study of the condition: (d) $Q/I \subset E(R/I)$; (i.e., Q/I is an essential extension of R/I) and the characterization of this condition is the key to the whole question. It is proved that $Q/I \subset E(R/I)$ if and only if $I \subset \mathfrak{J}(R)$ and the only ideals of R mapping onto R/I are the principal ideals of R. Another important tool in the investigation is the notion of a complemented extension A of R. Of great importance here is the proposition that if A is a complemented extension of R, and if I is the contraction of an ideal of A contained in $\mathfrak{J}(A)$, then $A = R_S$.

The main results of this paper are summarized in the following theorem.

MAIN THEOREM. (I) *The following statements are equivalent:*

(1) $E(R/I) = Q/I$.

(2) $\text{Inj.dim}_R I = 1$ *and* $I \subset \mathfrak{J}(R)$.

(3) *The canonical map:* $H \to R/I$ *is a torsion-free cover.*

(II) *The following statements are equivalent:*

(1) $\text{Inj.dim}_R I = 1$ (*i.e.* Q/I *is injective*).

(2) R_S *is a complemented extension of* R; $\text{inj.dim}_{R_S} I_S = 1$; *and* $\text{inj.dim}_{R'_S} R'_S \leqslant 1$, *where* $R'_S = \bigcap R_N\{N \in \text{max spec } R \mid I \not\subset N\}$ *is the complement of* R_S.

(3) *The canonical map:* $H \to R/rI$ *is a torsion-free lifting for all non-zero* $r \in R$.

(III) *The following statements are equivalent:*

(1) $E(R/I) \subset Q/I$.

(2) R_S *is a complemented extension of* R *and* $\text{inj.dim}_{R_S} I_S = 1$.

(3) *The canonical map* $H \to R/I$ *is a torsion-free lifting.*

In Section 1 complemented extensions of R are discussed. In Section 2 conditions (a), (b), (c), and (d) are related to the notion of complemented extensions of R. In Section 3 torsion-free covers and liftings are discussed and are related to conditions (a), (b) and (c). Finally, in Section 4 the results of the first three sections are applied to valuation rings, Noetherian domains, and h-local domains. There are examples given illustrating the first three sections, and counter-examples to possible conjectures.

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1. Complemented extensions of R.

DEFINITION. Let A be an R submodule of Q such that $R \subset A$. We shall say that A is a *complemented extension of* R if there exists an R-submodule A' of Q such that $A + A' = Q$ and $A \cap A' = R$. A' is called a *complement* of A. We shall prove that if A' exists, then it is unique. We shall call R and Q the trivial complemented extensions of R. It is immediate that A is a complemented extension of R if and only if A/R is a direct summand of K. Clearly we have $K = A/R \oplus A'/R$, $A/R \simeq Q/A'$, $A'/R \simeq Q/A$, and $K \simeq Q/A \oplus Q/A'$. In this section we shall attempt to characterize the complemented extensions of R.

PROPOSITION 1.1. *Let A be a complemented extension of R. Then the following statements are true:*

(1) *A is a flat commutative ring extension of R.*

(2) *If I is a non-zero ideal of R, then $IA \cap IA' = I$, $R/I \simeq A/IA \oplus A'/IA'$, and $Q/I \simeq Q/IA \oplus Q/IA'$.*

(3) *If J is a non-zero ideal of A, then $A = J + R$, $J = (J \cap R)A$, and $A/J \simeq R/(J \cap R)$. Therefore, J is a prime (or maximal) ideal of A if and only if $J \cap R$ is a prime (or maximal) ideal of R.*

Proof. (1) Let x be a non-zero element of A, and write $x = a/b$, where $a, b \in R$. Since A/R is a direct summand of K and $K = bK$, we have $A/R = b(A/R) = (bA + R)/R$. Thus $A = bA + R$ and so $xA = aA + xR \subset A$. This shows that A is closed under multiplication, and thus A is a subring of Q.

Since $A \cap A' = R$ and $A + A' = Q$, we have an exact sequence:

$$0 \to R \to A \oplus A' \to Q \to 0.$$

Because R and Q are flat R-modules, it follows that A is also a flat R-module.

(2) Let I be a non-zero ideal of R. The preceding exact sequence induces a map: $R/I \to A/IA \oplus A'/IA'$ whose kernel is equal to $(IA \cap IA')/I$. Because Q is flat and $Q = IQ$, this map is an isomorphism. Therefore, $IA \cap IA' = I$. Now $Q = IQ = IA + IA'$, and thus $Q/I \simeq Q/IA \oplus Q/IA'$.

(3) Let J be a non-zero ideal of A and $I = J \cap R$. Since $IK = K$, we have $A = IA + R$ and consequently $A = J + R$. If $x \in J$, then $x = y + r$, where $y \in IA$ and $r \in R$. Therefore, $r \in J \cap R = I$, and so $J = IA$. We also have $A/J = (J + R)/J \simeq R/(J \cap R)$, and this isomorphism is a ring isomorphism. Hence J is a prime (or maximal) ideal of A if and only if $J \cap R$ is a prime (or maximal) ideal of R. □

DEFINITION. An ideal of R is said to be *irreducible* if it is not the intersection of two properly larger ideals of R.

PROPOSITION 1.2. *If the Jacobson radical of R contains an irreducible non-zero ideal I of R, then R has only the trivial complemented extensions (i.e., K is indecomposable). In particular, this is the case if R is a quasi-local domain (i.e. has only a single maximal ideal).*

Proof. Let A be a complemented extension of R. Since I is irreducible, R/I is an indecomposable R-module. Hence by Proposition 1.1(2) either $A = IA$ or $A' = IA'$.

But by Proposition 1.1(3), IA is contained in every non-zero maximal ideal of A and IA' is contained in every non-zero maximal ideal of A'. Therefore either $A = Q$ or $A' = Q$. □

DEFINITION. Let $\operatorname{spec} R$ denote the set of prime ideals of R, and $\operatorname{max spec} R$ denote the set of maximal ideals of R. If B is a non-zero R-module, define $\Gamma(B) = \{P \in \operatorname{spec} R \mid B/PB \neq 0\}$ and $\Omega(B) = \{M \in \operatorname{max spec} R \mid B/MB \neq 0\}$.

PROPOSITION 1.3. *Let A be a complemented extension of R and let A' be a complement of A. Then the following statements are true:*

(1) $\Gamma(A) \cup \Gamma(A') = \operatorname{spec} R$ *and* $\Gamma(A) \cap \Gamma(A') = \{0\}$.

(2) *If* $0 \neq P \in \Gamma(A)$ (*or* $\Omega(A)$), *then* $PA \in \operatorname{spec} A$ (*or* $\operatorname{max spec} A$); $PA \cap R = P$, $A/PA \simeq R/P$, $A_P = R_P$, *and* $A'_P = Q$.

(3) *There is a* 1-1 *order preserving correspondence between* $\Gamma(A)$ *and* $\operatorname{spec} A$ *given by* $P \to PA$ *for* $P \in \Gamma(A)$ *and* $\mathcal{P} \to \mathcal{P} \cap R$ *for* $\mathcal{P} \in \operatorname{spec} A$.

(4) $A = \cap R_M,\ M \in \Omega(A)$.

(5) $A' = \cap R_N\ \{N \in \operatorname{max spec} R \mid A = NA\}$. *Thus A' is unique.*

Proof. (1) Let P be a non-zero prime ideal of R. By Proposition 1.1(2) we have a ring isomorphism: $R/P \to A/PA \oplus A'/PA'$. Because R/P is an integral domain, it follows that either $P \in \Gamma(A)$ or $P \in \Gamma(A')$ but not both simultaneously.

(2) Let $0 \neq P \in \Gamma(A)$ (or $\Omega(A)$). By (1) we have $R/P \simeq A/PA$, and hence PA is a prime (or maximal) ideal of A. By Proposition 1.1(3) we have $A/PA \simeq R/(PA \cap R)$, and thus $PA \cap R = P$. It is clear that A_P is a complemented extension of R_P and that A'_P is an R_P-complement of A_P. Since R_P has only the trivial complemented extensions by Proposition 1.2, we have $A_P = R_P$ and $A'_P = Q$.

(3) This follows from (2) and Proposition 1.1(3).

(4) A is the intersection of all of its localizations with respect to the maximal ideals of R. Hence (4) follows from (1) and (2).

(5) This follows from (4) and (1). □

REMARKS. Let A be a complemented extension of R. It follows from Proposition 1.3(3) that $\Omega(A)$ is empty if and only if $A = Q$. By convention if $\Omega(A)$ is empty, then the empty intersection $\cap R_M$, $\{M \in \Omega(A)\}$ is equal to Q, and hence $A = \cap R_M$ in this case also.

On the other hand if $M \in \Omega(A)$ and $N \in \Omega(A')$, then $M \cap N$ contains no non-zero prime ideal of R by Proposition 1.3(1), and hence $R_M \otimes_R R_N = Q$.

It can easily be shown that $\operatorname{Hom}_R(A/R, A'/R) = 0$, and this provides a different proof from the one given that A' is unique. Finally, it is clear from Proposition 1.3 that $\Gamma(A) - \{0\} = \operatorname{Supp}(A'/R)$.

DEFINITION. If B is a commutative ring, we shall let $\mathcal{J}(B)$ denote the *Jacobson radical of* B; i.e., $\mathcal{J}(B)$ is the intersection of all of the maximal ideals of B.

PROPOSITION 1.4. *Let I be a non-zero ideal of R and $S = \{1 - a \mid a \in I\}$. Then the following statements are true:*

(1) $I_S \subset \mathcal{J}(R_S)$ *and* $I_S \cap R = I$.

(2) $I_S + R = R_S$ *and* $R_S/I_S \simeq R/I$.

(3) $\Omega(R_S) = \{M \in \max\operatorname{spec} R \mid M \supset I\}$ *and* $R_S = \cap R_M$, $\{M \in \Omega(R_S)\}$.

(4) *If* R_S *is a complemented extension of* R, *then*

$$R_S' = \cap R_N, \qquad \{N \in \max\operatorname{spec} R \mid N \not\supset I\}.$$

Proof. (1) Let $b/s \in I_S$, where $b \in I$ and $s \in S$. Then $s = 1 - a$ where $a \in I$; and so $1 - b/s = (1/s)(1-(a+b)$ is a unit in R_S. It follows that $I_S \subset \mathfrak{J}(R_S)$. If $b/s = r \in R$, then $r = b + ar \in I$, and so $I_S \cap R = I$.

(2) Let $x = r/t$, where $r \in R$ and $t \in S$. Then $t = 1 - c$ where $c \in I$, and $x = cx + r \in I_S + R$. Thus $I_S + R = R_S$, and so $R_S/I_S \simeq R/(I_S \cap R) = R/I$.

(3) Let $M \in \max\operatorname{spec} R$; it is clear that $MR_S \neq R_S$ if and only if M does not meet S if and only if $M \supset I$; and thus $\Omega(R_S) = \{M \in \max\operatorname{spec} R \mid M \supset I\}$. Furthermore, if $M \supset I$, then $S \subset R - M$, and so $(R_S)_M = R_M$. Thus if $B = \cap R_M$, $M \in \Omega(R_S)$, then $R_S \subset B$. On the other hand let $x \in B$ and let $J = \{t \in R \mid tx \in R\}$. If $M \in \max\operatorname{spec} R$ and $M \supset I + J$, then $M \in \Omega(R_S)$ and so $B \subset R_M$. But this contradicts $J \subset M$, and so $I + J = R$. Hence J meets S, and so there exists $s \in S$ such that $sx = r \in R$. Therefore $x \in R_S$, and so $R_S = B$.

(4) If R_S is a complemented extension of R, then by Proposition 1.3(5) we have $R_S' = \cap R_N$, $\{N \in \max\operatorname{spec} R \mid NR_S = R_S\}$. But $NR_S = R_S$ if and only if N meets S if and only if $N \not\supset I$, and so we have the desired expression for R_S'. □

PROPOSITION 1.5. *Let* A *be a complemented extension of* R *and let* I *be a non-zero ideal of* R *such that* $IA \cap R = I$. *Then* $IA' = A'$, $Q/I \simeq Q/IA \oplus Q/A'$, *and* $A'/I \simeq Q/IA$.

Proof. We have $Q = IQ = IA + IA'$; hence, a fortiori, $Q = IA + A'$. We also have $A' \cap IA = (A' \cap A) \cap IA = R \cap IA = I$.

Let $x \in A'$; then $x = y + z$, where $y \in IA$ and $z \in IA'$. Hence $y \in A' \cap IA = I$, and so $x \in IA'$. Therefore, $A' = IA'$. Hence by Proposition 1.1(2) we have $Q/I \simeq Q/IA \oplus Q/IA' = Q/IA \oplus Q/A'$. We also have

$$Q/IA = (IA + A')/IA \simeq A'/(IA \cap A') = A'/I. \qquad \square$$

PROPOSITION 1.6. *Let* A *be a complemented extension of* R, I *a non-zero ideal of* R, *and* $S = \{1 - a \mid a \in I\}$. *Then* $A = R_S$ *if and only if* $IA \cap R = I$ *and* $IA \subset \mathfrak{J}(A)$, *the Jacobson radical of* A.

Proof. We have $IR_S \cap R = I$ *and* $IR_S \subset \mathfrak{J}(R_S)$ by Proposition 1.4(1). On the other hand suppose that $IA \cap R = I$ and $IA \subset \mathfrak{J}(A)$. By Proposition 1.4(3) $R_S = \cap R_M$, $M \in \Omega(R_S)$; and by Proposition 1.3(4) $A = \cap R_M$, $M \in \Omega(A)$. Thus it is sufficient to show that $\Omega(R_S) = \Omega(A)$.

If $M \in \Omega(R_S)$, then $M \supset I$ by Proposition 1.4(3). Since $IA' = A'$ by Proposition 1.5, we have $MA' = A'$. Therefore, $M \in \Omega(A)$ by Proposition 1.3(1). Conversely, suppose that $M \in \Omega(A)$. Then MA is a maximal ideal of A by Proposition 1.3(2). Therefore, $IA \subset \mathfrak{J}(A) \subset MA$; and so $I = IA \cap R \subset MA \cap R = M$. Thus $M \in \Omega(R_S)$ by Proposition 1.4(3). □

COROLLARY 1.7. *Let* A *be a complemented extension of* R *such that* $\mathfrak{J}(A) \neq 0$. *Let* J *be a non-zero ideal of* A *such that* $J \subset \mathfrak{J}(A)$, *and let* $I = J \cap R$ *and* $S = \{1 - a \mid a \in I\}$. *Then* $A = R_S$.

Proof. This is an immediate consequence of Propositions 1.1(3) and 1.6. □

COROLLARY 1.8. *Suppose that* $\mathfrak{J}(R) \neq 0$ *and that* $A \neq Q$ *is a complemented extension of* R. *Let* $I = \mathfrak{J}(A) \cap R$ *and* $S = \{1 - a \mid a \in I\}$. *Then* $A = R_S$.

Proof. Since A has non-zero maximal ideals, it follows from Proposition 1.1(3) that $\mathfrak{J}(R) \subset \mathfrak{J}(A)$; and thus $\mathfrak{J}(A) \neq 0$. The conclusion now follows from Corollary 1.7. □

DEFINITION. The R-topology of R is obtained by taking all of the non-zero ideals of R as a base of open neighborhoods of 0 in R. The *completion* of R in the R-topology is denoted by $H = H(R)$. The next lemma lists the properties of H that we shall need. Its component parts have been proved elsewhere.

LEMMA 1.9. (1) *H is a faithfully flat commutative ring extension of R.*

(2) *There is a canonical ring isomorphism* $H \simeq \mathrm{Hom}_R(K,K)$.

(3) $\mathrm{Hom}_R(H,H) = \mathrm{Hom}_H(H,H)$.

(4) *If I is a non-zero ideal of R, then* $H/IH \simeq R/I$.

(5) *If B is a subring of Q such that* $R \subset B$, *then*

$$\mathrm{Hom}_R(Q/B, Q/B) = \mathrm{Hom}_B(Q/B, Q/B) \simeq H(B),$$

the completion of B in the B-topology.

Proof. See [7, Chapter 2] and [9, Proof of Theorem 2.9]. □

PROPOSITION 1.10. (1) *Let A be a non-trivial complemented extension of R and let e be the element of* $H = \mathrm{Hom}_R(K,K)$ *that is the identity on A'/R and 0 on A/R. Then* $H(A) \simeq He$, $H(A') \simeq H(1-e)$, *and we have a ring direct sum decomposition* $H \simeq H(A) \oplus H(A')$. *If I is a non-zero ideal of R, then* $He/IHe \simeq H(A)/IH(A) \simeq A/IA \simeq R/(IA \cap R)$.

(2) *Conversely, if* $H = U \oplus U'$ *where U and U' are proper R-submodules of H, then there exists* $e \in H$ *such that* $e^2 = e$, $U = He$, *and* $U' = H(1-e)$. *If* $A/R = (1-e)(K)$, *then A is a non-trivial complemented extension of R,* $A'/R = e(K)$, $H(A) \simeq U$ *and* $H(A') \simeq U'$.

Proof. (1) It is easily seen that He is a ring isomorphic to $\mathrm{Hom}_R(A'/R, A'/R)$. Now $A'/R \simeq (A'+A)/A = Q/A$, and hence $\mathrm{Hom}_R(A'/R, A'/R)$ is ring isomorphic to $\mathrm{Hom}_R(Q/A, Q/A)$. By Lemma 1.9(5), $\mathrm{Hom}_R(Q/A, Q/A)$ is ring isomorphic to $H(A)$. Thus He is ring (and R-module) isomorphic to $H(A)$.

It follows that we have an induced isomorphism: $He/IHe \simeq H(A)/IH(A)$. Since $A \subset H(A)$, we have $IH(A) = (IA)H(A)$; and by Lemma 1.9(4) $H(A)/(IA)H(A) \simeq A/IA$. By Proposition 1.1(4), $A/IA \simeq R/(IA \cap R)$.

(2) Since U is a direct summand of H, there exists an R-homomorphism $\lambda: H \to H$ such that $\mathrm{Ker}\,\lambda = U$. By Lemma 1.9(3), λ is an H-homomorphism, and thus U is an ideal of H. Similarly U' is an ideal of H. Thus there exists $e \in H$, $e \neq 0, 1$ such that $e = e^2$, $U = He$ and $U' = H(1-e)$. Let $A/R = (1-e)(K)$ and $B/R = e(K)$. Then A/R and B/R are proper R-submodules of K and $K = A/R \oplus B/R$. Thus A is a non-trivial complemented extension of R; and by the uniqueness of A', we have $A' = B$. By (1), $H(A) \simeq U$ and $H(A') \simeq U'$. □

PROPOSITION 1.11. *Let A be a complemented extension of R and let C be an A-module. Then* $\text{inj.dim}_A C = \text{inj.dim}_R C$; *and thus* $\text{gl.dim} A \leqslant \text{gl.dim} R$.

Proof. It is sufficient to prove that C is A-injective if and only if C is R-injective. Since A is flat over R by Proposition 1.1(1), if C is A-injective then C is R-injective. Conversely, assume that C is R-injective. Let T be the R-torsion submodule of C. Then $C \simeq T \oplus C/T$. Now T is also the A-torsion submodule of C; and C/T is torsion-free and divisible, hence injective, over A. Thus we may assume that $C = T$ is a torsion A-module.

Let L be a non-zero ideal of A and $g: L \to C$ an A-homomorphism. Then g extends to an R-homomorphism $f: A \to C$. It is sufficient to prove that f is an A-homomorphism.

Because L is a torsion-free A-module and C is a torsion A-module, $\text{Ker}\, g$ is a non-zero ideal of A. Let $x \in A$ and $J = \text{Ker}\, g \cap \text{Ann}_A(f(x))$; then J is a non-zero ideal of A. Thus by Proposition 1.1(3) we have $A = J + R$. Thus if $a \in A$, then $a = b + r$, where $b \in J$ and $r \in R$. Hence $af(x) = bf(x) + rf(x) = f(rx)$ because $b \in \text{Ann}_A(f(x))$. On the other hand $f(ax) = f(bx) + f(rx) = f(rx)$ because $bx \in \text{Ker}\, g$ and $f(bx) = g(bx) = 0$. Therefore, f is an A-homomorphism. □

2. Ideals of injective dimension 1.

DEFINITION. If B is an R-module, then $E(B)$ denotes the injective envelope of B; and, as before, $\mathfrak{J}(R)$ denotes the Jacobson radical of R.

PROPOSITION 2.1. *Let I be an ideal of R. Then the following statements are equivalent:*

(1) *Q/I is an essential extension of R/I (i.e., $Q/I \subset E(R/I)$).*

(2) *If B is an R-submodule of Q such that $(R \cap B) \subset I$, then $B \subset I$.*

(3) *If A is an R-submodule of Q that maps onto R/I, then $A \simeq R$.*

(4) *$I \subset \mathfrak{J}(R)$; and if J is an ideal of R generated by two elements, then J maps onto R/I if and only if J is, in fact, a principal ideal of R.*

Proof. (1) implies (2). Suppose that B is an R-submodule of Q such that $(R \cap B) \subset I$. Then $(B + I) \cap R = I$, and hence $B + I = I$ since Q/I is an essential extension of R/I.

(2) implies (3). Let A be an R-submodule of Q and $f: A \to R/I$ a surjection. Then there exists $y \in A$ such that $A = Ry + \text{Ker} f$. Now $A/\text{Ker} f \simeq Ry/(Ry \cap \text{Ker} f) \simeq R/(R \cap y^{-1}\text{Ker} f)$; and since $A/\text{Ker} f$ maps onto R/I, we have $(R \cap y^{-1}\text{Ker} f) \subset I$. Hence by (2), $y^{-1}\text{Ker} f \subset I$, and so $\text{Ker} f \subset Iy$. Therefore $A = Ry \simeq R$.

(3) implies (4). Let $S = \{1 - a \mid a \in I\}$. By Proposition 1.4(2) we have $R_S/I_S \simeq R/I$, and so R_S maps onto R/I. Thus by (3), $R_S \simeq R$. Hence R is divisible by the elements of S, and therefore $R = R_S$. Thus $I \subset \mathfrak{J}(R)$.

Before we proceed with the proof of (4) implies (1) we shall prove the following technical lemma.

LEMMA. *Let I be an ideal of R; $a, b \in R$, $b \neq 0$, and let $x = (a/b + I) \in Q/I$. Then $Rx \cap (R/I) = 0$ if and only if $(Rb{:}a) = (Ib{:}a)$ if and only if $(Ra + Rb)/(Ra + Ib)$ maps onto R/I if and only if $(Ra + Rb)/(Ra + Ib) \simeq R/I$.*

Proof. Since $\mathrm{Ann}_R\, x = (Ib{:}a)$ and $\{r \in R \mid rx \in R/I\} = (Rb{:}a)$, we see that $Rx \cap (R/I) = 0$ if and only if $(Rb{:}a) = (Ib{:}a)$. Let $L = (Rb{:}a)ab^{-1}$; then L is an ideal of R, and it is immediate that $(Rb{:}a) = (Ib{:}a)$ if and only if $L \subset I$.

Now $(Ra + Ib) \cap Rb = (Rb{:}a)a + Ib = Lb + Ib = (L + I)b$. Hence

$$(Ra + Rb)/(Ra + Ib) \simeq Rb/[(Ra + Ib) \cap Rb] = Rb/(L + I)b \simeq R/(L + I).$$

Hence $L \subset I$ if and only if $(Ra + Rb)/(Ra + Ib)$ maps onto R/I if and only if $(Ra + Rb)/(Ra + Ib) \simeq R/I$. □

We now turn to the proof of (4) implies (1); hence assume (4). Let $x = (a/b + I)$ be an element of Q/I, where $a, b \in R$; and suppose that $Rx \cap (R/I) = 0$. By the Lemma $(Ra + Rb)/(Ra + Ib) \simeq R/I$; and hence by assumption there exists $c \in R$ such that $Ra + Rb = Rc$. Thus there exist $r, t, u, v \in R$ such that $a = rc$, $b = tc$, and $c = ua + vb$. Hence $1 = ur + vt$. Now $(Rb{:}a) = (Rtc{:}rc) = (Rt{:}r)$; and $(Ib{:}a) = (Itc{:}rc) = (It{:}r)$. Therefore, by the Lemma, $(Rt{:}r) = (It{:}r)$. Hence $t \in (It{:}r)$ and so $r \in I$. But $I \subset \mathfrak{J}(R)$, and so $vt = 1 - ur$ is a unit in R. Thus t is a unit in R and $Rb = Rtc = Rc$. Therefore, $a \in Rb$, $x \in R/I$, and so $x = 0$. This shows that Q/I is an essential extension of R/I. □

REMARKS. It follows easily from Proposition 2.1 that if Q/I is an essential extension of R/I, and $J \subset I$, then Q/J is an essential extension of R/J.

COROLLARY 2.2. *Let I be a non-zero ideal of R. Then the following statements are equivalent:*

(1) $E(R/I) = Q/I$.

(2) $\mathrm{Inj.dim}_R\, I = 1$ *and* $I \subset \mathfrak{J}(R)$.

Proof. (1) implies (2). We have $I \subset \mathfrak{J}(R)$ by Proposition 2.1; and from the exact sequence: $0 \to I \to Q \to Q/I \to 0$, we have $\mathrm{inj.dim}_R\, I = 1$.

(2) implies (1). The preceding exact sequence shows that Q/I is injective. It only remains to show that Q/I is an essential extension of R/I. Since $I \subset \mathfrak{J}(R)$, it is sufficient by Proposition 2.1 to show that if J is an ideal of R and $f: J \to R/I$ is a surjection, then J is a principal ideal of R.

We have an exact sequence:

$$0 \to I \to R \xrightarrow{\pi} R/I \to 0.$$

Now $\mathrm{Ext}^1_R(J, I) \simeq \mathrm{Ext}^2_R(R/J, I)$, and the latter module is 0 because $\mathrm{inj.dim}_R\, I = 1$. Thus we have an exact sequence:

$$0 \to \mathrm{Hom}_R(J, I) \to \mathrm{Hom}_R(J, R) \xrightarrow{\pi^*} \mathrm{Hom}_R(J, R/I) \to 0.$$

Thus there exists an R-homomorphism $g: J \to R$ such that $\pi g = f$. Since f is onto, $\pi(\mathrm{Im}\, g) = R/I$. Because $\mathrm{Ker}\, \pi = I$, we have $R = \mathrm{Im}\, g + I$. But $I \subset \mathfrak{J}(R)$, and so $R = \mathrm{Im}\, g$. Thus g maps J onto R. But g is multiplication by an element of Q, and thus g is an injection. Therefore g is an isomorphism, and so $J \simeq R$. Hence by Proposition 2.1, $E(R/I) = Q/I$. □

PROPOSITION 2.3. *Let R be a non-zero ideal of R. Then the following statements are equivalent:*

(1) $E(R/I) \subset Q/I$.

(2) *There exists a complemented extension A of R such that $IA \cap R = I$ and* $\text{inj.dim}_A IA = 1$ $(= \text{inj.dim}_R IA)$.

(3) *Let $S = \{1 - a \mid a \in I\}$. Then R_S is a complemented extension of R and* $\text{inj.dim}_{R_S} I_S = 1$ $(= \text{inj.dim}_R I_S)$.

In this case $Q/I_S \simeq E(R/I) = (\cap R_N)/I$, $\{N \in \text{max spec } R \mid I \not\subset N\}$. Hence $E(R/I)$ is a unique submodule of Q/I.

Proof. (1) implies (2). Let $E(R/I) = B/I$ where B is an R-submodule of Q such that $R \subset B \subset Q$. Since B/I is a direct summand of Q/I, there exists an R-submodule C of Q such that $B + C = Q$ and $B \cap C = I$. Let $A = R + C$; then $A + B = Q$. Clearly $R \subset A \cap B$; on the other hand let $x \in A \cap B$. Then $x = r + c$ where $r \in R$ and $c \in C$. Hence $c = x - r \in B \cap C = I$. Therefore $x \in R$, and so $A \cap B = R$. Therefore A is a complemented extension of R and $B = A'$.

Now $IA \cap R = I(R + C) \cap R = (I + IC) \cap R \subset C \cap R \subset I$; and thus $IA \cap R = I$. By Proposition 1.5 we have $Q/IA \simeq A'/I = B/I = E(R/I)$. Therefore Q/IA is an injective R-module. Hence by Proposition 1.11, Q/IA is an injective A-module. Therefore, $\text{inj.dim}_A IA = 1 = \text{inj.dim}_R IA$.

(1) implies (3). We shall use the notation and results of (1) implies (2). Since $A'/I \simeq E(R/I)$, A'/I is an essential extension of R/I. In the canonical isomorphism: $A'/I \to Q/IA$ of Proposition 1.5, R/I maps onto A/IA. Thus Q/IA is an essential extension of A/IA, and hence is the injective envelope over A of A/IA. Therefore, by Proposition 2.1, $IA \subset \mathfrak{J}(A)$. It now follows from Proposition 1.6 that $A = R_S$ where $S = \{1 - a \mid a \in I\}$. By Proposition 1.4,

$$A' = R'_S = \cap R_N, \qquad \{N \in \text{max spec } R \mid I \not\subset N\}.$$

(3) implies (2). Since $I_S \cap R = I$, the assertion is trivial.

(2) implies (1). Since Q/IA is A-injective, it is R-injective by Proposition 1.11. And since $Q/IA \simeq A'/I \subset Q/I$, we have $E(R/I) \subset Q/I$. □

PROPOSITION 2.4. *Let I be a non-zero ideal of R. Then the following statements are equivalent:*

(1) $\text{Inj.dim}_R I = 1$.

(2) *There exists a complemented extension A of R such that $IA \cap R = I$,* $\text{inj.dim}_A IA = 1$, *and* $\text{inj.dim}_{A'} A' \leqslant 1$.

(3) *Let $S = \{1 - a \mid a \in I\}$. Then R_S is a complemented extension of R,*

$$\text{inj.dim}_{R_S} I_S = 1, \quad \textit{and} \quad \text{inj.dim}_{R'_S} R'_S \leqslant 1$$

(*where* $R'_S = \cap R_N$, $\{N \in \text{max spec } R \mid I \not\subset N\}$).

Proof. (1) implies (3). Since Q/I is an injective R-module, we have $E(R/I) \subset Q/I$. Thus by Proposition 2.3 we only have to prove that $\text{inj.dim}_{R'_S} R'_S = 1$. But by Proposition 1.5, Q/R'_S is a direct summand of Q/I and hence is R-injective. Therefore, by Proposition 1.11, Q/R'_S is R'_S-injective.

(3) implies (2). This assertion is trivial.

(2) implies (1). By Proposition 1.11, $\text{inj.dim}_R IA = 1$ and $\text{inj.dim}_R A' \leqslant 1$. Since $Q/I \simeq Q/IA \oplus Q/A'$ by Proposition 1.5, we see that Q/I is also R-injective. □

COROLLARY 2.5. *Let I be a non-zero ideal of R such that $I \subset \mathfrak{J}(R)$. If $E(R/I) \subset Q/I$, then $E(R/I) = Q/I$ and so* $\text{inj.dim}_R I = 1$.

Proof. Let $S = \{1 - a \mid a \in I\}$. Then every element of S is a unit in R and thus $R = R_S$. Corollary 2.5 is now an immediate consequence of Proposition 2.3 and Corollary 2.2. □

3. Torsion-free liftings and covers.

DEFINITIONS. An R-module C is said to be a *cotorsion* R-module if

$$\text{Hom}_R(Q, C) = 0 = \text{Ext}^1_R(Q, C).$$

It is immediate that if C is a cotorsion R-module and V is a torsion-free and divisible R-module, then $\text{Hom}_R(V, C) = 0 = \text{Ext}^1_R(V, C)$.

We place the R-topology on an R-module B by taking the submodules $\{IB\}$, where I ranges over the non-zero ideals of R, as a base of open neighborhoods of 0 in B. Then B has a *completion* $\hat{B}$ in this topology and $\hat{B}$ is an H-module, where H is the completion of R in the R-topology.

The following lemma lists some of the properties of cotorsion and complete modules that we shall need.

LEMMA 3.1. (1) *A torsion-free R-module is complete in the R-topology if and only if it is a cotorsion R-module.*

(2) *If B is any R-module, then* $\text{Hom}_R(K, B)$ *is a torsion-free, cotorsion R-module, hence complete in the R-topology.*

(3) *If B is an R-module and* $\text{Ann}_R B \neq 0$, *then B is a cotorsion R-module.*

(4) *A cotorsion R-module C is an H-module and* $\text{Hom}_R(G, C) = \text{Hom}_H(G, C)$ *for any H-module G.*

Proof. See [7, Chapters 1 and 2]. □

DEFINITIONS. Let B be an R-module, D a torsion-free R-module, and $\theta: D \to B$ an R-homomorphism. We shall say that the pair (D, θ) is a *torsion-free lifting* of B if given any torsion-free R-module X and R-homomorphism $f: X \to B$, then there exists an R-homomorphism $\lambda: X \to D$ such that $\theta\lambda = f$. This is obviously equivalent to the assertion that the induced map $\theta_*: \text{Hom}_R(X, D) \to \text{Hom}_R(X, B)$ is surjective. Because there exists a free R-module mapping onto B, it is clear that a torsion-free lifting is surjective.

A torsion-free lifting (D, θ) of B is called a *torsion-free cover* of B if $\text{Ker}\,\theta$ contains no non-zero pure R-submodule of D. We remark that the definition of purity used is that C is *pure in D* if D/C is torsion-free. In [3] Enochs defined and proved the existence and uniqueness of the torsion-free cover for any R-module B. Subsequently, Banaschewski [1] gave a concrete construction of the torsion-free cover and an improved proof of its uniqueness. Because of the fundamental importance of Banaschewski's results to this paper, we shall state his theorem without proof.

BANASCHEWSKI'S THEOREM. *Let B be an R-module and $E=E(B)$ the injective envelope of B. Let $T=\{f\in \mathrm{Hom}_R(Q,E)\mid f(1)\in B\}$, and define $\phi: T\to B$ by $\phi(f)=f(1)$ for all $f\in T$. Then (T,ϕ) is a torsion-free cover of B.*

If (D,θ) is a torsion-free lifting of B, then $D=T_1\oplus D_1$, where $D_1\subset \mathrm{Ker}\,\theta$ and there exists an isomorphism $\lambda: T_1\to T$ such that if θ_1 is the restriction of θ to T_1, then (T_1,θ_1) is a torsion-free cover of B and $\theta_1=\phi\lambda$. Thus torsion-free covers are unique up to isomorphism.

Proof. See [1, Proposition 1 and Corollary]. □

We shall expand on Banaschewski's Theorem in the next proposition.

PROPOSITION 3.2. *Let B an R-module, let $E=E(B)$ be an injective envelope of B, and let (D,θ) be a torsion-free cover of B. Then:*

(1) $\mathrm{Ker}\,\theta\simeq \mathrm{Hom}_R(K,E)$ *is complete in the R-topology and* $\mathrm{inj.dim}_R(\mathrm{Ker}\,\theta)\leqslant 1$ *(and is 0 only if* $\mathrm{Ker}\,\theta=0$).

(2) *If B is a cotorsion R-module, then there exists an exact sequence:*

$$0\to \mathrm{Hom}_R(K,E)\to \mathrm{Hom}_R(K,E/B)\to B\to 0$$

that is a torsion-free cover of B. Therefore, $D\simeq \mathrm{Hom}_R(K,E/B)$ is complete in the R-topology.

Proof. (1) Let (T,ϕ) be the torsion-free cover of B given by Banaschewski's Theorem. Then without loss of generality we can assume that $D=T$ and $\theta=\phi$. Let $\pi: Q\to K$ be the canonical map. Then we have an exact sequence:

$$0\to \mathrm{Hom}_R(K,E)\xrightarrow{\pi^*}\mathrm{Hom}_R(Q,E)\xrightarrow{\alpha}E\to 0$$

where $\alpha(g)=g(1)$ for $g\in \mathrm{Hom}_R(Q,E)$. Then ϕ is the restriction of α to T and $\mathrm{Ker}\,\phi=\mathrm{Ker}\,\alpha=\mathrm{Im}\,\pi^*\simeq \mathrm{Hom}_R(K,E)$. Thus by Lemma 3.1(2), $\mathrm{Ker}\,\phi$ is complete in the R-topology. Since $\mathrm{Hom}_R(Q,E)$ is an injective R-module, the preceding exact sequence shows that $\mathrm{inj.dim}_R(\mathrm{Ker}\,\phi)\leqslant 1$ (and is 0 if and only if $\mathrm{Ker}\,\phi=0$).

(2) Now assume that B is a cotorsion R-module. We define

$$\lambda: T\to \mathrm{Hom}_R(K,E/B)$$

by $\lambda(f)(x+R)=f(x)+B$ for all $x\in Q$ and $f\in T$. Since $f(R)\subset B$, $\lambda(f)$ is a well-defined element of $\mathrm{Hom}_R(K,E/B)$, it is obvious that λ is an R-homomorphism. We shall prove that λ is an isomorphism.

If $f\in \mathrm{Ker}\,\lambda$, then $f(Q)\subset B$; but since $\mathrm{Hom}_R(Q,B)=0$, we have $f=0$. Thus λ is an injection. Now let $g\in \mathrm{Hom}_R(K,E/B)$. Then $g\pi\in \mathrm{Hom}_R(Q,E/B)$. Let $\beta: E\to E/B$ be the canonical map. Since $\mathrm{Hom}_R(Q,B)=0=\mathrm{Ext}^1_R(Q,B)$, β induces an isomorphism $\beta_*: \mathrm{Hom}_R(Q,E)\to \mathrm{Hom}_R(Q,E/B)$. Thus there exists

$$f\in \mathrm{Hom}_R(Q,E)\quad \text{such that}\quad g\pi=\beta f.$$

We have $\beta(f(1))=g(\pi(1))=g(0)=0$. Hence $f(1)\in \mathrm{Ker}\,\beta=B$, and so $f\in T$. We then have $\lambda(f)(x+R)=f(x)+B=\beta(f(x))=g(\pi(x))=g(x+R)$ for all $x\in Q$. Therefore $\lambda(f)=g$, and hence λ is an isomorphism of T onto $\mathrm{Hom}_R(K,E/B)$. Therefore, T is complete in the R-topology by Lemma 3.1(2).

Define $\gamma = \phi\lambda^{-1}$; then we have a commutative diagram with exact rows and vertical isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker}\phi & \longrightarrow & T & \xrightarrow{\phi} & B & \longrightarrow & 0 \\ & & \downarrow (\pi^*)^{-1} & & \downarrow \lambda & & \| & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(K,E) & \xrightarrow{\beta_*} & \operatorname{Hom}_R(K,E/B) & \xrightarrow{\gamma} & B & \longrightarrow & 0 \end{array}$$

Since the top row is a torsion-free cover, it follows readily that the bottom row is also. □

REMARKS. Let B be an R-module such that $\operatorname{Ann}_R B \neq 0$. Then B is a cotorsion R-module by Lemma 3.1(3), and hence Proposition 3.2 applies in this case. In particular, if B is a finitely generated torsion R-module, then Proposition 3.2 applies to B.

PROPOSITION 3.3. *Given an exact sequence of R-modules:*

$$0 \to C \to D \xrightarrow{\theta} B \to 0$$

where D is torsion-free, C is complete, and $\operatorname{inj.dim}_R C = 1$, *then* (D, θ) *is a torsion-free lifting of B and* $\operatorname{Ext}^1_R(X, C) = 0$ *for any torsion-free R-module X.*

Proof. Let X be a torsion-free R-module. Then it is sufficient to prove that $\operatorname{Ext}^1_R(X, C) = 0$. Now from the exact sequence: $0 \to X \to Q \otimes_R X \to K \otimes_R X \to 0$, we obtain the exact sequence:

$$\operatorname{Ext}^1_R(Q \otimes_R X, C) \to \operatorname{Ext}^1_R(X, C) \to \operatorname{Ext}^2_R(K \otimes_R X, C).$$

The first term of this sequence is 0 because C is a cotorsion R-module (Lemma 3.1(1)) and the last term is 0 because $\operatorname{inj.dim}_R C = 1$. Thus we have $\operatorname{Ext}^1_R(X, C) = 0$. □

We can now give a short direct proof of the theorem that appeared in [10, Theorem 1].

COROLLARY 3.4. *Let I be a non-zero ideal of R. Then* $R \to R/I$ *is a torsion-free cover if and only if R is complete and* $\operatorname{inj.dim}_R I = 1$.

Proof. We first observe that I is complete if and only if R is complete. We next observe that R has no proper non-zero pure submodules. The conclusion now follows from Propositions 3.2 and 3.3. □

REMARKS. Let R be a complete domain and I a non-zero ideal of R. Then $\operatorname{Hom}_R(Q, I) = 0 = \operatorname{Ext}^1_R(Q, I)$. It follows that $Q \simeq \operatorname{Hom}_R(Q, Q) \simeq \operatorname{Hom}_R(Q, Q/I)$, and thus Q/I is indecomposable. Therefore, $E(R/I) = Q/I$ if and only if $\operatorname{inj.dim}_R I = 1$ if and only if $E(R/I) \subset Q/I$. Thus, if $\operatorname{inj.dim}_R I = 1$, then $I \subset \mathfrak{J}(R)$ by Corollary 2.2.

LEMMA 3.5. *Let* $C \subset B \subset D$ *be torsion-free R-modules and suppose that the canonical map* $\pi : D \to D/C$ *is a torsion-free lifting. Then the canonical map* $\pi_1 : B \to B/C$ *is a torsion-free lifting.*

Proof. Let X be a torsion-free R-module and $f: X \to B/C$ an R-homomorphism. Then there exists an R-homomorphism $\lambda: X \to D$ such that $\pi\lambda = f$. It follows immediately that $\operatorname{Im}\lambda \subset B$, and hence $\pi_1: B \to B/C$ is a torsion-free lifting. □

REMARKS. We observe that it follows readily from the isomorphism $H \simeq \operatorname{Hom}_R(K, K)$ that H is a torsion-free R-module.

LEMMA 3.6. *Let I be a non-zero ideal of R and suppose that there exists a torsion-free lifting (or cover) $\phi: H \to R/I$. Then any surjection $\theta: H \to R/I$ is also a torsion-free lifting (or cover).*

Proof. By Lemma 1.9(4), $H/IH \simeq R/I$; and by Lemma 3.1(4) both ϕ and θ are H-homomorphisms. Let $J = \operatorname{Ker}\phi$; then J is an ideal of H. Since a torsion-free lifting is necessarily surjective, we have $H/J \simeq H/IH$, and so $J = IH$. Similarly $\operatorname{Ker}\theta = IH$. Because $\operatorname{Ker}\phi = \operatorname{Ker}\theta$, there exists an R-isomorphism $\nu: R/I \to R/I$ such that $\nu\phi = \theta$. The conclusion of the Lemma now follows readily from this. □

PROPOSITION 3.7. *Let I be a non-zero ideal of R. Then we have an exact sequence:*

$$(\star) \qquad 0 \to IH \to H \otimes_R Q \xrightarrow{\pi} Q/I \to 0.$$

Hence $\operatorname{inj.dim}_R I = \operatorname{inj.dim}_R IH$. *Furthermore, the following statements are equivalent:*

(1) $\operatorname{inj.dim}_R I = 1$.
(2) $\operatorname{Ext}^1_R(X, IH) = 0$ *for any torsion-free R-module X.*
(3) *The exact sequence $(\star)$ is a torsion-free lifting.*
(4) *The canonical exact sequence:*

$$0 \to (rI)H \to H \to R/rI \to 0$$

is a torsion-free lifting for any non-zero $r \in R$.

Proof. Since H is a flat R-module by Lemma 1.9(1), we have an exact sequence:

$$0 \to H \otimes_R I \to H \otimes_R Q \to H \otimes_R Q/I \to 0.$$

Now $H \otimes_R I \simeq IH$ by [7, Theorem 13], and $H \otimes_R Q/I \simeq Q/I$ by [7, Theorem 11]. Thus we obtain exact sequence $(\star)$. Since $H \otimes_R Q$ is a Q-module, and hence R-injective, it follows from $(\star)$ that $\operatorname{inj.dim}_R I = \operatorname{inj.dim}_R IH$.

(1) implies (2). We have $\operatorname{inj.dim}_R IH = 1$ by the preceding paragraph. Furthermore, IH is complete in the R-topology by [7, Theorem 13]. Therefore, by Proposition 3.3, $\operatorname{Ext}^1_R(X, IH) = 0$ for any torsion-free R-module X.

(2) if and only if (3). Let X be a torsion-free R-module. Because $H \otimes_R Q$ is R-injective, we derive from $(\star)$ an exact sequence:

$$\operatorname{Hom}_R(X, H \otimes_R Q) \xrightarrow{\pi_*} \operatorname{Hom}_R(X, Q/I) \to \operatorname{Ext}^1_R(X, IH) \to 0.$$

Thus $(\star)$ is a torsion-free lifting if and only if π_* is surjective if and only if $\operatorname{Ext}^1_R(X, IH) = 0$.

(2) implies (4). Let X be a torsion-free R-module and $0 \neq r \in R$. Since $(rI)H \simeq IH$, condition (2) guarantees that $\operatorname{Hom}_R(X, -)$ is exact on the exact sequence of (4).

(4) implies (1). Let J be a non-zero ideal of R and $0 \neq r \in J$. By Lemma 1.9(4) we have $H/(rI)H \simeq R/rI$; and thus by assumption the canonical map $\alpha: H \to H/(rI)H$ is a torsion-free lifting. Hence by Lemma 3.5 the canonical map $\beta: IH \to IH/(rI)H$ is a torsion-free lifting. Thus from the exact sequence:

$$0 \to IH \xrightarrow{r} IH \xrightarrow{\beta} IH/(rI)H \to 0$$

we derive the exact sequence: $0 \to \mathrm{Ext}^1_R(J, IH) \xrightarrow{r} \mathrm{Ext}^1_R(J, IH)$. But $\mathrm{Ext}^1_R(J, IH) \simeq \mathrm{Ext}^2_R(R/J, IH)$ is annihilated by $r \in J$. Therefore, $\mathrm{Ext}^2_R(R/J, IH) = 0$. Therefore, $\mathrm{inj.dim}_R\, IH = 1$. Hence by the first paragraph of the Proposition, $\mathrm{inj.dim}_R\, I = 1$. □

PROPOSITION 3.8. *Let I be a non-zero ideal of R. Then the following statements are equivalent:*

(1) $\mathrm{Inj.dim}_R\, I = 1$ *and* $I \subset \mathfrak{J}(R)$.

(2) $E(R/I) = Q/I$.

(3) *The exact sequence* $0 \to IH \to H \to R/I \to 0$ *is a torsion-free cover.*

Proof. (1) if and only if (2). This is Corollary 2.2.

(2) implies (3). Let $B = R/I$ and $E = Q/I = E(B)$; then $E/B = (Q/I)/(R/I) \simeq Q/R = K$. Hence $\mathrm{Hom}_R(K, E/B) \simeq \mathrm{Hom}_R(K, K) \simeq H$. Therefore, by Proposition 3.2(2), there exists a map: $H \to R/I$ that is a torsion-free cover. Hence by Lemma 3.6, the canonical map: $H \to R/I$ is a torsion-free cover.

(3) implies (2). Let $B = R/I$ and $E = E(B)$, and assume that $H \to R/I$ is a torsion-free cover. Then by Proposition 3.2(1), $\mathrm{inj.dim}_R\, IH = 1$; and hence by Proposition 3.7, Q/I is R-injective. Thus $E = A/I$ where A is an R-submodule of Q such that $R \subset A$; and $E/B = (A/I)/(R/I) \simeq A/R$. Since A/I is a direct summand of Q/I, there exists an R-submodule D of Q such that $A + D = Q$ and $A \cap D = I$. If we let $A' = D + R$, then $A + A' = Q$ and $A \cap A' = R$. Therefore A/R is a direct summand of K. Suppose that $A \neq Q$. Let $T = \mathrm{Hom}_R(K, E/B) \simeq \mathrm{Hom}_R(K, A/R)$. Then T is a proper direct summand of $\mathrm{Hom}_R(K, K) \simeq H$; and hence T is isomorphic to an ideal direct summand of H by Lemma 3.1. Thus $\mathrm{Ann}_H\, T \neq 0$. But $T \simeq H$ by Proposition 3.2(2); and this isomorphism is an H-isomorphism by Lemma 3.1. Therefore $\mathrm{Ann}_H\, T = 0$. This contradiction shows that $E = Q/I$. □

PROPOSITION 3.9. *Let I be a non-zero ideal of R and $S = \{1 - a \mid a \in I\}$. Then the following statements are equivalent:*

(1) $E(R/I) \subset Q/I$.

(2) *The exact sequence* $0 \to IH \to H \to R/I \to 0$ *is a torsion-free lifting over* R.

(3) *The canonical map:* $H(R_S) \to R_S/I_S$ *is a torsion-free cover over* R_S *and* $H(R_S)$ *is a direct summand of* H.

(4) *The map* $\theta: H(R_S) \to R/I$ *induced by the isomorphism* $R/I \simeq R_S/I_S$ *is a torsion-free cover over* R, *and* $H(R_S)$ *is a direct summand of* H.

Proof. (1) implies (3). By Proposition 2.3, $\mathrm{inj.dim}_{R_S}\, I_S = 1$ and R_S is a complemented extension of R. Hence by Proposition 3.8 $\phi: H(R_S) \to R_S/I_S$ is a torsion-free cover over R_S; and by Proposition 1.10, $H(R_S)$ is a direct summand of H.

(3) implies (4). Since $R/I \simeq R_S/I_S$ (by Proposition 1.4) we have an R_S-homomorphism $\theta: H(R_S) \to R/I$, which by (3) we can assume is a torsion-free cover over R_S.

Let X be a torsion-free R-module and $f: X \to R/I$ an R-isomorphism. Then f extends to an R_S-homomorphism $f_S: X_S \to R/I$, and hence there exists an R_S-homomorphism $\lambda: X_S \to H(R_S)$ such that $\theta\lambda = f_S$. If we let ν be the restriction of λ to X, then it is clear that we have $\theta\nu = f$. Thus θ is a torsion-free lifting over R.

Let C be an R-submodule of $\operatorname{Ker}\theta$ such that C is R-pure in $H(R_S)$. Then C_S is an R_S-submodule of $\operatorname{Ker}\theta$ and C_S is R_S-pure in $H(R_S)$. Since θ is a torsion-free cover over R_S, it follows that $C_S = 0$. Hence $C = 0$, showing that θ is a torsion-free cover over R.

(4) implies (2). By assumption there exists an R-submodule W of H such that $H = W \oplus H(R_S)$. We extend $\theta: H(R_S) \to R/I$ to all of H by defining it to be 0 on W. The extension obviously is a torsion-free lifting: $H \to R/I$. Hence by Lemma 3.6 the canonical map $H \to R/I$ is a torsion-free lifting over R.

(2) implies (1). Assume that the canonical map $\pi: H \to R/I$ is a torsion-free lifting over R. By Banaschewski's Theorem there is an R-module direct sum decomposition $H = U \oplus U'$, where $U' \subset \operatorname{Ker}\pi = IH$; and $\pi_1: U \to R/I$ is a torsion-free cover over R (where π_1 is the restriction of π to U). By Proposition 1.10 U and U' are ideals of H and there exists a complemented extension A of R such that $H(A) = U$ and $H(A') = U'$.

Now $IH = IU \oplus IU'$ and $U' \subset IH$. Hence we have $U' = IU'$. By Lemma 1.9(4) we have $U'/IU' \simeq A'/IA'$, and hence $IA' = A'$. Thus by Proposition 1.1(2) $I = IA \cap IA' = IA \cap A' = IA \cap R$.

We have $\operatorname{Ker}\pi_1 = U \cap IH = IU = IH(A)$. Hence

$$0 \to IH(A) \to H(A) \xrightarrow{\pi_1} R/I \to 0$$

is a torsion-free cover over R. Thus by Proposition 3.2(1) $\operatorname{inj.dim}_R(IH(A)) = 1$. But $IH(A) = (IA)H(A)$, and so $\operatorname{inj.dim}_A(IA)H(A) = 1$ by Proposition 1.11. Hence by Proposition 3.7 applied to A, $\operatorname{inj.dim}_A(IA) = 1$. Since $IA \cap R = I$, we conclude from Proposition 2.3 that $E(R/I) \subset Q/I$. □

4. Applications to valuation rings, Noetherian domains, and *h*-local domains. If R is a valuation ring, then the R-submodules of Q are linearly ordered, and hence Q/I is an essential extension of R/I for any ideal I of R. Therefore, this condition is close to being a valuation type of condition. This closeness is emphasized by the following proposition.

PROPOSITION 4.1. (1) *R has a maximal ideal M such that Q/M is an essential extension of R/M if and only if R is a valuation ring.*

(2) *R has a non-zero prime ideal P such that Q/P is an essential extension of R/P if and only if $P = PR_P$ and R_P is a valuation ring.*

(3) *If R is a Noetherian domain, then R has a non-zero ideal I such that Q/I is an essential extension of R/I if and only if R is a semi-local domain of Krull dimension* 1.

Proof. (1) Suppose that M is a maximal ideal of R and that Q/M is an essential extension of R/M. Then by Proposition 2.1, $M \subset \mathfrak{J}(R)$; and hence M is the only maximal ideal of R. Let J be any finitely generated ideal of R. By the Nakayama Lemma, $J/MJ \neq 0$; and so there exists a surjection of J onto R/M. Therefore, by Proposition 2.1, J is a principal ideal of R. It follows easily that R is a valuation ring.

(2) Let P be a non-zero prime ideal of R such that Q/P is an essential extension of R/P. Since $PR_P \cap R = P$, it follows from Proposition 2.1(2) that $PR_P = P$. Hence $Q/P = Q/PR_P$ is an essential extension of R_P/PR_P. Therefore, by (1), R_P is a valuation ring.

Conversely, suppose that P is a non-zero prime ideal of R such that $P = PR_P$ and that R_P is a valuation ring. Then Q/P is an essential extension of R_P/P over both R_P and R. Clearly R_P/P is an essential extension of R/P over R. Therefore, Q/P is an essential extension of R/P.

(3) Let R be a Noetherian domain and I a non-zero ideal of R such that Q/I is an essential extension of R/I. Let $P_1, \ldots, P_k$ be the prime ideals of R belonging to I. Then by [5, Theorem 2.3 and Proposition 3.1], $E(R/I) = E_1^{n_1} \oplus \cdots \oplus E_k^{n_k}$, where $E_i = E(R/P_i)$ and $E_i^{n_i}$ is the direct sum of $n_i < \infty$ copies of E_i. Thus $Q/I \subset E_1^{n_1} \oplus \cdots \oplus E_k^{n_k}$.

Let b be a non-zero element of R and $x = (1/b + I) \in Q/I$. Since the annihilator of a non-zero element of E_i is P_i-primary, we have $bI = \mathrm{Ann}_R\, x = J_1 \cap \cdots \cap J_k$, where either J_i is P_i-primary or $J_i = R$. It follows that every rank 1 prime ideal of R is equal to one of the P_i's, and hence R has only a finite number of rank 1 prime ideals. This, of course, implies that R is a semi-local of Krull dimension 1.

Conversely, suppose that R is a semi-local Noetherian domain of Krull dimension 1.

Case 1: *R is a local domain*

Let M be the maximal ideal of R and $0 \neq a \in M$. By [9, Theorem 5.5], Q/Ra is an Artinian R-module, and hence Q/Ra is an essential extension of its socle (the socle of a module is the sum of all of its simple submodules). The socle of Q/Ra is equal to B/Ra, where $B = \{q \in Q \mid qM \subset Ra\}$. Let $M^{-1} = \{q \in Q \mid qM \subset R\}$; then $B = M^{-1}a \subset R$. Thus $0 \neq B/Ra \subset R/Ra$, and hence, a fortiori, Q/Ra is an essential extension of R/Ra.

Case 2: *General case*

Let $M_1, \ldots, M_n$ be the maximal ideals of R and $J = \mathfrak{J}(R) = M_1 \cap \cdots \cap M_n$. Let a be a non-zero element of J, and let $q \in Q - Ra$. Let $M_1, \ldots, M_k$ be the maximal ideals of R such that $q \notin aR_{M_i}$. By Case 1, Q/aR_{M_1} is an essential extension of R_{M_1}/aR_{M_1}, and hence there exists $t_1 \in R$ such that $t_1 q \in R_{M_1} - aR_{M_1}$. Working successively on each i, we see that there exists $t \in R$ such that $tq \in \bigcap_{i=1}^n R_{M_i} = R$ and $tq \notin \bigcap_{i=1}^n aR_{M_i} = Ra$. Thus Q/Ra is an essential extension of R/Ra. □

REMARKS. The proof of Proposition 4.1 shows that if R is a Noetherian semi-local domain of Krull dimension 1, and if a is a non-zero element of $\mathfrak{J}(R)$, then Q/Ra is an essential extension of R/Ra.

DEFINITION. Let I and J be ideals of R. Then we define

$$[I{:}J] = \{q \in Q \mid qJ \subset I\}.$$

PROPOSITION 4.2. *Let I be an ideal of R such that* $\mathrm{inj.dim}_R\, I = 1$. *Then the following statements are true:*

(1) *If J is a non-zero flat ideal of R, then* $\mathrm{inj.dim}_R [I{:}J] = 1$.

(2) *If I is a flat ideal of R and* $B = \mathrm{Hom}_R(I, I)$, *then* $\mathrm{inj.dim}_R\, B = 1$.

Proof. (1) Since Q/I is R-injective, we have an exact sequence:

$$0 \to \mathrm{Hom}_R(R/J, Q/I) \to Q/I \to \mathrm{Hom}_R(J, Q/I) \to 0.$$

Because J is flat, $\mathrm{Hom}_R(J, Q/I)$ is R-injective. Thus $\mathrm{inj.\,dim}_R(\mathrm{Hom}_R(R/J, Q/I)) \leqslant 1$. Now

$$\mathrm{Hom}_R(R/J, Q/I) \simeq (\text{Annihilator of } J \text{ in } Q/I) = [I{:}J]/I.$$

Hence $\mathrm{inj.\,dim}_R[I{:}J] = 1$.

(2) Since $B \simeq [I{:}I]$, the conclusion follows from (1). □

DEFINITION. A valuation ring R is said to be an *almost maximal* valuation ring if $\mathrm{inj.\,dim}_R R = 1$. It is said to be a *maximal* valuation ring if in addition R is complete in the R (or valuation) topology.

REMARKS. Enochs has proved [4, Corollary 1] that there exists a maximal ideal M of R such that $R \to R/M$ is a torsion-free cover if and only if R is a maximal valuation ring. The next proposition generalizes this theorem as well as others.

PROPOSITION 4.3. *The following statements are equivalent:*

(1) *R is an almost maximal valuation ring.*

(2) *R is a valuation ring and* $\mathrm{inj.\,dim}_R I = 1$ *for every non-zero ideal I of R.*

(3) *R has only one maximal ideal M and* $\mathrm{inj.\,dim}_R M = 1$.

(4) *There exists a maximal ideal M of R such that $E(R/M) = Q/M$.*

(5) *There exists a maximal ideal M of R such that $H \to R/M$ is a torsion-free cover.*

Proof. (4) if and only if (5) is given by Proposition 3.8; (2) implies (3) is trivial; and (3) implies (4) is given by Corollary 2.2.

(4) implies (1). By Proposition 4.1, R is a valuation ring with maximal ideal M. Then M is flat and $[M{:}M] = R$. Hence $\mathrm{inj.\,dim}_R R = 1$ by Proposition 4.2.

(1) implies (2). Let I be a non-zero ideal of R.

Case I: I is not isomorphic to M, the maximal ideal of R.

Let $I^{-1} = [R{:}I]$; then $[R{:}I^{-1}] = I$. For clearly $I \subset [R{:}I^{-1}] \subset R$. Suppose there exists $a \in [R{:}I^{-1}] - I$. Then $I \subset Ma$ and so $a^{-1} \in I^{-1}$. On the other hand $aI^{-1} \subset R$, and so $I^{-1} \subset Ra^{-1}$. Thus $I^{-1} = Ra^{-1}$ and $[R{:}I^{-1}] = Ra$. If there exists $b \in Ma - I$, then by the preceding argument $[R{:}I^{-1}] = Rb$. Therefore $a \in Rb \subset Ma$. This contradiction shows that $I = Ma$. But this contradicts the assumption that I is not isomorphic to M. Hence we have $[R{:}I^{-1}] = I$. Since I^{-1} is flat and $\mathrm{inj.\,dim}_R R = 1$, we have $\mathrm{inj.\,dim}_R I = 1$ by Proposition 4.2.

Case II: I is isomorphic to M.

Let J be a non-zero ideal of R. It is sufficient to prove that $\mathrm{Ext}^1_R(J, M) = 0$; hence we can assume that J is not a principal ideal of R. We have an exact sequence:

$$\mathrm{Hom}_R(J, R/M) \to \mathrm{Ext}^1_R(J, M) \to \mathrm{Ext}^1_R(J, R).$$

The last term of this sequence is 0 because $\operatorname{inj.dim}_R R = 1$. Hence it is sufficient to prove that $\operatorname{Hom}_R(J, R/M) = 0$. Suppose there exists a surjection $f: J \to R/M$. Then there exists $a \in J$ such that $J = Ra + \operatorname{Ker} f$. Since $Ra \not\subset \operatorname{Ker} f$, we have $\operatorname{Ker} f \subset Ra$. Thus $J = Ra$ is a principal ideal of R. This contradiction proves that $\operatorname{Hom}_R(J, R/M) = 0$, and hence $\operatorname{inj.dim}_R M = 1$. □

The next proposition generalizes [6, Theorem 5].

PROPOSITION 4.4. *Let R be a Noetherian domain and suppose that R has an ideal I such that* $\operatorname{inj.dim}_R I = 1$. *Then R has Krull dimension* 1.

Proof. Let M be a maximal ideal of R. Then M contains an isomorphic copy of I, and hence we can assume that $I \subset M$. Since R is Noetherian, $\operatorname{inj.dim}_{R_M} I_M \leqslant \operatorname{inj.dim}_R I$, and thus we can assume that R is local with maximal ideal M. But then by Corollary 2.2 we have $Q/I \simeq E(R/I)$. Hence by Proposition 4.1(3), R has Krull dimension 1. □

COROLLARY 4.5. *Let R be a Noetherian domain and suppose that I is a non-zero ideal of R such that $E(R/I) \subset Q/I$. Then every prime ideal of R containing I has rank* 1 *and is a maximal ideal of R.*

Proof. Let $S = \{1 - a \mid a \in I\}$; then by Proposition 2.3, we have $\operatorname{inj.dim}_{R_S} I_S = 1$. Hence by Proposition 4.4, R_S has Krull dimension 1. Let P be a prime ideal of R containing I. Then PR_S is a maximal ideal of R_S. If M is a maximal ideal of R containing P, then MR_S is also a maximal ideal of R_S, and so $MR_S = PR_S$. Therefore $P = M$ is a maximal ideal of R. Since $\operatorname{rank}(PR_S) = 1$ in R_S, it follows that $\operatorname{rank} P = 1$ in R. □

REMARKS. Let R be a Noetherian local domain of Krull dimension 1. Let M be the maximal ideal of R and I a non-zero ideal of R. I is said to be a *canonical ideal* of R if $[I:[I:J]] = J$ for every non-zero ideal J of R. It is well known (see [9, Chapter 15]) that a canonical ideal, if it exists, is unique up to isomorphism; and that I is a canonical ideal of R if and only if $(I:M)/I \simeq R/M$ if and only if $\operatorname{inj.dim}_R I = 1$. Hence by Proposition 3.8, I is a canonical ideal of R if and only if $Q/I = E(R/I)$ if and only if $H \to R/I$ is a torsion-free cover.

Now while it is true by Proposition 4.1(3) that $Q/I \subset E(R/I)$ for every principal ideal I of R, it is not true in general that R has a canonical ideal; i.e., an ideal I such that $Q/I = E(R/I)$. Nevertheless, if the integral closure of R is a finitely generated R-module, then R has a canonical ideal. Hence in particular, if R is complete, then R has a canonical ideal.

R is said to be a (1-dimensional) *Gorenstein ring* if $\operatorname{inj.dim}_R R = 1$; i.e., if R has a canonical ideal that is a principal ideal. Moreover R is a Gorenstein ring if and only if $M^{-1}/R \simeq R/M$, where $M^{-1} = \{q \in Q \mid qM \subset R\}$ (see [9, Theorem 13.1]).

PROPOSITION 4.6. *Let R be a complete Noetherian local domain of Krull dimension* 1. *Let I be a canonical ideal of R and M the maximal ideal of R. Then the following statements are true:*

(1) $0 \to I \to (I:M) \to R/M \to 0$ *is a torsion-free cover.*

(2) *R is a Gorenstein ring if and only if I is a principal ideal of R if and only if* $0 \to R \to M^{-1} \to M^{-1}/R \to 0$ *is a torsion-free cover.*

(3) *R is a complete discrete valuation ring if and only if $(I:M)$ is a principal ideal of R if and only if $0 \to M \to R \to R/M \to 0$ is a torsion-free cover.*

Proof. (1) We have $(I:M)/I \simeq R/M$ as indicated in the remarks preceding this proposition. Since $\text{inj.}\dim_R I = 1$, and I is complete in the R-topology, and $(I:M)$ has no non-zero pure submodules, it follows from Proposition 3.3 that $(I:M) \to (I:M)/I$ is a torsion-free cover.

(2) By the preceding remarks R is a Gorenstein ring if and only if I is a principal ideal of R. Since R is complete and M^{-1} has no non-zero pure submodules, the canonical map $M^{-1} \to M^{-1}/R$ is a torsion-free cover if and only if $\text{inj.}\dim_R R = 1$ by Propositions 3.2 and 3.3.

(3) If R is a complete discrete valuation ring, then every ideal of R is a principal ideal of R. On the other hand assume that $(I:M) = Ra$, where $a \in (I:M)$. Then there exists an ideal J of R such that $I = Ja$. Then $R/M \simeq (I:M)/I = Ra/Ja \simeq R/J$, and hence $J = M$. Therefore $\text{inj.}\dim_R M = 1$, and so R is a valuation ring by Proposition 4.3. These remarks and Proposition 4.3 also show that R is a valuation ring if and only if $R \to R/M$ is a torsion-free cover. □

REMARKS. (1) Proposition 4.6(1) shows that there is a local domain R with maximal ideal M and a torsion-free cover $T \to R/M$ such that T is an ideal of R, but T is not isomorphic to R.

(2) Let R be a complete Noetherian local domain of Krull dimension 1 with maximal ideal M, and let I be the canonical ideal of R. Since R is a Gorenstein ring if and only if $M^{-1}/R \simeq R/M$, it might be conjectured from Proposition 4.6(2) that R is a Gorenstein ring if and only if $(I:M) \simeq M^{-1}$, or equivalently if and only if there exists a map $M^{-1} \to R/M$ that is a torsion-free cover. The falsity of this conjecture is proved by the following counter-example. Let R be a complete Noetherian local domain of Krull dimension 1 such that R is not Gorenstein, but such that M^{-1} is a complete discrete valuation ring (such an example is constructed in [8, (4) on p. 287]). Now $(I:M)$ is an ideal of M^{-1}, and hence in this case is a principal ideal of M^{-1}. Therefore, $M^{-1} \simeq (I:M)$ and so by Proposition 4.6(1) there exists a map $M^{-1} \to R/M$ that is a torsion-free cover.

DEFINITION. R is said to be an *h-local* if each non-zero element of R is contained in only a finite number of maximal ideals of R, and each non-zero prime ideal of R is contained in only one maximal ideal of R. An example of an h-local domain is a Noetherian domain of Krull dimension 1.

PROPOSITION 4.7. *Let R be an h-local domain, I a non-zero ideal of R, and $S = \{1 - a \mid a \in I\}$. Then the following statements are true:*

(1) *R_S is a complemented ring extension of R.*

(2) *$H(R_S) \simeq H(R_{M_1}) \oplus \cdots \oplus H(R_{M_n})$, where $M_1, \ldots, M_n$ are the maximal ideals of R that contain I.*

(3) *$H(R_S)$ is a direct summand of H.*

Proof. (1) Let $M_1, \ldots, M_n$ be the maximal ideals of R that contain I. Then $R_S = \bigcap_{i=1}^{n} R_{M_i}$ by Proposition 1.4(4). Let $A = \bigcap R_N$, $\{N \in \max\operatorname{spec} R \mid I \not\subset N\}$. Then

$R_S \cap A$ is the intersection of all of the localizations of R with respect to the maximal ideals of R, and hence $R_S \cap A = R$.

Let M be a maximal ideal of R and let $[M] = \cap R_P$, $\{P \in \max\operatorname{spec} R \mid P \neq M\}$. Then by [7, Theorem 2.2] we have $[M] \otimes_R R_M \simeq Q$. Thus since $[M_i] \subset A$, we have $A_{M_i} = Q$ for $i = 1, \ldots, n$; and since $[N] \subset R_S$ for $N \in \max\operatorname{spec} R$ such that $I \not\subset N$, we have $(R_S)_N = Q$. Thus $(R_S + A)_M = Q$ for all $M \in \max\operatorname{spec} R$. Hence $R_S + A = Q$, showing that R_S is a complemented extension of R and that $R_S' = A$.

(2) R_S is an h-local ring with only a finite number of maximal ideals:

$$M_1 R_S, \ldots, M_n R_S; \qquad \text{and} \qquad (R_S)_{M_i} = R_{M_i} \quad \text{for all} \quad i = 1, \ldots, n.$$

Therefore by [7, Theorem 2.2] $H(R_S) \simeq H(R_{M_1}) \oplus \cdots \oplus H(R_{M_n})$.

(3) Since R_S is a complemented extension of R, we have that $H(R_S)$ is a direct summand of H by Proposition 1.10. □

PROPOSITION 4.8. *Let R be an h-local domain, I a non-zero ideal of R and $S = \{1 - a \mid a \in I\}$. Let $M_1, \ldots, M_n$ be the maximal ideals of R that contain I. Then the following statements are equivalent:*

(1) $E(R/I) \subset Q/I$.

(2) $\operatorname{Inj.dim}_{R_S} I_S = 1$.

(3) $\operatorname{Inj.dim}_{R_{M_i}} I_{M_i} = 1$ *for all* $i = 1, \ldots, n$.

(4) *The canonical map* $H(R_{M_1}) \oplus \cdots \oplus H(R_{M_n}) \to R/I$ *is a torsion-free cover.*

Proof. (2) if and only if (3). We have $(I_S)_{M_i} = I_{M_i}$ for $i = 1, \ldots, n$; and since R_S is an h-local ring with $\max\operatorname{spec} R_S = \{M_1 R_S, \ldots, M_n R_S\}$, we have $\operatorname{inj.dim}_{R_S} I_S = \sup_i(\operatorname{inj.dim}_{R_{M_i}} I_{M_i})$ by [7, Theorem 24].

Since R_S is a complemented extension of R and since

$$H(R_S) \simeq H(R_{M_1}) \oplus \cdots \oplus H(R_{M_n})$$

by Proposition 4.7, the equivalence of (1), (2) and (4) follows from Propositions 2.3 and 3.9. □

REMARKS. Let Z be the ring of integers and p a non-zero prime integer. Then Proposition 4.8 shows that the torsion-free cover of Z/pZ is the ring of p-adic integers.

Proposition 4.8 also proves that the equivalence over h-local domains of the statements $E(R/I) \subset Q/I$ and $\operatorname{inj.dim}_{R_S} I_S = 1$ without the additional assumption required in Proposition 2.3 that R_S is a complemented extension of R. This is because R_S is always a complemented extension of R when R is an h-local domain, as we have seen in Proposition 4.7.

It might be conjectured that the assumption that R_S is a complemented extension of R is redundant in general, and might be a consequence of $\operatorname{inj.dim}_{R_S} I_S = 1$. The falsity of this conjecture is demonstrated in Proposition 4.9 where we produce an ideal I such that $\operatorname{inj.dim}_{R_S} I_S = 1$, but $E(R/I) \not\subset Q/I$ and R_S is not a complemented extension of R. A concrete example of a domain that satisfies the properties assumed in Proposition 4.9 may be found in [7, (2), p. 154].

PROPOSITION 4.9. *Let R be an integral domain that satisfies the following three properties:*

(*a*) *R has exactly two maximal ideals M and N.*

(*b*) *$M \cap N$ contains a non-zero ideal P of R.*

(*c*) *R_M and R_N are maximal valuation rings.*

Then the following statements are true:

(1) *Let $I = M$ and $S = \{1 - a \mid a \in I\}$ so that $R_S = R_M$. Then* $\text{inj.dim}_{R_S} I_S = 1$, *but $E(R/I) \not\subset Q/I$ and R_S is not a complemented extension of R. Moreover, R_S is complete and the canonical map $R_S \to R/I$ is a torsion-free cover.*

(2) *We have $P = PR_P$, R_P is a maximal valuation ring, $E(R/P) = Q/P$, R is complete, and $R \to R/P$ is a torsion-free cover.*

(3) *$0 \to MR_M \oplus NR_N \to R_M \oplus R_N \to R/(M \cap N) \to 0$ is a torsion-free cover.*

(4) $\text{inj.dim}\, R = 2$.

Proof. (1) It is easy to see that $R_S = R_M$, and thus R_S is a maximal valuation ring by assumption. Thus $\text{inj.dim}_{R_S} I_S = 1$ by Proposition 4.3. By Proposition 1.2 R_S is not a complemented extension of R. Therefore, by Proposition 2.3, $E(R/I) \not\subset Q/I$. Now R_S is complete in the R_S-topology since it is a maximal valuation ring. Hence it is easily seen that R_S is also complete in the R-topology. Therefore I_S is also complete in the R-topology. Now $\text{inj.dim}_R I_S = \text{inj.dim}_{R_S} I_S = 1$ and $R_S/I_S \simeq R/I$. Hence by Proposition 3.3, $0 \to I_S \to R_S \to R/I \to 0$ is a torsion-free cover.

(2) Since R_M is a maximal valuation ring, it follows easily from Proposition 4.3 that every localization of R_M is also a maximal valuation ring. Now $R_P = (R_M)_P$, and thus R_P is a maximal valuation ring. The same argument as that used in (1) shows that $\text{inj.dim}_R PR_P = 1$, PR_P is complete in the R-topology, and $0 \to PR_P \to R_P \to R_P/PR_P \to 0$ is a torsion-free cover over R.

Since R_M and R_P are valuation rings and $R_M \subset R_P$, we have $P_M = PR_P$. Similarly, $P_N = PR_P$. Thus $P = P_M \cap P_N = PR_P$, and so $\text{inj.dim}_R P = 1$ and P is complete in the R-topology. Therefore, R is also complete in the R-topology. By Proposition 3.4, $R \to R/P$ is a torsion-free cover. Since $P \subset \mathfrak{J}(R)$, we have $E(R/P) = Q/P$ by Corollary 2.2.

(3) By the Chinese Remainder Theorem we have $R/(M \cap N) \simeq R/M \oplus R/N \simeq R_M/MR_M \oplus R_N/NR_N$. Thus we have an exact sequence

$$0 \to MR_M \oplus NR_N \to R_M \oplus R_N \to R/(M \cap N) \to 0.$$

By (1) $\text{inj.dim}_R(MR_M \oplus NR_N) = 1$ and $MR_M \oplus NR_N$ is complete in the R-topology. Thus by Proposition 3.3, the preceding exact sequence is a torsion-free lifting of $R/(M \cap N)$ over R.

Suppose that this exact sequence is not a torsion-free cover of $R/(M \cap N)$, and let $T \to R/(M \cap N)$ be a torsion-free cover. By Banaschewski's Theorem, T is a proper direct summand of $R_M \oplus R_N$, and hence $\text{rank}\, T = 1$. Now Q/MR_M is an injective R-module and is an essential extension of R_M/MR_M. Therefore

$$E(R/M) = E(R_M/MR_M) \simeq Q/MR_M.$$

Similarly, $E(R/N) \simeq Q/NRN$. Thus if $E = E(R/(M \cap N))$, then

$$E = E(R/M) \oplus E(R/N) \simeq Q/MR_M \oplus Q/NR_N.$$

Thus $\mathrm{Hom}_R(K,E)$ has rank $\geqslant 2$. But $\mathrm{Hom}_R(K,E) \subset T$ by Proposition 3.2(1), and hence $\mathrm{Hom}_R(K,E)$ has rank 1. This contradiction proves that

$$R_M \oplus R_N \to R/(M \cap N)$$

is a torsion-free cover over R.

(4) It is clear that $R_M + R_N \subset (R_M)_N \subset R_P \neq Q$. On the other hand, let $s \in R - M$ and $t \in R - N$. Then $Rs + Rt = R$, and hence there exist $u, v \in R$ such that $1 = us + vt$. Therefore $1/(st) = u/t + v/s \in R_N + M_N$. Thus $R_M + R_N = (R_M)_N$. Since $(R_M)_N$ is a maximal valuation ring, we have as before that $\mathrm{inj.dim}_R (R_M)_N = 1$. From the exact sequence:

$$0 \to R \to R_M \oplus R_N \to R_M + R_N \to 0$$

and the fact that the last two terms of this sequence have injective dimension 1, we see that $\mathrm{inj.dim}_R R \leqslant 2$.

Suppose that $\mathrm{inj.dim}_R R = 1$. Choose $a \in N$, $a \notin M$. Then $\mathrm{inj.dim}_R Ra = 1$ and Ra is complete in the R-topology by (3). Thus by Proposition 3.3,

$$0 \to Ra \to R \to R/Ra \to 0$$

is a torsion-free cover. Hence by Proposition 3.8, $Ra \subset \mathfrak{J}(R) = M \cap N$. This contradiction shows that $\mathrm{inj.dim}_R R = 2$. □

REMARKS. If we drop the assumption in Proposition 4.9 that $M \cap N$ contains a non-zero prime ideal of R, then R is an h-local domain. Examples of this kind of ring exist (see [7, Example p. 119]). In this case we have $R_M + R_N = Q$ and $\mathrm{inj.dim}_R R = 1$. The part of the argument in the preceding paragraph that fails is that R is now not complete. In fact, we have $H = R_M \oplus R_N$; and if $0 \neq b \in R$, then the exact sequence $0 \to R_M b \oplus R_N b \to R_M \oplus R_N \to R/Rb \to 0$ is a torsion-free lifting; and by Proposition 3.8 it is a torsion-free cover if and only if $b \in M \cap N$. By Proposition 4.8 $\mathrm{inj.dim}_R (M \cap N) = 1$ and $R_M \oplus R_N \to R/(M \cap N)$ is a torsion-free cover.

It is of some interest that the kinds of rings described in Proposition 4.9 and in the preceding remarks, together with maximal valuation rings, are the only integrally closed domains that have the property that every torsion-free R-module of finite rank is a direct sum of modules of rank 1 (see [7, Theorem 95]).

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