LACUNARY SERIES AND THE BOUNDARY BEHAVIOR OF BLOCH FUNCTIONS

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1. Introduction. In this paper we are concerned with two open problems involving the boundary behavior of Bloch functions and the derivatives of schlicht analytic functions. Lohwater conjectured (see also [5]) that certain Bloch functions which are represented by lacunary series in $|z|<1$ and having radial (angular) limits only on a set of measure zero on $|z|=1$, must possess radial (angular) limits on a set whose logarithmic capacity is positive. Recently, Jefferson [5] obtained some preliminary results in this direction. In the present note we establish the validity of this conjecture. To this end, we shall first provide here a refinement of Beardon's construction of Cantor sets [1] which have positive logarithmic capacity (Theorem 1.5). This result, in conjunction with some theorems of J.-P. Kahane, M. Weiss, and G. Weiss (see [6]), enables us to show that certain lacunary trigonometric series converge (to $+\infty$) on a set which is of measure zero but of positive logarithmic capacity. The method of Abel summability then leads to a proof of the aforementioned conjecture (Theorem 2.15).

The second and larger open problem of Lohwater (see Lohwater [7]) asks whether the derivative of a schlicht analytic function can have radial limits only on a set of (logarithmic) capacity zero. It is known [9] that the derivative $f'(z)$ of a schlicht function $f(z)$ can have radial limits only on a set E of measure zero on the circle $|z|=1$. Today, there are several results known [7] about the topological nature of this set E. But in spite of the sophisticated techniques that have been developed in this area of investigation, the problem of determining the capacity of E seems very difficult. Thus, while this question is still open, in Section 2 we provide some partial results in this direction. Indeed, using our previous results about Bloch functions and a representation theorem of Pommerenke [11], we establish the existence of a schlicht analytic function in $|z|<1$ whose derivative possesses radial limits only on a set E of measure zero, while the (logarithmic) capacity of E is positive.

In [1] Beardon provided a condition which is both necessary and sufficient for a Cantor set to have positive capacity with respect to a generalized capacity function (cf. R. Nevanlinna [10]). The function-theoretic applictions we shall present in the sequel require that we relax some of the conditions in the usual construction of Cantor sets which have positive capacity. Beardon's theorem remains valid under weaker hypotheses.

We now construct a Cantor set $C\subseteq[0,2\pi]$ as follows. Let

$$C=\bigcap_{j=1}^{\infty}\left(\bigcup S_j\right), \tag{1.1}$$

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where S_j is a set of 2^j closed disjoint intervals contained in $[0, 2\pi]$. Moreover, we impose on these intervals the following conditions.

(1.2) Each interval in S_j contains exactly two intervals of S_{j+1}.

(1.3) The intervals of S_j have equal length d_j and $d_j \to 0$ as $j \to \infty$.

(1.4) For some $\alpha > 0$ the inequalities $d_{j+1}/d_j \geqslant \alpha > 0$ hold for $j = 1, 2, 3, \ldots$.

We remark that the only difference between this construction and the one given in [1] is that here we have disposed of the requirement that two intervals of S_{n+1} be symmetrically placed in an interval of S_n. Since the argument used in [1] applies *verbatim*, even in this general situation, we omit the proof of the following theorem.

THEOREM 1.5 (Beardon [1]). *Every Cantor set of the above type, which satisfies* (1.1)-(1.3) *and* (1.4), *has positive logarithmic capacity.*

2. Lacunary series. Let $\{\lambda_k\}$ be a lacunary sequence of positive integers satisfying

$$\lambda_{k+1}/\lambda_k > q > 1. \tag{2.1}$$

Let

$$f(z) = \sum_{k=0}^{\infty} c_k z^{\lambda_k}, \qquad |z| < 1, \tag{2.2}$$

where $\{c_k\}$ is a sequence of complex numbers such that

$$|c_k| \leqslant M, \qquad k = 0, 1, 2, 3, \ldots, \tag{2.3}$$

and

$$\sum^{\infty} |c_k| = \infty. \tag{2.4}$$

Then it is known (see, for example, [5]) that

$$|f'(z)| = O\left(\frac{1}{1-|z|}\right), \tag{2.5}$$

so that $f(z)$ is in the Bloch class. (For the various properties of functions in the Bloch class see Pommerenke [11] and the survey article of Cima [3]). Since a Bloch function is also normal, the concepts of asymptotic value, radial limit and angular limit are equivalent (see, for example, Lohwater [8]). If we impose some other condition on the coefficients $\{c_k\}$, to wit,

$$\liminf_{k \to \infty} |c_k| > 0, \tag{2.6}$$

then the function $f(z)$ in (2.2) can have only the radial limit ∞ (see, for example, [2, Corollary 1]). But an analytic function having angular limit ∞ on a set of positive measure on $|z| = 1$, must be identically ∞ (see, for example, [4, p. 145]). Therefore, the function $f(z)$ defined by (2.2) and satisfying (2.1), (2.3) and (2.6) possesses radial limits only on a set, E, of measure zero. In this section we shall show that this set E

is, nevertheless, "thick" in the sense that the logarithmic capacity of E is positive (Theorem 2.15). This result thus solves the first open problem to which we had alluded in the Introduction. Finally, we conclude this paper with a result which provides a partial answer to the larger problem involving the derivatives of schlicht analytic functions (see Section 1).

Our investigation of the boundary behavior of certain Bloch functions will be based on the following theorem concerning lacunary trigonometric series.

THEOREM 2.7. *Suppose that the sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *of positive integers satisfies* (2.1) *and that the sequence* $\{c_k\}_{k=1}^{\infty}$ *of complex numbers satisfies the conditions* (2.3) *and* (2.4). *Then there is a Cantor set* $C \subseteq [0, 2\pi]$ *of positive logarithmic capacity such that for* x *in* C

$$\lim_{n\to\infty} \operatorname{Re}\left\{\sum_{k=1}^{n} c_k e^{i\lambda_k x}\right\} = +\infty.$$

Moreover, if n *is large enough, then there is a constant* K *such that for every* x *in* C

$$\operatorname{Re}\left\{\sum_{k=1}^{n} c_k e^{i\lambda_k x}\right\} \geqslant K \sum_{k=1}^{n} |c_k|.$$

In order to prove Theorem 2.7 we will borrow several results from [6].

LEMMA 2.8 ([6, p. 6]). *Let* $Q(x) = \sum_{k=1}^{N} c_k e^{i\mu_k x}$, *where* $\{\mu_k\}_{k=1}^{N}$ *is a lacunary sequence of positive real numbers with* $\mu_{k+1}/\mu_k > q > 1$. *Then there exist two constants* $A = A_q$ *and* $B = B_q \geqslant 1$, *which depend only on* q, *such that every interval* I *of length* A/μ_1 *contains a subinterval* J *of length* $2/(B\mu_N)$ *such that*

$$B \operatorname{Re}\{Q(x)\} \geqslant \sum_{k=1}^{N} |c_k|$$

for each x *in* J.

The next lemma provides a rather useful decomposition of a general lacunary power series into successive lacunary blocks.

LEMMA 2.9 ([6, p. 17]). *Let* $\epsilon > 0$ *and let* s *be a positive integer. Let*

$$S(x) = \sum_{k=1}^{\infty} c_k e^{i\mu_k x}, \qquad \mu_k \in \mathbf{R}^+, \qquad \mu_{k+1}/\mu_k > q > 1,$$

be a lacunary power series. Then there exists a constant R, *depending on* q, ϵ *and* s, *such that* $S(x)$ *can be written as a sum of successive lacunary blocks* (*corresponding to the same* q)

$$S(x) = Q_1(x) + Q_1^*(x) + Q_2(x) + Q_2^*(x) + \cdots$$

with the following properties.

(i) $Q_j(x)$ *and* $Q_j^*(x)$ *are blocks of consecutive terms of the series.*

(ii) Q_j^* *has at most* s *terms and* Q_j *has at most* $3(m+1)s$ *terms, where* $m = [1/\epsilon] + 1$ *and* $[x]$ *denotes the greatest integer less than or equal to* x.

(iii) $\mu_j''/\mu_j' < R$, *where* μ_j' *and* μ_j'' *denote the lowest and highest frequency of* $Q_j(x)$.

(iv) $r < \mu_{j+1}'/\mu_j'' \leqslant r^2$, *where* $r = q^{s+1}$.

(v) $\Delta_j^* \leqslant \epsilon(\Delta_j + \Delta_{j+1})$, *where* Δ_j *and* Δ_j^* *denote the sums of the absolute values of the coefficients of* $Q_j(x)$ *and* $Q_j^*(x)$, *respectively*.

We remark that the decomposition in Lemma 2.9 is based on the following simple observation. A lacunary power series $\sum c_k e^{i\mu_k x}$, $\mu_{k+1}/\mu_k > q > 1$, $\mu_k \in \mathbf{R}^+$, can be modified by introducing terms of the form $0 \cdot e^{inx}$, if necessary, so that the ratio of consecutive frequencies satisfies

$$q < \frac{\mu_{k+1}}{\mu_k} \leqslant q^2, \qquad k = 1, 2, 3, \ldots .$$

(For the sake of simplicity of notation we have labeled the new frequencies also by μ_k.)

We now proceed to the proof of Theorem 2.7.

Proof of Theorem 2.7. Consider the lacunary series $\sum_{k=1}^{\infty} c_k e^{i\lambda_k x}$, where the frequencies λ_k are positive integers and satisfy the gap condition $\lambda_{k+1}/\lambda_k > q > 1$. By introducing zero-terms (see above remark) we can find a sequence $\{\mu_k\}_{k=1}^{\infty}$ of positive real numbers such that $\{\lambda_k\} \subseteq \{\mu_k\}$, $1 < q < \mu_{k+1}/\mu_k \leqslant q^2$ and for each positive integer n

$$\sum_{k=1}^{n} c_k e^{i\lambda_k x} = \sum_{\mu_k \leqslant \lambda_n} c_k' e^{i\mu_k x}, \qquad 0 \leqslant x \leqslant 2\pi, \tag{2.10}$$

where

$$c_k' = \begin{cases} c_p & \text{if } k \text{ is such that } \mu_k = \lambda_p \\ 0 & \text{otherwise} \end{cases} .$$

By (2.10) it suffices to show that

$$\lim_{n \to \infty} \operatorname{Re}\left\{ \sum_{k=1}^{n} c_k' e^{i\mu_k x} \right\} = +\infty \tag{2.11}$$

on a set of positive logarithmic capacity.

Now let $A = A_q$ and $B = B_q$ be the constants given by Lemma 2.8. Let s be a positive integer so large that

$$r = q^{s+1} > AB. \tag{2.12}$$

Finally, let $\epsilon = 1/4B$. Then by Lemma 2.9 we obtain the decomposition

$$\sum_{k=1}^{\infty} c_k e^{i\mu_k x} = Q_1(x) + Q_1^*(x) + Q_2(x) + Q_2^*(x) + \cdots,$$

where Q_j, $Q_j^*(x)$, etc. satisfy the properties (i)–(v) of Lemma 2.9. Furthermore, we may assume, without loss of generality, that μ_1' is so large that $2\pi > 2(A/\mu_1')$. (Recall that μ_1' is the first frequency in Q_1.) Because $2\pi > 2(A/\mu_1')$, there are two disjoint closed subintervals of $[0, 2\pi]$ each of length A/μ_1'. Now Lemma 2.8 gives us a closed subinterval of each, labeled $I_1(1)$ and $I_2(1)$, of length $2/B\mu_1''$, such that for x in $I_1(1) \cup I_2(1)$, $\operatorname{Re}\{Q_1(x)\} \geq (1/B)\Delta_1$.

We next define inductively sets of closed disjoint intervals $\{I_j(k)\}_{k=1}^{2^j}$ such that

(i) $I_{j-1}(k) \supseteq I_j(2k) \cup I_j(2k-1)$,

(ii) $\operatorname{Re}\{Q_j(x)\} \geq (1/B)\Delta_j$ on $I_j(k)$, $1 \leq k \leq 2^j$,

and

(iii) the length of $I_j(k)$ is $d_j = 2/(B\mu_j'')$.

The induction step applies Lemma 2.8 to $Q_{j+1}(x)$ and $I_j(k)$ for each k, $1 \leq k \leq 2^j$. It is necessary, however, to check that $I_j(k)$ contains two disjoint closed intervals each of length A/μ_{j+1}'. But the length of $I_j(k)$ is $d_j = 2/(B\mu_j'')$ and so by our choice of r, (2.12), the inequality $AB < r < \mu_{j+1}'/\mu_j''$ implies that $d_j = 2/(B\mu_j'') > 2(A/\mu_{j+1}')$. Consequently, by Lemma 2.8 $I_j(k)$ contains two intervals $I_{j+1}(2k)$ and $I_{j+1}(2k-1)$ which satisfy the above properties (i), (ii) and (iii).

With the aid of the intervals $I_j(k)$ we now define the Cantor set

$$C = \bigcap_{j=1}^{\infty} \left[\bigcup_{k=1}^{2^j} I_j(k) \right].$$

Now by Lemma 2.9 parts (iii) and (iv) we have

$$\frac{d_{j+1}}{d_j} = \frac{\mu_j''}{\mu_{j+1}''} = \frac{\mu_j''}{\mu_{j+1}'} \cdot \frac{\mu_{j+1}'}{\mu_{j+1}''} > r^{-2}R^{-1} = \alpha > 0,$$

so that condition (1.4) is fulfilled. Hence, by Theorem 1.5 the Cantor set C has positive logarithmic capacity.

We now proceed to show that (2.11) holds for each x in C. Let N be a positive integer. Then there is a positive integer j_0 such that $\mu_{j_0}'' \leq \mu_N < \mu_{j_0+1}''$. With the aid of the lacunary blocks of Lemma 2.9, we obtain

$$\operatorname{Re}\left\{ \sum_{k=1}^{N} c_k' e^{i\mu_k x} \right\} \geq \operatorname{Re}\left\{ \sum_{j=1}^{j_0} Q_j(x) \right\} - \sum_{j=1}^{j_0} \Delta_j^* - \Delta_{j_0+1}.$$

By our choice of the intervals $I_j(k)$, $\operatorname{Re} Q_j(x) \geq (1/B)\Delta_j$ for every x in $I_j(k)$ and *a fortiori* for every x in C. Consequently, if x is in C, then

$$\begin{aligned}
\operatorname{Re}\left\{ \sum_{k=1}^{N} c_k' e^{i\mu_k x} \right\} &\geq \frac{1}{B} \sum_{j=1}^{j_0} \Delta_j - \sum_{j=1}^{j_0} \Delta_j^* - \Delta_{j_0+1} \\
&= \frac{1}{2B}(\Delta_1 + \Delta_{j_0}) + \sum_{j=1}^{j_0-1} \left[\frac{1}{2B}(\Delta_j + \Delta_{j+1}) - \Delta_j^* \right] - \Delta_{j_0}^* - \Delta_{j_0+1}, \\
&\geq \frac{1}{2B}(\Delta_1 + \Delta_{j_0}) + \frac{1}{4B} \sum_{j=1}^{j_0-1} (\Delta_j + \Delta_{j+1}) - \Delta_{j_0}^* - \Delta_{j_0+1},
\end{aligned}$$

where the last estimate holds by virtue of the inequality $\Delta_j^* \leq (1/4B)(\Delta_j + \Delta_{j+1})$ (see Lemma 2.9 part (v)). Since $|c_k'| \leq |c_k| \leq M$ by assumption, and since Q_j^* has at most

s terms, while Q_j has at most $3(m+1)s$ terms we have that $\Delta_{j_0}^* \leqslant sM$ and $\Delta_{j_0+1} \leqslant 3(m+1)sM$. Thus, if x is in C, then

$$\operatorname{Re}\left\{\sum_{k=1}^{N} c_k' e^{i\mu_k x}\right\} \geqslant \frac{1}{2B} \sum_{j=1}^{j_0} \Delta_j - sM - 3(m+1)sM.$$

Clearly, as $N \to \infty$ the right-hand side tends to $+\infty$. (Note that this is the only place where we have used the assumption that $\sum_{k=1}^{\infty}|c_k| = \infty$.) Finally, the second conclusion of Theorem 2.7 follows from the above inequality and thus the proof of the theorem is complete. □

It is clear that Theorem 2.7 remains valid if we replace $[0, 2\pi]$ by any other closed interval $[a, b]$. Thus, as a direct consequence of this theorem, we obtain the following corollary.

COROLLARY 2.13. *Suppose that the sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *of positive integers satisfies* (2.1) *and that the sequence* $\{c_k\}_{k=1}^{\infty}$ *of complex numbers satisfies the conditions* (2.3) *and* (2.4). *Then there is a set* $E \subseteq [0, 2\pi]$ *with the following properties:*

(a) *E is of type F_σ, that is, E is the union of a countable set of closed sets;*
(b) *the logarithmic capacity of E is positive;*
(c) *E is dense in $[0, 2\pi]$; and*
(d) *for each x in E,* $\lim_{n\to\infty} \operatorname{Re}\{\sum_{k=1}^{n} c_k e^{i\lambda_k x}\} = +\infty$.

Before we will turn to some of the applications of the foregoing results we will include here, for the sake of completeness, the following lemma concerning Abel summability.

LEMMA 2.14. *Let* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, $c_k \in \mathbf{C}$, *be analytic in* $|z| < 1$. *If at* $\theta = \theta_0$ $\operatorname{Re}\{\sum_{k=0}^{\infty} c_k e^{ik\theta_0}\} = +\infty$, *then* $\lim_{r\to 1^-} f(re^{i\theta_0}) = \infty$.

Proof. It suffices to show that $\lim_{r\to 1^-} \operatorname{Re}\{\sum_{k=0}^{\infty} c_k r^k e^{ik\theta_0}\} = +\infty$. For the sake of simplicity of notation set $a_k = \operatorname{Re}\{c_k e^{ik\theta_0}\}$, so that $\sum_{k=0}^{\infty} a_k = +\infty$. Let $S_k = a_0 + \cdots + a_k$, $k = 0, 1, 2, \ldots$. Then the partial summation formula, when applied to the (partial sums of the) convergent power series $\sum_{k=0}^{\infty} a_k r^k$, $0 \leqslant r < 1$, implies that

$$\sum_{k=0}^{\infty} a_k r^k = (1-r) \sum_{k=0}^{\infty} S_k r^k, \qquad 0 \leqslant r < 1.$$

We will now show that for any $K > 0$ an r_0 can be found such that $\sum_{k=0}^{\infty} a_k r^k > K$ for all $r \geqslant r_0$. Since by hypothesis $S_n \to \infty$ as $n \to \infty$, we may assume, without loss of generality, that $S_n \geqslant 0$ for all n. Furthermore, there is a positive integer N such that $S_n \geqslant 2K$ for all $n \geqslant N$. Now fix r_0 such that $r_0^N \geqslant \frac{1}{2}$. Then for all $r \geqslant r_0$ we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k r^k &= (1-r) \sum_{k=0}^{\infty} S_k r^k \\
&\geqslant (1-r) \sum_{k=0}^{N-1} S_k r^k + (1-r)2K \sum_{k=N}^{\infty} r^k \\
&> (1-r)2K r_0^N \left(\frac{1}{1-r}\right) \geqslant K.
\end{aligned}$$

This completes the proof of the lemma. □

As an immediate consequence of Theorem 2.7 and Lemma 2.14, we obtain the following result.

THEOREM 2.15. *Let* $f(z)=\sum_{k=0}^{\infty} c_k z^{\lambda_k}$, $|z|<1$, *where the sequence* $\{\lambda_k\}_{k=0}^{\infty}$ *of positive integers satisfies* (2.1) *and the sequence* $\{c_k\}_{k=0}^{\infty}$ *of complex numbers satisfy* (2.3) *and* (2.4). *Let E denote the set of points on* $|z|=1$ *at which* $\lim_{r\to 1^-} f(re^{i\theta})=\infty$. *Then the logarithmic capacity of E is positive.*

Theorem 2.15 thus establishes the validity of Lohwater's conjecture concerning lacunary power series (see Introduction). The condition that $|c_k|\leqslant M$ $(k=0,1,2,\dots)$, implies, as we have noted above, that the function $f(z)$ of Theorem 2.15 is in the Bloch class. Therefore, the following stronger conclusion follows: the set of points on $|z|=1$ at which the function $f(z)$ of Theorem 2.15 has angular limit ∞ is of positive logarithmic capacity (see also [5, Corollary 4.1 and Theorem 4.2]).

We are still unable to settle the larger problem whether the derivative of a schlicht analytic function can have radial limit *only* on a set of capacity zero. On the other hand, it is known (Lohwater [9]) that the derivative $h'(z)$ of a schlicht analytic function $h(z)$ has angular limits on a set which is uncountable and everywhere dense on $|z|=1$, even though $h'(z)$ possesses radial (angular) limits only on a set of measure zero.

REMARK. One can construct a schlicht analytic function $h(z)$ in $|z|<1$ whose derivative has radial limits *only* on a set of measure zero but of positive logarithmic capacity. Let $f(z)$ be a function as in Theorem 2.15 and suppose that $f(z)$ satisfies the additional requirement (2.6). Then condition (2.3) implies that $f(z)$ is in the Bloch class. Now by Pommerenke's representation theorem for Bloch functions [11], there is a schlicht analytic function $h(z)=z+a_2z^2+\cdots$ in $|z|<1$ and a real constant $\alpha>0$ such that $f(z)=\alpha\log h'(z)$. Then this function $h(z)$ has the desired properties. Moreover, the radial limits of $h(z)$ are only zero and infinity.

Added in proof: With great sorrow we note here that our esteemed colleague, teacher and advisor, Professor A. J. Lohwater passed away on June 10, 1982.

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