SURFACE SYMMETRY I

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1. Introduction. Let G be a finite group and consider a free, orientation-preserving action ϕ of G on a closed, oriented, connected surface M. There is one well-known invariant of such an action, namely its class $[M,\phi]$ in the free, oriented G-bordism group $\mathcal{O}_2^{\text{free}}(G)$. This invariant, and other concepts introduced here, are explained in more detail in Section 2. For the present, $\mathcal{O}_2^{\text{free}}(G)$ can be identified with the homology group $H_2(G;\mathbf{Z})$ or the bordism group $\Omega_2(K(G,1))$.

THEOREM 1.1. *Two free orientation-preserving actions ϕ_1 and ϕ_2 of a finite abelian group G on a closed, oriented, connected surface M are equivalent by an equivariant, orientation-preserving homeomorphism if and only if* $[M,\phi_1]=[M,\phi_2]$ *in* $\mathcal{O}_2^{\text{free}}(G)$.

Theorem 1.1 is proved by classifying appropriate covering spaces using a purely algebraic result of independent interest. Let V denote a symplectic inner product space over the integers $\mathbf{Z}$. In the absence of mention to the contrary we adhere to the basic definitions about inner product spaces found in [5; Chapter 1]. Then Aut V acts on the set Epi(V,G) of epimorphisms from V to G.

THEOREM 1.2. *If G is a finite abelian group and V is a symplectic inner product space over $\mathbf{Z}$, then there is an injection* Epi$(V,G)/$Aut $V\to H_2(G;\mathbf{Z})$.

To prove Theorem 1.2 we extend the result known as Witt's Theorem, which states that an isometric embedding $U\to V$ of a free summand U of a symplectic inner product space V over field R is the restriction of an isometry of V [1; p. (2)], to the analogous statement when R is any local ring. This is undoubtedly known to some, but we are aware of no clear statement in the literature. The reason is that such results are usually given in the more difficult context of symmetric inner product spaces; and in this case the analogous extension to a local ring is valid only if 2 is a unit [5; p. 9].

Now consider the case of an arbitrary, effective, orientation-preserving action ϕ of a finite group G on the surface M. Another obvious invariant is the set $\mathfrak{D}$ of *fixed point data* of ϕ. In the case of smooth actions, $\mathfrak{D}$ consists of the local representations of the isotropy groups G_x, for each x in the singular set, which consists of all points with nontrivial isotropy group. In the general case of an oriented surface with orientation-preserving action, the isotropy groups G_x are all cyclic, and the set $\mathfrak{D}$ of fixed point data is given as a set (with multiplicities) of conjugacy classes in G of generators for these isotropy groups G_x, one for each singular orbit. The point is that $\mathfrak{D}$ describes the action of G in a neighborhood of the singular set for the action. When G is abelian, $\mathfrak{D}$ becomes an unordered set of nontrivial, not necessarily distinct,

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elements of G with product the identity. In this case the elements of $\mathfrak{D}$ generate a well-defined subgroup $H(\mathfrak{D})$, and ϕ induces a free action $\bar{\phi}$ of $G/H(\mathfrak{D})$ on the orbit surface $M/(\phi \mid H(\mathfrak{D}))$. Then there is the following general classification theorem.

THEOREM 1.3. *An effective, orientation-preserving action ϕ of a finite abelian group G on a connected, closed, oriented surface M is determined, up to orientation-preserving, equivariant homeomorphism by two invariants:*

(i) *the fixed point data $\mathfrak{D}$ of ϕ and*

(ii) *the free cobordism class of the induced free action of $G/H(\mathfrak{D})$ on $M/(\phi \mid H(\mathfrak{D}))$.*

Invariant (ii) can be identified with an element of $H_2(G/H(\mathfrak{D}); \mathbf{Z})$. Alternatively it can be interpreted as a restricted G-cobordism class of ϕ itself, where each point of the singular set of a G-cobordism is required to have isotropy group lying within $H(\mathfrak{D})$.

When G is a finite cyclic group $\mathbf{Z}/n$, then, since $H_2(\mathbf{Z}/n; \mathbf{Z}) = 0$, Theorem 1.3 specializes to give the result of J. Nielsen [6] that an action of $\mathbf{Z}/n$ is determined by its fixed point data.

When $G = (\mathbf{Z}/p)^r$, p prime, and the genus of M is sufficiently big with respect to r, Theorem 1.3 can be inferred from the computational results of P. A. Smith in [7].

In outline the remainder of this paper is as follows. In Section 2 we translate the problem of classifying free G-actions on a given surface into that of classifying the connected G-covering spaces of a given surface, as in [7]. We also define and interpret the basic invariant in $\mathcal{O}_2^{\text{free}}(G) \simeq H_2(G; \mathbf{Z})$ in a purely algebraic way. In Section 3 we prove the version of Witt's Theorem for symplectic inner product spaces over a local ring. In Section 4, Theorem 1.2 is proved, completing the proof of Theorem 1.1. In Section 5, these results are extended to prove Theorem 1.3. Finally, in Section 6 we briefly discuss the case of nonabelian groups G, to be considered in more detail in a sequel to this paper.

2. Free actions and covering spaces. Let $\mathfrak{F}\mathfrak{A}(G,M)$ denote the set of free, orientation-preserving actions of the finite group G on the connected, closed, oriented surface M. Thus $\mathfrak{F}\mathfrak{A}(G,M)$ consists of injective homomorphisms $\phi: G \to \mathfrak{H}(M)$, where $\mathfrak{H}(M)$ denotes the group of all orientation-preserving homeomorphisms of M, and each $\phi(g)$, $g \neq e$, has no fixed points. The action of $\mathfrak{H}(M)$ on itself by conjugation induces an action of $\mathfrak{H}(M)$ on $\mathfrak{F}\mathfrak{A}(G,M)$. The collection $\mathfrak{F}\mathfrak{A}(G,M)^*$ of $\mathfrak{H}(M)$-orbits is the set of equivalence classes of free actions of G on M.

It follows from the classification of surfaces and the multiplicativity of the Euler characteristic in a covering that the orbit spaces M/ϕ, $\phi \in \mathfrak{F}\mathfrak{A}(G,M)$, are all oriented and orientation-preserving homeomorphic to a given closed, oriented surface N. Therefore, modulo the action of $\mathfrak{H}(M)$, one can assign to each $\phi \in \mathfrak{F}\mathfrak{A}(G,M)$ a connected G-covering of N. Let $\mathrm{Cov}(G,N)$ denote the set of all such connected G-coverings of N. The pullback construction defines an action of $\mathfrak{H}(N)$ on $\mathrm{Cov}(G,N)$; let $\mathrm{Cov}(G,N)^*$ denote the set of $\mathfrak{H}(N)$-orbits, the set of equivalence classes of connected, regular coverings of N with group G.

LEMMA 2.1. *There is a bijection* $\mathfrak{F}\mathfrak{A}(G,M)^* \longleftrightarrow \mathrm{Cov}(G,N)^*$.

The proof is not difficult (cf. [7; pp. 257–259]).

Fix a base point $x_0 \in N$. Let $\mathrm{Epi}(A,B)$ denote the group of epimorphisms of the group A onto the group B.

LEMMA 2.2. *There is a bijection* $\mathrm{Cov}(G,N) \Longleftrightarrow \mathrm{Epi}(\pi_1(N,x_0),G)$.

The proof is an exercise in covering space theory and is omitted.

If two G-coverings of N are equivalent, it follows from the homogeneity of surfaces and the homotopy lifting property of coverings that the two coverings are equivalent via a base point preserving homeomorphism of N.

Let $\mathcal{H}(N,x_0)$ denote the group of base point preserving, orientation-preserving homeomorphisms of N. Then there is an induced homomorphism

$$\mathcal{H}(N,x_0) \to \mathrm{Aut}(\pi_1(N,x_0))$$

which is surjective. (Since $N=S^2$ or is aspherical, any automorphism of $\pi_1(N,x_0)$ can be realized by a based homotopy equivalence; by results due originally to Nielsen, this homotopy equivalence is homotopic to a homeomorphism. See [3; 13.1] for a reasonably simple proof of this fact.) Now $\mathrm{Aut}(\pi_1(N,x_0))$ acts on $\mathrm{Epi}(\pi_1(N,x_0),G)$ by pre-composition: $\alpha\cdot\phi=\phi\circ\alpha^{-1}$. Let $\mathrm{Epi}(\pi,G)^*$ denote the set of $\mathrm{Aut}(\pi)$-orbits. These remarks provide a proof of the following lemma.

LEMMA 2.3. *There is a bijection* $\mathrm{Cov}(G,N)^* \Longleftrightarrow \mathrm{Epi}(\pi_1(N,x_0),G)^*$.

Let $\mathcal{O}_2^{\mathrm{free}}(G)$ denote the set of free, oriented G-cobordism classes of free oriented G-surfaces. (See [2], for example.) There is then the bordism invariant

$$\mathbf{B}: \mathcal{FA}(G,M) \to \mathcal{O}_2^{\mathrm{free}}(G),$$

which factors through $\mathcal{FA}(G,M)^*$. From the point of view of covering spaces one obtains a similar bordism invariant $\mathbf{B}:\mathrm{Cov}(G,N)\to\Omega_2(K(G,1))$ where $K(G,1)$ denotes an Eilenberg–MacLane space of type $(G,1)$. Here $\mathbf{B}$ assigns to a G-covering the oriented cobordism class of the classifying map $N\to K(G,1)$ of the covering.

Now one can see directly or via the bordism spectral sequence that

$$\Omega_2(K(G,1)) \simeq H_2(K(G,1);\mathbf{Z}) \simeq H_2(G;\mathbf{Z}).$$

In terms of Epi, then, we obtain

$$\mathbf{B}: \mathrm{Epi}(\pi_1(N,x_0),G) \to H_2(G;\mathbf{Z}).$$

If $\phi:\pi_1(N,x_0)\to G$ and $\hat{\phi}:N\to K(G,1)$ is the corresponding map of spaces, then $\mathbf{B}(\phi)=(\hat{\phi})_*[N]$, where $[N]\in H_2(N;\mathbf{Z})$ is the fundamental class of the oriented manifold N.

All three versions of $\mathbf{B}$ factor through the corresponding sets of equivalence classes, yielding for example, $\mathbf{B}^*:\mathrm{Epi}(\pi_1(N,x_0),G)^*\to H_2(G;\mathbf{Z})$.

Of course, in this context $\mathbf{B}$ and $\mathbf{B}^*$ are well-defined on the larger domains $\mathrm{Hom}(\pi_1(N,x_0),G)$ and $\mathrm{Hom}(\pi_1(N,x_0),G)^*$.

Now suppose that G is a finite *abelian* group. Then

$$\mathrm{Epi}(\pi_1(N,x_0),G) \simeq \mathrm{Epi}(H_1(N),G),$$

since the Hurewicz homomorphism identifies $H_1(N)$ as the abelianization of $\pi_1(N,x_0)$. (Throughout this paper all homology groups are understood to have integer coefficients.) Then $H_1(N)$ is a symplectic inner product space over $\mathbf{Z}$, using intersection numbers. The Hurewicz homomorphism induces a surjection

$$\mathrm{Aut}(\pi_1(N,x_0)) \twoheadrightarrow \mathrm{Aut}(H_1(N)),$$

where $\mathrm{Aut}(H_1(N))$ denotes the group of isometries of $H_1(N)$ and consists of those isomorphisms of the group $H_1(N)$ which are induced by homeomorphisms. (Essentially, one constructs explicit generators for $\mathrm{Aut}(H_1(N))$, and then one shows that these generators can be realized by Dehn twist homeomorphisms about appropriate simple closed curves. Cf. [4; p. 178].) Then $\mathrm{Aut}(H_1(N))$ acts on $\mathrm{Epi}(H_1(N),G)$, and there is an induced bijection

$$\mathrm{Epi}(\pi_1(N,x_0),G)^* \longleftrightarrow \mathrm{Epi}(H_1(N),G)^*.$$

Since symplectic inner product spaces V over the integers correspond exactly to the inner product spaces $H_1(N)$, N a closed oriented surface, we obtain the following precise version of Theorem 1.2 stated in the introduction.

THEOREM 2.4. *Let V be a symplectic inner product space over $\mathbf{Z}$. Then the bordism invariant $\mathbf{B}$ induces an injection $\mathbf{B}^*:\mathrm{Epi}(V,G)^* \rightarrowtail H_2(G)$, provided G is a finite abelian group.*

REMARK 2.5. We shall also see that $\mathbf{B}^*$ is surjective if and only if $\dim V \geqslant 2\,\mathrm{rank}\,G$, where $\mathrm{rank}\,G$ is the minimum cardinality of generating sets for G.

The remainder of this section is devoted to characterizing when $\mathbf{B}(\phi_1)=\mathbf{B}(\phi_2)$ in $H_2(G)$, for $\phi_1,\phi_2 \in \mathrm{Epi}(H_1(N),G)$ and to reformulating Theorem 2.4 one last time into the statement which will be proved in Section 4. We note that for this part of the discussion N denotes a closed, oriented surface which need not be connected.

Let $G=\mathbf{Z}/m_1\times\mathbf{Z}/m_2\times\cdots\times\mathbf{Z}/m_r$ where $m_{i+1}\mid m_i$ for $i=1,\ldots,r-1$ (and $r=\mathrm{rank}\,G$). Then $\mathrm{Hom}(V,G)\simeq\prod_{i=1}^r\mathrm{Hom}(V,\mathbf{Z}/m_i)$. Therefore any homomorphism $\phi: V\to G$ can be uniquely expressed as (ϕ^i) in terms of its r coordinate functions.

Now V/m_i is an inner product space over $\mathbf{Z}/m_i$ and as such is canonically isomorphic to $\mathrm{Hom}(V/m_i,\mathbf{Z}/m_i)\simeq\mathrm{Hom}(V,\mathbf{Z}/m_i)$, by the correspondence $v \longrightarrow \phi_v$, where $\phi_v(w)=v\cdot w$. Thus $\mathrm{Hom}(V,\mathbf{Z}/m_i)$ becomes an inner product space over $\mathbf{Z}/m_i$ in a natural way. If $\phi=(\phi^i): V\to G$, then the expression $\phi^i\cdot\phi^j$ makes sense, by appropriate reduction of coefficients, in $\mathbf{Z}/m_j$, if $i\leqslant j$.

PROPOSITION 2.6. *Let $\phi_1,\phi_2 \in \mathrm{Hom}(H_1(N),G)$, where N is a closed, oriented surface and $G=\mathbf{Z}/m_1\times\cdots\times\mathbf{Z}/m_r$ as above. If $\mathbf{B}(\phi_1)=\mathbf{B}(\phi_2)$ in $H_2(G)$, then for $i<j$, $\phi_1^i\cdot\phi_1^j=\phi_2^i\cdot\phi_2^j$ in $\mathbf{Z}/m_j$.*

REMARK 2.7. In particular, for $\phi\in\mathrm{Hom}(H_1(N),G)$, $\mathbf{B}(\phi)$ determines the $\binom{r}{2}$ numbers $\phi^i\cdot\phi^j$ in $\mathbf{Z}/m_j$, $i<j$.

Proof of Proposition 2.6. First let $\phi\in\mathrm{Hom}(H_1(N),G)$ such that $\mathbf{B}(\phi)=0$. We shall show that $\phi^i\cdot\phi^j=0$ in $\mathbf{Z}/m_j$. Since $\mathbf{B}(\phi)=0$, the corresponding map $\hat{\phi}: N\to K(G,1)$ extends to a map $\hat{\theta}: V\to K(G,1)$ where V is a compact, oriented

3-manifold with oriented boundary N. It is a standard consequence of Poincaré duality that there is a symplectic basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ for $H_1(N)$ such that $i_*(a_k)=0$, $k=1,\ldots,n$, where $i:N\to V$ is the inclusion. Then each ϕ^i is a linear combination of $b_1^*, \ldots, b_n^*$ only, and hence $\phi^i\cdot\phi^j=0$ in $\mathbf{Z}/m_j$.

More generally, suppose $\phi_1, \phi_2 \in \mathrm{Hom}(H_1(N), G)$ such that $\mathbf{B}(\phi_1)=\mathbf{B}(\phi_2)$. Let $\bar{\phi}_2$ denote ϕ_2 in $\mathrm{Hom}(H_1(\bar{N}), G)$, where $\bar{N}$ denotes N with the opposite orientation. Then $\bar{\phi}_2^i\cdot\bar{\phi}_2^j=-\phi_2^i\cdot\phi_2^j$. Consider $\phi_1\oplus\bar{\phi}_2 \in \mathrm{Hom}(H_1(N\amalg\bar{N}), G)$. It follows that $\mathbf{B}(\phi_1\oplus\bar{\phi}_2)=\mathbf{B}(\phi_1)-\mathbf{B}(\phi_2)=0$, so that the first case shows that

$$(\phi_1\oplus\bar{\phi}_2)^i\cdot(\phi_1\oplus\bar{\phi}_2)^j=0.$$

But $(\phi_1\oplus\bar{\phi}_2)^k=\phi_1^k\oplus\bar{\phi}_2^k$, and since $N\cap\bar{N}=\varnothing$, $\phi_1^k\cdot\bar{\phi}_2^j=0$ and $\bar{\phi}_2^i\cdot\phi_1^j=0$, it follows that $\phi_1^i\cdot\phi_1^j=\phi_2^i\cdot\phi_2^j$ in $\mathbf{Z}/m_j$, as required. □

Using Proposition 2.6 we can at last formulate Theorems 1.2 and 2.4 in the version to be proved in Section 4, without any reference to homology.

THEOREM 2.8. *Let V be a symplectic inner product space over $\mathbf{Z}$; let*

$$G=\mathbf{Z}/m_1\times\mathbf{Z}/m_2\times\cdots\times\mathbf{Z}/m_r,$$

where $m_{i+1}\mid m_i$ for $i<r$; and let $\phi_1, \phi_2 \in \mathrm{Epi}(V, G)$. Then there exists $\alpha\in\mathrm{Aut}\,V$ such that $\phi_2=\phi_1\circ\alpha$ if and only if, for $1\leqslant i<j\leqslant r$, $\phi_1^i\cdot\phi_1^j=\phi_2^i\cdot\phi_2^j$ in $\mathbf{Z}/m_j$.

REMARK 2.9. It follows from Theorem 2.8 that the converse of Proposition 2.6 is true.

3. Witt's Theorem over a local ring. In this section we prove the following generalization of Witt's Theorem.

THEOREM 3.1. *Let V be a symplectic inner product space over a local ring R, and let U_1 and U_2 be free R-module summands of V. Then any isometry $U_1\to U_2$ extends to an isometry of V.*

When R is a field this is proved, for example, in Artin [1; p. 121]. We shall establish some formalism for reducing the general case to that of a field. This formalism will be used in Section 4 as well.

A *Witt problem* in the symplectic inner product space V consists of r independent vectors $x_1, x_2, \ldots, x_r$ spanning a free summand of V of dimension r and $r+1$ independent vectors $y_1, y_2, \ldots, y_{r+1}$ spanning a free summand of V of dimension $r+1$ such that $x_i\cdot x_j=y_i\cdot y_j$ for $i,j\leqslant r$. A *solution* of the Witt problem $\{x_i, y_j\}$ is a vector $x_{r+1}\in V$ such that

(i) $x_i\cdot x_{r+1}=y_i\cdot y_{r+1}$, $i\leqslant r$, and

(ii) $x_1, \ldots, x_{r+1}$ spans a free summand of V of dimension $r+1$.

If x_{r+1} satisfies (i), then x_{r+1} is called a *partial solution*.

Now Theorem 3.1 is equivalent to the assertion that all Witt problems are solvable.

In the following lemmas all rings are understood to be commutative and have a unit element. All ring homomorphisms must preserve the unit elements.

LEMMA 3.2. *Let V be a symplectic inner product space over a ring R, and let $R \to S$ be a ring homomorphism (mapping unit element to unit element). Then the induced homomorphism $V \to S \otimes_R V$ maps Witt problems in V to Witt problems in $S \otimes_R V$, where $S \otimes_R V$ is given the induced inner product.*

The proof is clear, since $S \otimes_R -$ preserves free summands.

LEMMA 3.3. *Let W be a free summand of a symplectic inner product space V over a ring R. Let $\rho : R \to S$ be a ring homomorphism. Then $S \otimes_R (W^\perp) = (S \otimes_R W)^\perp$ in $S \otimes_R V$.*

Proof. By definition $W^\perp$ is given by a short exact sequence

$$0 \to W^\perp \to V \to \mathrm{Hom}_R(W, R) \to 0$$

which splits since $\mathrm{Hom}_R(W,R)$ is free over R. This leads to a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & S\otimes W^\perp & \to & S\otimes V & \to & S\otimes \mathrm{Hom}_R(W,R) & \to 0 \\ & \downarrow \alpha & & = \downarrow \beta & & \downarrow \gamma & \\ 0 \to & (S\otimes W)^\perp & \to & S\otimes V & \to & \mathrm{Hom}_S(S\otimes W, S) & \to 0 \end{array}$$

The middle vertical arrow β is the identity and the left one α is an inclusion. The right one γ is $s\otimes f \mapsto g$ where $g(t\otimes w) = st\rho(f(w))$.

Now $S\otimes \mathrm{Hom}_R(W,R)$ and $\mathrm{Hom}_S(S\otimes W, S)$ are both free S-modules, and γ is easily seen to induce a bijection between the bases constructed from an R-basis of W. Thus γ is an isomorphism. Therefore α is also an isomorphism, as required. □

LEMMA 3.4. *Let $R \to S$ be a ring epimorphism, let V be a symplectic inner product space over R, and let $\{x_i, y_i\}$ be a Witt problem in V which has a partial solution. Then the set of partial solutions in V maps onto the set of partial solutions of the induced Witt problem $\{\bar{x}_i, \bar{y}_i\}$ in $S \otimes_R V$.*

Proof. Any two partial solutions in V of the given Witt problem differ by an element of $\langle x_1, \dots, x_r\rangle^\perp = \{v \in V : v\cdot x_i = 0,\ i = 1, \dots, r\}$, which thus parametrizes the set of partial solutions. The image $\bar{x}_{r+1}$ in $S\otimes_R V$ of a partial solution x_{r+1} in V is clearly a partial solution of the induced Witt problem $\{\bar{x}_i, \bar{y}_i\}$. Therefore the set of partial solutions of the induced Witt problem is parametrized by $\langle \bar{x}_1, \dots, \bar{x}_r\rangle^\perp$. But under $V \to S\otimes_R V$, $\langle x_1, \dots, x_r\rangle^\perp$ maps *onto* $\langle \bar{x}_1, \dots, \bar{x}_r\rangle^\perp$ by Lemma 3.3. The result follows. □

Proof of Theorem 3.1. It suffices to show that any Witt problem $\{x_1, \dots, x_r; y_1, \dots, y_{r+1}\}$ in the symplectic inner product space V over the local ring R has a solution. Let $R \to F$ be the projection of R onto the field obtained by dividing out the maximal ideal of nonunits.

We first show that the Witt problem has partial solutions. We seek x_{r+1} such that $x_i \cdot x_{r+1} = y_i \cdot y_{r+1}$ for $i \leq r$. If V has dimension $n = 2m$ and we fix a symplectic basis $e_1, \dots, e_n$ with respect to which the form has matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, this amounts to r equations in n unknowns. Let $x_i = \sum a_{ij} e_j$ and $c_i = y_i \cdot y_{r+1}$. Then we have the system of equations $Ax = c$, where

$$A = \begin{pmatrix} -a_{1,m+1} & -a_{1,m+2} & \dots & -a_{1n} & a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -a_{r,m+1} & -a_{r,m+2} & \dots & -a_{rn} & a_{r1} & a_{r2} & \dots & a_{rm} \end{pmatrix}.$$

The matrix A has rank r, since it is equivalent to the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{r1} & \dots & a_{rn} \end{pmatrix},$$

and $\{x_1, \dots, x_r\}$ forms a basis for a free summand of V. Therefore $Ax = c$ has solutions, and the given Witt problem has partial solutions.

On the other hand, the induced Witt problem $\{\bar{x}_i; \bar{y}_j\}$ in $F \otimes_R V$ has solutions by the classical Witt Theorem. By Lemma 3.4 there is a partial solution x_{r+1} of the original problem whose image $\bar{x}_{r+1}$ is a solution of the induced problem. We claim that x_{r+1} actually solves the given Witt problem. It suffices to show that the $2n \times (r+1)$ matrix given by $(x_1, x_2, \dots, x_{r+1})$ has rank $r+1$. The induced matrix $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{r+1})$ has rank $r+1$ since $\bar{x}_{r+1}$ solves the induced Witt problem. That is, some $(r+1) \times (r+1)$ submatrix has nonzero determinant in F. But then the corresponding submatrix over R has unit determinant, completing the proof. □

We shall use the following slightly stronger statement.

COROLLARY 3.5. *Let $\{x_1, \dots, x_r; y_1, \dots, y_{r+1}\}$ be a Witt problem in a symplectic inner product space V over a local ring R. Let $R \twoheadrightarrow S$ be an epimorphism of local rings. Then the given Witt problem has solutions, and the set of solutions maps onto the set of solutions of the induced Witt problem in $S \otimes V$.*

The proof of the first clause is explicitly given above. The proof of the second clause is obtained from the final paragraph of the proof of Theorem 3.1 by replacing references to the field F with references to S.

4. Proof of the free classification theorem. After two lemmas, the proof of Theorem 2.8 is given in three stages.

In this section, V denotes a symplectic inner product space over $\mathbf{Z}$ of dimension $2n$ and V/m denotes $\mathbf{Z}/m \otimes V$.

LEMMA 4.1. *Reduction modulo an integer m, $\rho: V \to V/m$, induces a surjection* $\operatorname{Aut}(V) \twoheadrightarrow \operatorname{Aut}(V/m)$—i.e. $\mathrm{Sp}_{2n}(\mathbf{Z}) \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbf{Z}/m)$.

Proof. It suffices to show that any symplectic basis $a_1, \dots, a_n, b_1, \dots, b_n$ of V/m is the image of some symplectic basis $a_1', \dots, a_n', b_1', \dots, b_n'$ of V. Choose a_1' to be any indivisible element of V such that $\rho(a_1') = a_1$. Let $c_1 \in V$ such that $a_1' \cdot c_1 = 1$ and let $d_1 \in V$ be any indivisible element such that $\rho(d_1) = b_1$. Then $a_1' \cdot d_1 = km + 1$ for some k. Set $b_1' = d_1 - kmc_1$. Then $\rho(b_1') = \rho(d_1) = b_1$ and $a_1' \cdot b_1' = 1$. Since the orthogonal splittings $V = \langle a_1', b_1' \rangle \oplus \langle a_1', b_1' \rangle^{\perp}$ and

$$V/m = \langle a_1, b_1 \rangle \oplus \langle a_2, \dots, a_n, b_2, \dots, b_n \rangle$$

respect reduction modulo m, an induction on dimension completes the proof. □

LEMMA 4.2. *If the positive integer m has prime factorization* $m = p_1^{k_1} \dots p_s^{k_s}$, *then the reductions* $\rho_i : V/m \to V/p_i^{k_i}$ *induce an isomorphism* $\operatorname{Aut}(V/m) \simeq \bigoplus_i \operatorname{Aut}(V/p_i^{k_i})$ —i.e. $\mathrm{Sp}_{2n}(\mathbf{Z}/m) \simeq \bigoplus_i \mathrm{Sp}_{2n}(\mathbf{Z}/p_i^{k_i})$.

Proof. Certainly on the level of finite abelian groups, V_m splits as the direct sum of its primary components: $V/m \simeq \bigoplus V/p_i^{k_i}$. Any automorphism of V/m respects this splitting and induces automorphisms of the primary components. On the other hand the inner product on V/m is identified with the sum of the inner products of the primary components. Thus the direct sum of isometries $V/p_i^{k_i} \to V/p_i^{k_i}$ yields an isometry of V/m, as required. □

I. *Reduction of the general case to that of a p-group.* Let G be an arbitrary finite abelian group of order $m = p_1^{k_1} \dots p_s^{k_s}$. Let $G_i \subset G$ be the p_i-primary component of G, so that $G = G_1 \times \cdots \times G_s$. Then there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Epi}(V, G) & \xrightarrow{\simeq} & \prod \operatorname{Epi}(V, G_i) \\ \simeq \downarrow & & \simeq \downarrow \\ \operatorname{Epi}(V/m, G) & \xrightarrow{\simeq} & \prod \operatorname{Epi}(V/p_i^{k_i}, G_i) \end{array}$$

in which the vertical arrows are bijections induced by reduction of coefficients, and the horizontal arrows are bijections induced by the projections of G and V/m onto their primary components.

There is an induced commutative diagram obtained by dividing out the actions of $\operatorname{Aut} V$, $\operatorname{Aut} V/m$, and $\operatorname{Aut} V/p_i^{k_i}$, $i = 1, \dots, s$:

$$\begin{array}{ccc} \operatorname{Epi}(V, G)^* & \to & \prod \operatorname{Epi}(V, G_i)^* \\ \downarrow & & \downarrow \\ \operatorname{Epi}(V/m, G)^* & \to & \prod \operatorname{Epi}(V/p_i^{k_i}, G_i)^* \end{array}$$

By Lemma 3.1 the vertical arrows are bijections. By Lemma 3.2 the lower horizontal arrow is a bijection. It follows that $\operatorname{Epi}(V, G)^* \simeq \prod_i \operatorname{Epi}(V, G_i)^*$.

On the other hand, if $G = \mathbf{Z}/m_1 \times \cdots \times \mathbf{Z}/m_r$, where each $m_{i+1} \mid m_i$, and $\phi_1, \phi_2 \in \operatorname{Epi}(V, G)$, then there are the components $\phi_1^1, \dots, \phi_1^r$ and $\phi_2^1, \dots, \phi_2^r$ determined by the projections of G onto each $\mathbf{Z}/m_i$. Also each ϕ_k^i has its p_j-primary component ϕ_k^{ij}.

Now $\phi_k^{ij} \cdot \phi_k^{lj}$ is the p_j-primary component of $\phi_k^i \cdot \phi_k^l$ in $\mathbf{Z}/m_l$, for $i < l$. It follows that, for $i < l$, $\phi_1^i \cdot \phi_1^l = \phi_2^i \cdot \phi_2^l$ in $\mathbf{Z}/m_l$ if and only if $\phi_1^{ij} \cdot \phi_1^{lj} = \phi_2^{ij} \cdot \phi_2^{lj}$ in the p_j-primary part of $\mathbf{Z}/m_l$ for all j. In particular, the left vertical arrow in the following commutative diagram of natural transformations is an isomorphism.

$$\begin{array}{ccc} \operatorname{Epi}(V, G)^* & \xrightarrow{\mathbf{B}^*} & H_2(G) \\ \simeq \downarrow & & \simeq \downarrow \\ \prod_i \operatorname{Epi}(V, G_i)^* & \xrightarrow{\mathbf{B}^*} & \prod_i H_2(G_i) \end{array}$$

The right vertical arrow is an isomorphism by the Kunneth formula. Therefore the lower arrow is injective if and only if the upper arrow is injective, which completes the proof of the reduction to the p-primary case. □

II. *Reduction of the p-group case to that of* $G=(\mathbf{Z}/p^k)^r$. Suppose that $G=\mathbf{Z}/p^{k_1}\times\cdots\times\mathbf{Z}/p^{k_r}$ where p is a prime and $k_1\geqslant k_2\geqslant\cdots\geqslant k_r$. Let

$$\tilde{G}=\mathbf{Z}/p^{k_1}\times\mathbf{Z}/p^{k_1}\times\cdots\times\mathbf{Z}/p^{k_1}=(\mathbf{Z}/p^{k_1})^r.$$

Then there is a natural surjective homomorphism $\tilde{G}\twoheadrightarrow G$. Given any $\phi\in\mathrm{Epi}(V,G)$, there is always a lift $\tilde{\phi}:V\to\tilde{G}$ since V is free. Such a lift $\tilde{\phi}$ is automatically an epimorphism in this situation: one must verify that the induced homomorphism $V/p^{k_1}\to\tilde{G}$ has rank r over $\mathbf{Z}/p^{k_1}$ and this follows since the further induced homomorphism $V/p^{k_r}\to\mathbf{Z}/p^{k_r}\times\cdots\times\mathbf{Z}/p^{k_r}$ has rank r over $\mathbf{Z}/p^{k_r}$.

It suffices to show that if $\phi_1,\phi_2\in\mathrm{Epi}(V,G)$, with $\mathbf{B}(\phi_1)=\mathbf{B}(\phi_2)$, and $\tilde{\phi}_1\in\mathrm{Epi}(V,\tilde{G})$ is a lift of ϕ_1, then there is a lift $\tilde{\phi}_2\in\mathrm{Epi}(V,\tilde{G})$ of ϕ_2 such that $\mathbf{B}(\tilde{\phi}_1)=\mathbf{B}(\tilde{\phi}_2)$.

A straightforward inductive argument, lifting one coordinate function one step at a time reduces one to the case when $G=\mathbf{Z}/p^k\times\cdots\times\mathbf{Z}/p^k\times\mathbf{Z}/p^l$, $l<k$, and $\tilde{G}=\mathbf{Z}/p^k\times\cdots\times\mathbf{Z}/p^k\times\mathbf{Z}/p^k$, each of s factors, say. Given ϕ_1, ϕ_2 and $\tilde{\phi}_1$, the problem of finding $\tilde{\phi}_2$ is a Witt problem in the inner product space $\mathrm{Hom}(V,\tilde{G})$, of the sort discussed in Section 3. Each of ϕ_1, ϕ_2, and $\tilde{\phi}_1$ has s components. The first $s-1$ components of $\tilde{\phi}_2$ are necessarily those of ϕ_2. To find $\tilde{\phi}_2^s$ is a Witt problem over $\mathbf{Z}/p^k$ which has a solution ϕ_2^s over $\mathbf{Z}/p^l$. By Corollary 3.5, there is a choice of $\tilde{\phi}_2^s$ which maps to ϕ_2^s and solves the Witt problem, as required. □

III. *Completion of the proof of Theorem* 2.8 *when* $G=(\mathbf{Z}/p^k)^r$. Let

$$\phi_1,\phi_2\in\mathrm{Epi}(V,G)$$

with components ϕ_m^i $(m=1,2;\ 1\leqslant i\leqslant r)$ and suppose that $\phi_1^i\cdot\phi_1^j=\phi_2^i\cdot\phi_2^j$ for $1\leqslant i<j\leqslant r$. Then $\phi_1^1,\ldots,\phi_1^r$ and $\phi_2^1,\ldots,\phi_2^r$ each span free r-dimensional summands of the symplectic inner product space $\mathrm{Hom}(V,\mathbf{Z}/p^k)\cong\mathrm{Hom}(V/p^k,\mathbf{Z}/p^k)$ over the local ring $\mathbf{Z}/p^k$. Let $\bar{\phi}_m^i:V/p^k\to\mathbf{Z}/p^k$ denote the epimorphism induced by ϕ_m^i for $m=1,2$ and $1\leqslant i\leqslant r$. By Theorem 3.1 there is an automorphism γ of $\mathrm{Hom}(V/p^k,\mathbf{Z}/p^k)$ such that $\gamma(\bar{\phi}_2^i)=\bar{\phi}_1^i$ for $i=1,\ldots,r$. Since V/p^k and $\mathrm{Hom}(V/p^k,\mathbf{Z}/p^k)$ are naturally isomorphic as inner product spaces over $\mathbf{Z}/p^k$, γ is induced by an isometry β such that $\bar{\phi}_2^i\circ\beta=\bar{\phi}_1^i$ for $i=1,\ldots,r$. By Lemma 4.1, β is induced by an isometry α of V. Then $\phi_2^i\circ\alpha=\phi_1^i$ for $1\leqslant i\leqslant r$. This completes the proof of Step III and of Theorem 2.8. □

The remainder of this section is devoted to determining when the bordism invariant $\mathbf{B}:\mathrm{Epi}(V,G)\to H_2(G)$ is surjective.

PROPOSITION 4.3. *If the zero element* $0\in H_2(G)$ *lies in the image of* $\mathbf{B}$, *then* $\dim V\geqslant 2\operatorname{rank} G$.

Proof. Let $\phi:V\twoheadrightarrow G$ such that $\mathbf{B}(\phi)=0$. Realize ϕ by a map $\hat{\phi}:N\to K(G,1)$, where N is a closed, oriented surface with $H_1(N)=V$. Since $\mathbf{B}(\phi)=0$, $\hat{\phi}$ extends to a map $\phi:W\to K(G,1)$ where W is a compact, oriented 3-manifold with oriented boundary N. It is a standard consequence of Poincaré–Lefschetz duality that there is a symplectic basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ for $H_1(N)$ such that $i_*(a_j) = 0$, $1 \leqslant j \leqslant n$. Identifying V with $H_1(N)$, it follows that $\phi(a_j) = 0$, $1 \leqslant j \leqslant n$. Therefore G is generated by $\phi(b_1), \ldots, \phi(b_n)$. Thus $n \geqslant \operatorname{rank} G$, as required. □

PROPOSITION 4.4. *If* $\dim V \geqslant 2 \operatorname{rank} G$, *then* $\mathbf{B}: \operatorname{Epi}(V, G) \to H_2(G)$ *is surjective.*

Proof. By the commutative diagram in Step I of the proof of Theorem 2.8 one can assume that G is p-primary for some prime p. Suppose $\tilde{G}$ is the p-group considered in that proof, Step II. Then $H_2(\tilde{G}) \to H_2(G)$ is surjective. Therefore it suffices to prove the result for $G = \tilde{G}$. That is, we may assume $G = (\mathbf{Z}/p^k)^r$.

Given a collection $\{t_{ij} \in \mathbf{Z}/p^k : 1 \leqslant i < j \leqslant r\}$ of $\binom{r}{2}$ integers modulo p^k, one must construct an epimorphism $\phi: V \to G$ such that $\phi^i \cdot \phi^j = t_{ij}$ for all i, j. Let $\sigma_1, \ldots, \sigma_n$, $\tau_1, \ldots, \tau_n$ be a symplectic basis for $\operatorname{Hom}(V, \mathbf{Z}/p^k)$ over $\mathbf{Z}/p^k$. For $i = 1, \ldots, r \leqslant n$, set $\phi^i = \tau_i + \sum_{i<k} t_{ik}\sigma_k$. Then if $i < j$,

$$\phi^i \cdot \phi^j = \left(\tau_i + \sum_{k>i} t_{ik}\sigma_k\right) \cdot \left(\tau_j + \sum_{l>j} t_{jl}\sigma_l\right) = t_{ij}\sigma_j \cdot \tau_j = t_{ij}.$$

Matrix considerations show $\phi^1, \ldots, \phi^r$ span a free r-dimensional summand of $\operatorname{Hom}(V, \mathbf{Z}/p^k)$ and define the desired element of $\operatorname{Epi}(V, G)$. □

REMARK 4.5. The action of $\operatorname{Aut}(V)$ on $\operatorname{Epi}(V, G)$ extends to an action of $\operatorname{Aut}(V) \times \operatorname{Aut}(G)$. Similarly $\operatorname{Aut}(G)$ acts on $H_2(G)$. The naturality of $\mathbf{B}$ implies that there is an induced injection $\operatorname{Aut}(V) \backslash \operatorname{Epi}(V, G)/\operatorname{Aut}(G) \rightarrowtail H_2(G)/\operatorname{Aut}(G)$. The term $\operatorname{Aut}(V) \backslash \operatorname{Epi}(V, G)/\operatorname{Aut}(G)$ corresponds to the "weak" equivalence classes of free G actions on the appropriate surface. In any specific case one can compute $H_2(G)/\operatorname{Aut}(G)$. For example, if $G = (\mathbf{Z}/p)^r$, p prime, then $H_2(G)/\operatorname{Aut}(G)$ consists of exactly 2 elements, detected by bordism. Thus, in this case, a given surface admits 0, 1, or 2 weak equivalence classes of free G actions.

5. The general classification theorem. In this section we reformulate, prove, and interpret the general classification of finite abelian group actions on surfaces via fixed point data and appropriate equivariant cobordism class.

Let G be an arbitrary finite group and let ϕ be an effective, orientation-preserving action of G on a connected, closed, oriented surface M. The *singular set* $S = S_\phi$ is the set of points of M with nontrivial isotropy group; the *branch set* $B = B_\phi$ is the image of S in the orbit surface $N = M/\phi$. The *G-branched covering* $\pi: M \to N$ is determined by a surjective homomorphism $\rho: \pi_1(N - B, x_0) \to G$, where (M, ϕ) corresponds to the end compactification of the corresponding G-covering space of $N - B$.

Let $B = \{x_1, \ldots, x_n\}$ and let $C_1, \ldots, C_n$ be small simple closed curves in $N - B$ such that C_i bounds a disk D_i and $D_i \cap B = \{x_i\}$, $i = 1, \ldots, n$. Each C_i inherits an orientation from the orientation on D_i which is induced from that on M. If each C_i is connected to the base point x_0 by a path, then elements $\gamma_i \in \pi_1(N - B, x_0)$ are determined. The set of conjugacy classes of the elements $\rho(\gamma_1), \ldots, \rho(\gamma_n)$ of G (counted with multiplicities) is the set of *fixed point data* $\mathfrak{D} = \mathfrak{D}(\phi)$ and depends (up to order) only on the action ϕ and the orientation of M.

Note that it follows from the Riemann-Hurwitz formula and the classification of surfaces that two actions of G on M with the same fixed point data have homeomorphic orbit surfaces.

From now on assume that G is abelian. Then we may view ρ as defined on homology and suppress base point reference: $\rho: H_1(N-B) \to G$. If $[C_i]$ denotes the homology class of the oriented curve C_i, then the set of fixed point data $\mathfrak{D}$ becomes the set (with multiplicities) $\{\rho[C_1], \ldots, \rho[C_n]\}$ of nonzero elements of G. Note that $\rho[C_1]\ldots\rho[C_n] = \rho([C_1]+\cdots+[C_n]) = 0$.

Let $H = H(\mathfrak{D})$ be the subgroup of G generated by the fixed point data. H is well-defined since G is abelian. Let $\pi: G \to G/H$ denote the projection. Then $\pi\rho$ factors through $H_1(N)$:

$$\begin{array}{ccc} H_1(N-B) & \xrightarrow{\rho} & G \\ i_* \downarrow & & \downarrow \pi \\ H_1(N) & \xrightarrow{\bar{\rho}} & G/H. \end{array}$$

Corresponding to the induced homomorphism $\bar{\rho}$ there is a *free* action $\bar{\phi}$ induced on $M/(\phi \mid H)$.

For a given set $\mathfrak{D} = \{g_1, \ldots, g_n\}$ of fixed point data consisting of nontrivial elements of G with product the identity, let $\mathfrak{A}(G,M;\mathfrak{D})$ denote the set of effective, orientation-preserving actions of G on M with fixed point data $\mathfrak{D}$. Then the homeomorphism group $\mathfrak{H}(M)$ acts on $\mathfrak{A}(G,M;\mathfrak{D})$ with orbit space $\mathfrak{A}(G,M;\mathfrak{D})^*$, and we have the following reformulation of the Classification Theorem 1.3.

THEOREM 5.1. *Let G be a finite abelian group, let $\mathfrak{D}$ be a set of fixed point data, and let M be a connected, closed, oriented surface. Then there is an injection* $\mathbf{B}^*: \mathfrak{A}(G,M;\mathfrak{D})^* \longmapsto H_2(G/H(\mathfrak{D}))$, *induced by* $\phi \longmapsto \mathbf{B}^*(\bar{\phi})$.

REMARK 5.2. The target $H_2(G/H(\mathfrak{D}))$ of $\mathbf{B}$ in Theorem 5.1 may be geometrically interpreted here as the equivariant cobordism group of G-actions modulo G-cobordisms in which all isotropy subgroups lie in $H(\mathfrak{D})$.

Proof of Theorem 5.1. Let ϕ_1 and ϕ_2 be effective actions of G on M with fixed point data $\mathfrak{D}$ such that $\mathbf{B}(\bar{\phi}_1) = \mathbf{B}(\bar{\phi}_2)$ where $\bar{\phi}_k$ denotes the induced free action of G/H on $M/(\phi_k \mid H)$, $H = H(\mathfrak{D})$. We may identify the two orbit surfaces $M/\phi_1 \cong N \cong M/\phi_2$ in such a way that B_{ϕ_1} is identified with B_{ϕ_2}, and corresponding points have the same fixed point data. Then for $k=1,2$, $\bar{\phi}_k$ is determined by an epimorphism $\bar{\rho}_k: H_1(N) \to G$. By the free classification Theorem 1.1, there is a homeomorphism $f: N \to N$ such that $\bar{\rho}_2 = \bar{\rho}_1 \circ f_*$. It may be assumed, by the homogeneity of manifolds, that f is the identity on $B = B_{\phi_1} = B_{\phi_2} \subset N$. It follows that, changing ϕ_1 by an equivalence if necessary, we may identify $M/(\phi_1 \mid H)$ and $M/(\phi_2 \mid H)$ (by the lift of f) and assume that $\bar{\phi}_1 = \bar{\phi}_2$. In particular $\pi\rho_1 = \pi\rho_2$ where $\pi: G \to G/H$ is the projection and $\rho_k: H_1(N-B) \to G$ determines ϕ_k, $k=1,2$. Then the algebraic difference $\rho_1 - \rho_2: H_1(N-B) \to G$, defined by $(\rho_1-\rho_2)(x) = \rho_1(x) - \rho_2(x)$, has image in H.

Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be a symplectic basis for $H_1(N)$ represented by simple closed curves $A_1, \ldots, A_g, B_1, \ldots, B_g$ in $N-B$ such that

$$A_i \cap A_j = A_i \cap B_j = B_i \cap B_j = \varnothing$$

if $i \neq j$, and $A_i \cap B_i$ consists of a single point of transverse intersection. We shall show how to alter these curves into a similar new set $A_1', \ldots, A_g'$, $B_1', \ldots, B_g' \subset N-B$, representing a new symplectic basis $a_1', \ldots, a_g'$, $b_1', \ldots, b_g'$ for $H_1(N)$ such that $\rho_2(a_i') = \rho_1(a_i)$ and $\rho_2(b_i') = \rho_1(b_i)$, $1 \leqslant i \leqslant g$. There is then a homeomorphism of N which takes a_i' to a_i and b_i' to b_i, is the identity on B, and induces the desired equivalence between ρ_1 and ρ_2, hence between ϕ_1 and ϕ_2.

Let $C_1, \ldots, C_n$ be the small simple closed curves in $N-B$ considered earlier, chosen to miss each A_i and B_i.

Consider the following operation which alters A_i (or B_i) and leaves the remaining curves unchanged.

(5.3) Replace A_i (or B_i) by its band connected sum with a parallel copy of C_j.

The effect on the corresponding basis of $H_1(N)$ is to replace a_i by $a_i \pm c_j$ (or b_i by $b_i \pm c_j$), while all other entries in the basis of $H_1(N)$ remain unchanged. Both the + sign and the − sign may be realized by appropriate choice of the connecting band.

By repeated application of operation (5.3) one simply reduces the lengths of each $\rho_1(a_i)\rho_2(a_i)^{-1}$ and $\rho_1(b_i)\rho_2(b_i)^{-1}$ when expressed as a product of the fixed generators $\rho_1[C_1], \ldots, \rho_1[C_n]$ of H. This completes the proof. □

REMARK 5.4. The operations of coning off boundary components or deleting the interiors of invariant disks centered at singular points indicate how to convert Theorem 5.1 to a classification of actions on surfaces with boundary.

6. Concluding remarks. The main classification results of this paper definitely do not extend as given to actions of nonabelian finite groups. It remains a problem of some interest to decide just how far the results do extend and to construct new invariants of actions of nonabelian groups.

In Part II of this work we shall show that free actions of metacyclic groups are classified by cobordism. In addition, we shall analyze "indecomposable" actions—actions with exactly 3 singular orbits and orbit space the sphere. For the metacyclic groups and finite matrix groups $SL_2(F_q)$, it will be shown that indecomposable actions are determined up to *weak* equivalence by their fixed point data. On the other hand, two indecomposable actions of the symmetric group on 7 letters will be constructed which have the same fixed point data and the same cobordism class in the sense of Section 5, but which are not weakly equivalent.

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