# SOME ORDERINGS INDUCED BY SPACES OF ANALYTIC FUNCTIONS

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We consider here some orderings on subsets of the complex plane (or the Riemann sphere) that are induced by certain spaces of analytic (or meromorphic) functions, and write $E \leq_S E'$, where $S$ is the space of functions under consideration. The problem that arises is to give geometrical conditions on $E$ and $E'$ that $E \leq_S E'$. We give complete solutions when $S = N$, the space of normal meromorphic functions in the open unit disc $\mathbf{D}$, and when $S = H^\infty(\mathbf{D})$, the space of bounded analytic functions in $\mathbf{D}$. We briefly discuss the problem when $S = B$, the Bloch space, and when $S = H^p(\mathbf{D})$, $0 < p < \infty$, the Hardy space in $\mathbf{D}$. These problems have a family resemblance, but are different in detail.

*Definition* N. Let $E$ and $E'$ be subsets of the extended complex plane $\mathbf{C}^\wedge$. Suppose that for every meromorphic function $f$ in $\mathbf{D}$,

$$[\sup\{f^\#(z)(1-|z|^2) : f(z) \in E'\} < \infty] \Rightarrow [\sup\{f^\#(z)(1-|z|^2) : f(z) \in E\} < \infty]$$

We then say $E \leq_N E'$.

*Problem* N. (Solved in this paper.) Find geometrical necessary and sufficient conditions on $E$ and $E'$ that $E \leq_N E'$.

*Remark* N. We emphasize that in this and our other order definitions, $z$ runs over $f^{-1}(E')$ and $f^{-1}(E)$ respectively, and not $z$ runs over $E'$ amd $E$ respectively, which would give the problems an entirely different flavor. In [6], it was proved (in other language) that if card $E' \geq 5$, then $E \leq_N E'$ for $E = \mathbf{C}^\wedge$ and hence for every set $E$. The letter N is used here for "Normal"—$f$ is normal [1] if and only if $\sup\{f^\#(z)(1-|z|^2) : z \in \mathbf{D}\} < \infty$. where $f^\#$ is the spherical derivative

$$f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

Our main theorem gives necessary and sufficient geometrical conditions that $E \leq_N E'$.

*Definition* B. Let $E$ and $E'$ be subsets of the complex plane $\mathbf{C}$. Suppose that for every analytic function $f$ in $\mathbf{D}$,

$$[\sup\{|f'(z)|(1-|z|^2) : f(z) \in E'\} < \infty] \Rightarrow [\sup\{|f'(z)|(1-|z|^2) : f(z) \in E\} < \infty].$$

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We then say $E \leq_B E'$.

*Problem* B. Find geometrical necessary and sufficient conditions on $E$ and $E'$ *that* $E \leq_B E'$.

*Remark* B. In [11], it was proved (in other language) that $\mathbf{C} \leq_B E$ if and only if there is an upper bound on the radii of discs contained in the complement of $E$; $E = \mathbf{Z}^2$, the Gaussian integers, is a good example. The letter $B$ is used for "Bloch"—$f$ is a Bloch function [1] if and only if $\sup\{|f'(z)|(1-|z|^2) : z \in \mathbf{D}\} < \infty$. We will later give a necessary (but not sufficient) geometrical condition that $E \leq_B E'$.

*Definition* ∞. Let $E$ and $E'$ be subsets of $\mathbf{C}$. Suppose that for every analytic function $f$ in $\mathbf{D}$

$$[\sup\{|f(z)| : f(z) \in E'\} < \infty] \Rightarrow [\sup\{|f(z)| : f(z) \in E\} < \infty].$$

We then say $E \leq_\infty E'$.

*Problem* ∞. Find necessary and sufficient geometrical conditions on $E$ and $E'$ that $E \leq_\infty E'$.

*Remark* ∞. Some people may find the definition confusing at first sight, but it really induces a nontrivial ordering. We later give a solution of Problem ∞ due to R. Timoney, whom we thank for his permission to use it here.

*Definition* p. Let $0 < p < \infty$ and let $E$ and $E'$ be subsets of $\mathbf{C}$. Suppose that whenever $f$ is an analytic function in $\mathbf{D}$, for which there exists a nonnegative harmonic function $\varphi$ in $\mathbf{D}$ so that $|f(z)|^p \leq \varphi(z)$ whenever $f(z) \in E'$, there must exist a nonnegative harmonic function $\psi$ in $\mathbf{D}$ so that $|f(z)|^p \leq \psi(z)$ whenever $f(z) \in E$. We then say $E \leq_p E'$.

*Problem* p. Find geometrical necessary and sufficient conditions that $E \leq_p E'$.

*Remark* p. Here $p$ refers to $H^p$, the Hardy space of analytic functions in $\mathbf{D}$, with norm

$$\|f\|_p = \lim_{r\to 1-} \left\{ \frac{1}{2\pi} \int_{\pi}^{\pi} |f(re^{i\theta}|^p \, d\theta \right\}^{1/p}.$$

It has been shown by Parreau [7] and by Rudin [8] that $f \in H^p$ if and only if $|f(z)|^p$ has a harmonic majorant $\varphi$ in $\mathbf{D}$. We have no clear idea of an answer to Problem p, but for relevant material to our inequality $E \leq_p E'$, see [5] and [3].

*Definition* BC. Let $E$ and $E'$ be subsets of $\mathbf{C}$. Suppose that whenever $f$ is an analytic function in $\mathbf{D}$ for which there exists a nonnegative harmonic function $\psi$ in $\mathbf{D}$ so that $\log^+|f(z)| \leq \psi(z)$ whenever $f(z) \in E'$, there must exist a nonnegative harmonic function $\varphi$ in $\mathbf{D}$ so that $\log^+|f(z)| \leq \varphi(z)$ whenever $f(z) \in E$. We then say $E \leq_{BC} E'$.

*Problem* BC. Find geometrical conditions on $E$ and $E'$ that $E \leq_{BC} E'$.

*Remark* BC. The letters "BC" stand here for "bounded characteristic". An analytic function $f$ in $\mathbf{D}$ is of bounded characteristic when $\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|d\theta$ is a bounded function of $r$ for $r < 1$. Equivalently, $f$ is of bounded characteristic when $f$ is the quotient of two bounded analytic functions. A third formulation (see [4]) is that $\log^+|f|$ have a harmonic majorant in $\mathbf{D}$—whence our ordering.

THEOREM N. *If* card $E' \geq 5$, *then* $E \leq_N E'$ *for all sets* $E$. *If* card $E' \leq 4$, *then* $E \leq E'$ *if and only if* $E \subseteq E'$.

*Proof.* For the first part, it was shown in [6] that if card $E' \geq 5$ then $\mathbf{C} \leq_N E'$. The second part clearly follows from the next result, which incidentally answers a question left open in [6] about the sharpness of its main theorem.

MAIN THEOREM. *If* $E$ *is any subset of the extended complex plane such that* card $E \geq 4$, *then there exists a meromorphic function* $f(z)$ *in* $\mathbf{D}$ *so that both*

i) $\sup\{f^{\#}(z)(1-|z|^2) : f(z) \in E\} < \infty$,

*and, for each value* $\lambda \notin E$,

ii) $\sup\{f^{\#}(z)(1-|z|^2) : f(z) = \lambda\} = \infty$.

*Proof.* It is clear that we need consider only the cases $E \subseteq \{0,1,\infty\}$ and $E = \{0,1,\infty,d\}$, where $d$ is arbitrary but different from 0,1, and $\infty$. This is because an appropriate linear fractional transformation will transform these into any other set of appropriate cardinality. We will consider the cases in order of ease, mixing up the usual numerical order, skipping over the more routine computations.

*Case* 1. Card $E = 2$ ($E = \{0,\infty\}$). Let $f(z) = \exp\left(\frac{1+z}{1-z}\right)^2$. Here $f$ omits 0 and $\infty$ so i) is trivial, and ii) is obtained by an easy computation.

*Case* 2. Card $E = 3$ ($E = \{0,1,\infty\}$). Let $f(z) = \frac{1}{2} + \frac{1}{2}\sin\left(\frac{1+z}{1-z}\right)$. Here $f$ omits $\infty$, and each solution to $f(z) \in \{0,1\}$ is a double solution ($f'(z) = 0$), so that i) is easy, and ii) comes from the fact that if $f(z) = \lambda$ ($\lambda \neq 0,1$), then

$$f'(z) = K(1-z)^{-2},$$

where $K = K(\lambda) \neq 0$, as $z$ approaches 1. Letting $z \to 1$ through a sequence of values on which $f(z) = \lambda$, within a symmetric Stolz angle (so that $\frac{1-|z|}{1-z}$ is bounded away from zero), we get the result in this case.

*Case* 3. Card $E = 1$ ($E = \{\infty\}$). Let $\wp(z)$ be the Weierstrass $\wp$-function (see [10]) with periods 1 and $i$, taking $z \in \mathbf{C}$ for now. Then

$$\wp''(z) = 12 \sum_{m,n,\in \mathbf{Z}} \left(\frac{1}{z-(m+in)}\right)^4,$$

where $\mathbf{Z}$ is the set of all integers. An easy calculation shows that

$$\mathscr{P}''\left(\frac{1+i}{2}+iz\right)=\mathscr{P}''\left(\frac{1+i}{2}+z\right)=\mathscr{P}''\left(\frac{1+i}{2}-iz\right)=\mathscr{P}''\left(\frac{1+i}{2}-z\right).$$

Hence the zeros of $\mathscr{P}''$ are located at four distinct points of a period parallelogram (e.g. the square with vertices 0, 1, $1+i$ and $i$). The zeros of $\mathscr{P}'$ are located at $i/2$, $1/2$, and $(1+i)/2$, and all are simple zeros, so $\mathscr{P}''$ is not zero at any of these points. Since $\mathscr{P}'(z)$ assumes each value exactly three times in a period parallelogram, the equation $\mathscr{P}'(z)-\lambda=0$ has at least one *simple* zero for each finite $\lambda$. Further, each solution to $\mathscr{P}'(z)=\infty$ is a triple pole. Setting

$$f(z)=\mathscr{P}'\left(\frac{1+z}{1-z}\right),\qquad z\in \mathbf{D},$$

i) and ii) follow easily, on again (for (ii)) letting $z\to 1$ through a symmetric Stolz angle and taking on only simple solutions to $f(z)=\lambda$.

*Case* 4. Card $E=0\,(E=\emptyset)$. Let $\mathscr{P}(z)$ be as in Case 3, and let

$$g(x)=\frac{x^3+2x^2-2x-1}{x^3-2x^2+4x-1}.$$

Consider the equation $g(x)=\lambda$—it has a simple solution for each $\lambda$. (Of course, there may be some double solutions for some choices of $\lambda$, but no triple solutions—hence at least one simple solution.) For $\lambda=1$, the solutions are $x=\infty$, 0, and $3/2$. For $\lambda\neq 1$, the equation $g(x)=\lambda$ is equivalent to

$$x^3+2\frac{1+\lambda}{1-\lambda}x^2-2\frac{1+2\lambda}{1-\lambda}x-1=0,$$

and the left side cannot be a perfect cube for any choice of $\lambda$ (even for $\lambda=\infty$).

Let $f(z)=g\left(\mathscr{P}'\left(\frac{1+z}{1-z}\right)\right)$. It is easy to see that $f(z)=\lambda$ has infinitely many simple solutions in each symmetric Stolz angle at $z=1$, for we may take $\lambda$ and $z$ so that both $g(x)=\lambda$ and $\mathscr{P}'\left(\frac{1+z}{1-z}\right)=x$ have simple solutions. There is such a $z$ with $f(z)=\lambda$ corresponding to each period parallelogram in the right half plane, and consequently ii) is established for each $\lambda$.

*Case* 5. Card $E=4$ ($E=\{0,1,\infty,d\}$, where $d\neq 0,1,\infty$ is arbitrary.) Let

$$T(z)=\begin{cases} d\dfrac{z-1}{z-a}, & \text{where } a=\dfrac{2d-1}{2-d} \quad \text{for } d\neq 2\\ z+1, & \text{if } d=2.\end{cases}$$

If $d=2$, we set $a=0$.

Now when $d \neq 2$, $T(a) = \infty$, $T(1) = 0$, $T(-a-1) = 1$, and $T(\infty) = d$ so that $T(\{a,1,-a-1,\infty\}) = E$. Also, if $d = 2$, $T(\{a,1,-a-1,\infty\}) = E$. Since $d$ is different from 0, 1, and $\infty$, the values $a$, 1, and $(-a-1)$ are all different, and they are all finite in any event. Let

$$z(w) = \int_{\infty}^{w} \frac{dt}{\sqrt{(t-a)(t-1)(t+a+1)}}$$

and let $w(z)$ denote its inverse function. Then $w(z)$ is a Weierstrass $\wp$ function (whose periods we leave unspecified), having $a$, 1, $-a-1$, and $\infty$ as its completely ramified values. Hence

$$f(z) = T\left(w\left(\frac{1+z}{1-z}\right)\right)$$

is the desired function. The argument for i) is simple, and that for ii) follows the lines of the previous cases.

THEOREM $\infty$. (R. M. Timoney) *Let $E$ and $E'$ be subsets of* $\mathbf{C}$. *Then* $E \leq_\infty E'$ *if and only if* $E \cap G$ *is bounded for every connected component* $G$ *of* $(\mathbf{C} \setminus \overline{E}') \cup D(0,R)$ *for each* R > 0, *where* $D(0,\mathrm{R}) = \{z \in \mathbf{C} : |z| < R\}$.

(*Second*) *Remark* $\infty$. To illustrate this result, let $E' = E_1' \cup E_2'$ where

$$E_1' = \bigcup_{n=1}^{\infty} \left\{ re^{i\theta} : \frac{2\pi}{n+1} \leq \theta \leq \frac{2\pi}{n} \quad \text{and} \quad r \geq n \right\}$$

$$E_2' = \bigcup_{n=1}^{\infty} \left\{ re^{i2\pi/n} : r \geq 0 \right\},$$

so that $E'$ is closed and $\mathbf{C} \setminus E'$ is an infinite union of disjoint open circular sectors $S_n$ whose common apex is 0, whose open angle shrinks to zero as $n \to \infty$, but whose radii increase to $\infty$ as $n \to \infty$. It is tempting to think, since each connected component of $\mathbf{C} \setminus E'$ is bounded, that therefore $\mathbf{C} \leq_\infty E'$, but this is not so, since $(\mathbf{C} \setminus \overline{E}') \cup D(0,R)$ is an unbounded connected open set for $R = 1/2$ (actually for *any* $R > 0$), so that $\mathbf{C} \leq_\infty E'$ is false, by our theorem, and the proof makes it clear why.

*Proof of Theorem* $\infty$. There is clearly no harm in supposing that both $E$ and $E'$ are closed. Suppose first that $E \leq_\infty E'$ and that there exists $R > 0$ and a connected component $G$ of $(\mathbf{C} \setminus E') \cup D(0,R)$ with $E \cap G$ unbounded. Clearly, $G$ is open and connected. Therefore there exists an analytic function $f$ with $f(\mathbf{D}) = G$, since we may consider $\mathbf{D}$ as the universal covering surface of $G$. (Those $G$ whose universal covering surface is $\mathbf{C}$ are trivial in our context.) Now observe that

$$f(\mathbf{D}) \cap E' = G \cap E' \subseteq D(0,R)$$

is bounded, while $f(D) \cap E = G \cap E$ is not. This contradicts the supposition that $E \leq_\infty E'$.

In the reverse direction, suppose the above condition holds for every $R > 0$. Let $f$ be an analytic function on $\mathbf{D}$ with

$$\sup \{|f(z)| : f(z) \in E'\} < R < \infty.$$

Then $f(\mathbf{D}) \subseteq (\mathbf{C} \setminus E') \cup D(0,R)$. Assuming $f$ is not constant (a trivial case), we see that $f(\mathbf{D})$ is open.

Now $f(\mathbf{D})$ is connected, so that the condition implies that $f(\mathbf{D}) \cap E$ is bounded, and the theorem is proved.

PROPOSITION B. *If $E \leq_B E'$, then the radii of discs in $\mathbf{C} \setminus E'$ whose centers lie in E are bounded above, or, equivalently,*

$$\sup \{\text{dist}\,(z,E') : z \in E\} < \infty.$$

The converse is false.

(*Second*) *Remark* B. This result and similar proofs were found simultaneously by R. M. Timoney and the authors. No simple necessary and sufficient condition is apparent at this time.

*Proof of Proposition* B. (Sketch). If the conclusion were false, then the method of [11] would produce a schlicht counterexample. To see that the converse is false, just take $E = \{0,1\}$ and $E' = \{0\}$ and apply Theorem N or reason directly. Note that if $f(z) = c$, a fixed constant, then $f^{\#}(z)$ and $|f'(z)|$ have a ratio that depends only on $c$.

(*Third*) *Remark* B. If $E'$ contains three or more finite points and if $E$ is any bounded set then $E \leq_B E'$. As a proof, we note that $\infty$ is omitted by an analytic function, and omitted values count twice in the 5-point theorem of [6]. (For details, see [9, p. 143].) Thus,

$$\sup \{|f'(z)|(1 - |z|^2) : f(z) \in E'\} < \infty$$

implies that $f(z)$ is normal, and the result follows since

$$(|f'(z)| / f^{\#}(z)) = (1 + |f(z)|^2)$$

is bounded above and below as $f(z)$ ranges over any given bounded set $E$.

*Final Remark.* Without going further into details, one could formulate orderings, that follow the above lines, for other classes of functions, like $B_0$ (see [1]) or the hypernormal or absolutely hypernormal functions (see [2]). An interesting question is whether the absolutely hypernormal ordering and the Bloch ordering ($\leq_B$) coincide. The absolutely hypernormal functions themselves coincide with the Bloch functions, but their defining properties are quite different.

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