ERGODIC HARDY SPACES AND DUALITY

Paul S. Muhly

1. INTRODUCTION

Suppose X is a measure space with probability measure m and that $\{T_t\}_{t\in\mathbb{R}}$ is an ergodic, measurable action of $\mathbb{R}$ on X preserving m. Via composition, $\{T_t\}_{t\in\mathbb{R}}$ acts on functions over X, $(T_t f)(x) = f(T_t x)$, and when restricted to $L^p(X)$, $p < \infty$, $\{T_t\}_{t\in\mathbb{R}}$ is strongly continuous. On $L^\infty(X)$, $\{T_t\}_{t\in\mathbb{R}}$ is only continuous in the weak-* topology. *Ergodic* H^∞, $H^\infty(X)$, is defined to be the subalgebra of $L^\infty(X)$ consisting of those functions f such that for each x in X, with the exception possibly of a null set, the function of t, $(T_t f)(x)$, admits a bounded analytic extension to $\operatorname{Im} z > 0$; *i.e.*, $(T_t f)(x)$ lies in $H^\infty(\mathbb{R})$ as a function of t for almost all x. For p in the range $0 < p < \infty$, *ergodic* H^p, $H^p(X)$, is defined to be the closure of $H^\infty(X)$ in $L^p(X)$; equivalently (*cf.* [10]), at least when $1 \le p$, $H^p(X)$ is the space of all functions f on X such that with the exception possibly of a null set, the function of t, $(T_t f)(x)$, when divided by $t + i$, lies in the usual Hardy space $H^p(\mathbb{R})$.

The measure m is multiplicative on $H^\infty(X)$ and $H^\infty(X)$is a weak-* Dirichlet algebra in $L^\infty(X)$, [8], [13]. Consequently, Jensen's inequality is valid[12] and so the linear functional determined by m on $H^\infty(X)$ extends to a continuous linear functional on each of the spaces $H^p(X)$, $0 < p < 1$ [5, p. 124]. Our objective in this note is to prove the following.

THEOREM. *If* $\{T_t\}_{t\in\mathbb{R}}$ *is not periodic, then for* $0 < p < 1$ *each continuous linear functional on* $H^p(X)$ *is a constant multiple of the linear functional determined by* m.

This result shows that there is a striking difference between the Hardy spaces based on a periodic flow and those based on a properly erogodic flow. In the periodic case, of course, the Hardy spaces are just the classical ones and it is well known that H^p, $p < 1$, has a rich dual even though the space fails to be locally convex [3]. If $\{T_t\}_{t\in\mathbb{R}}$ is properly ergodic and has pure point spectrum, an assumption which is tantamount to assuming that X is a group dual to a dense subgroup of $\mathbb{R}$, then the spaces $H^p(X)$ are those discussed in [6]. In this case, Shapiro proved the theorem (and a lot more) in [11]. His techniques are entirely different from ours, based, as they are, on deep results in harmonic analysis. Our approach is to use a famous theorem of Ambrose [1] to establish an analogue of the Poisson summation formula and then to use this analogue to identify the kernel of m with a quotient of a space whose dual is easily seen to be trivial.

Results in [9] show that if $\{T_t\}_{t\in\mathbb{R}}$ is not periodic, then when m is regarded as a point in the maximal ideal space of $H^\infty(X)$, the Gleason part it determines

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is simply $\{m\}$. Our theorem leads us to conjecture, therefore, that if ϕ is a point in the maximal ideal space of a function algebra and if the Gleason part through ϕ is the singleton $\{\phi\}$, then for any Jensen measure representing ϕ, the H^p spaces based on it have one-dimensional duals when $p < 1$.

In this note we abuse notation and terminology a little and refer to the expression $\int |f|^p dm$, which defines the metric on L^p when $p < 1$, as the norm of f and write $\|f\|$. Likewise we will refer to the norm of a continuous linear transformation between two L^p-spaces; this quantity is defined by the formula

$$\|T\| = \sup \{\|Tf\| : \|f\| \leq 1\}.$$

There ought not be any confusion caused by this and it makes the exposition easier.

We would like to take this opportunity to thank Joel Shapiro for several stimulating and informative conversations which led to our result and we would also like to thank Don Sarason who helped us correct an egregious error in an earlier draft.

Thanks are due to the referee as well for calling our attention to the paper "Maximal functions and H^p spaces defined by ergodic transformations" by R. Coifman and G. Weiss [Proc. Nat. Acad. Sci., 70 (1973), 1761–1763] which shows that the dual of $H^1(X)$ is an ergodic theoretic generalization of B.M.O. Of course $H^1(X)$ has a rich dual because it's a Banach space. The point, however, is that in a vague sense, the dual of H^1, B.M.O., is a limit of the duals of H^p, $p < 1$. On account of this, one would expect that the duals of $H^p(X)$, $p < 1$, are substantial and "cluster" at ergodic B.M.O. Our theorem shows that this is simply not the case.

An additional remark is worth making here. In the classical setting, H^1 not only has a rich dual, it itself is a dual space by the F. and M. Riesz theorem. Joel Shapiro has pointed out to us that when the flow has pure point spectrum, $H^1(X)$ is not a dual space. We suspect that this is the case quite generally, but we have no idea of a proof.

2. THE PROOF

The proof is broken into a series of lemmas, but first we describe how the theorem of Ambrose allows us to assume that X has a special form.

Suppose that μ is a probability on a space Ω and that τ is an ergodic, invertible, measurable transformation on Ω preserving μ. Suppose in addition that F is a bounded measurable function on Ω, bounded away from zero, and normalized to have integral one. With this data it is possible to construct two spaces, several transformation groups and a variety of maps. Extend F to a function ϕ on $\mathbb{Z} \times \Omega$ by the formula

$$\phi(n,\omega) = \begin{cases} \sum_{k=0}^{n-1} F(\tau^k, \omega), & n > 0 \\ 0, & n = 0 \\ -\phi(\tau^n \omega, -n), & n < 0. \end{cases}$$

The important thing about ϕ is that it satisfies the *cocyle identity*

$$\phi(n + m, \omega) = \phi(n, \omega) + \phi(m, \tau^n \omega), \qquad m, n \in \mathbb{Z}, \omega \in \Omega.$$

Define the measure space $\tilde{X}$ to be $\Omega \times \mathbb{R}$ with measure $\tilde{m}$ equal to μ times Lebesgue measure on $\mathbb{R}$. Next define X to be the region under the graph of F, $X = \{(\omega,r) \in \tilde{X}: 0 \leq r < F(\omega)\}$, and let m be the restriction of $\tilde{m}$ to X. Since $\int F d\mu = 1$, m is a probability. On $\tilde{X}$, let $\{S_t\}_{t\in\mathbb{R}}$ denote the group of $\tilde{m}$-preserving transformations defined by the formula $S_t(\omega,r) = (\omega,r + t)$, and define σ to be the measurable, invertible, $\tilde{m}$-preserving transformation by the formula

$$\sigma(\omega,r) = (\tau\omega, r + F(\omega)).$$

Observe that σ commutes with $\{S_t\}_{t\in\mathbb{R}}$ and that its powers are given by the formula $\sigma^n(\omega,r) = (\tau^n\omega, r + \phi(n,\omega))$, $n \in \mathbb{Z}$, $(\omega,r) \in \tilde{X}$. Observe too that the hypotheses on F imply that $\tilde{X}$ is the *disjoint* union $\bigcup_{n\in\mathbb{Z}} \sigma^n(X)$. Now define π mapping $\tilde{X}$ onto X by the formula $\pi(\omega,r) = (\tau^n\omega, r - \phi(n,\omega))$, if $\phi(n,\omega) \leq r < \phi(n + 1,\omega)$, and note that by definition $\pi \circ \sigma = \pi$ and that $\pi((\omega_1,r_1)) = \pi((\omega_2, r_2))$ if and only if some power of σ maps (ω_1,r_1) to (ω_2,r_2). Finally, the m-preserving transformation group $\{T_t\}_{t\in\mathbb{R}}$ is defined on X by the formula

$$T_t(\omega,r) = (\tau^n \omega, (r + t - \phi(n,\omega)), \qquad \phi(n,\omega) \leq r + t < \phi(n + 1,\omega),$$

and the relation $\pi S_t = T_t\pi$ is easily verified. Note that a set E in X is null for m if and only if $\pi^{-1}(E)$ is null for $\tilde{m}$ and, consequently, $\{T_t\}_{t\in\mathbb{R}}$ is ergodic. Indeed, if E is invariant under $\{T_t\}_{t\in\mathbb{R}}$, then $\pi^{-1}(E)$ is invariant under $\{S_t\}_{t\in\mathbb{R}}$. It follows that $\pi^{-1}(E)$ is either null or, up to a null set, of the form $B \times \mathbb{R}$ for some $B \subseteq \Omega$. But $\pi^{-1}(E)$ is also invariant under σ and so B is invariant under τ. Since τ is ergodic by hypothesis, it follows that B is either null or co-null and the assertion follows.

Summing up then, even though X is defined as a subset of $\tilde{X}$, we may quite properly think of the pair $(\tilde{X},\{S_t\}_{t\in\mathbb{R}})$ as covering $(X,\{T_t\}_{t\in\mathbb{R}})$ with π as the equivariant covering map and $\{\sigma^n\}_{n\in\mathbb{Z}}$ as the group of covering transformations. It is perhaps instructive to note too that if Ω is the singleton $\{\omega\}$ and if $F(\omega) = 1$, then what we just described is how one identifies the torus $\mathbb{R}/\mathbb{Z}$ with $[0,1)$. The theorem of Ambrose [1] asserts that if $\mathbb{R}$ acts ergodically in a measure preserving fashion on some space X' then it is possible to find Ω, μ, τ, and F so that the given action of $\mathbb{R}$ on X' is isomorphic, in a measure preserving fashion, to the action of $\mathbb{R}$ on X just constructed from Ω, μ, τ, and F. Thus, there is no loss of generality in assuming, as we do from now on, that the action of $\mathbb{R}$ on X in our theorem is of this special kind.

We refer the reader to articles of Ambrose [1] and Mackey [7] for further details and amplification of the discussion just presented.

If f is a bounded measurable function on $\tilde{X}$ which vanishes when $|r|$ is sufficiently large independently of ω, define $\mathbb{P}f$ on X by the formula

$$(\mathbb{P}f)(\omega,r) = \sum_{n\in\mathbb{Z}} f\circ\sigma^n(\omega,r), \qquad (\omega,r) \in X.$$

The hypothesis that F is bounded away from zero implies that the sum defining $\mathbb{P}f$ is finite for each fixed (ω,r). Observe, too, that for such functions, which are manifestly dense in each $L^p(\tilde{X})$, the equation $\mathbb{P}S_t f = T_t \mathbb{P}f$ is satisfied, for all $t \in \mathbb{R}$.

LEMMA 1. *For each* p, $0 < p \le 1$, *the map* $\mathbb{P}$ *extends to a bounded linear transformation of norm one mapping* $L^p(\tilde{X})$ *onto* $L^p(X)$ *and satisfying the equation* $\mathbb{P}S_t = T_t\mathbb{P}$ *for all* t *in* $\mathbb{R}$.

Proof. Since $(a+b)^p \le a^p + b^p$ when $0 < p \le 1$ and $a, b \ge 0$, and since X is a "fundamental domain" for σ, we see that for f of the form above,

$$\|\mathbb{P}f\|_p = \int_X \left|\sum_{n\in\mathbb{Z}} f\circ\sigma^n\right|^p dm \le \sum_{n\in\mathbb{Z}} \int_X |f\circ\sigma^n|^p\, d\tilde{m} = \sum_{n\in\mathbb{Z}} \int_{\sigma^{-n}X} |f|^p\, d\tilde{m} = \|f\|_p.$$

This proves that $\mathbb{P}$ is contractive and that its unique extension to all of $L^p(\tilde{X})$, which we continue to denote by $\mathbb{P}$, is given by the sum used originally to define $\mathbb{P}$. It is also clear that $\mathbb{P}S_t = T_t\mathbb{P}$ for all t in $\mathbb{R}$. Finally, the facts that $\mathbb{P}$ is onto and that $\mathbb{P}$ has norm one are immediate from the observation that if f is in $L^p(X)$ and if $\tilde{f}$ is defined on $\tilde{X}$ by the formula

$$\tilde{f}(\omega,r) = f(\omega,r),\ (\omega,r) \in X, \quad \text{and} \quad \tilde{f}(\omega,r) = 0 \text{ otherwise},$$

then $\mathbb{P}\tilde{f} = f$.

It is perhaps instructive and helpful to note that for $p = 1$ the adjoint of $\mathbb{P}$ is the injection i of $L^\infty(X)$ into $L^\infty(\tilde{X})$ defined by the formula $i(\varphi) = \varphi\circ\pi$, $\varphi \in L^\infty(X)$. We note, too, that $\mathbb{P}$ can not be extended to any of the spaces $L^p(\tilde{X})$, $p > 1$, because, as was pointed out to us by Don Sarason, the sum defining $\mathbb{P}f$ is identically $+\infty$ when f is the function which is $1/n$ on $\sigma^n(X)$, $n = 1, 2, \ldots$, and is zero elsewhere.

For $f \in L^p(\tilde{X})$(respectively $L^p(X)$), $1 \le p \le \infty$, and ξ in $L^1(\mathbb{R})$ we define

$$f *_S \xi = \int_{\mathbb{R}} (S_{-t}f)\xi(t)\, dt$$

(respectively $f *_T \xi = \int_{\mathbb{R}} (T_{-t}f)\xi(t)\, dt$). By Fubini's theorem, these integrals may be thought of as defined in a pointwise sense or, if one wishes, they may be taken in a vector-valued sense. From either perspective, this process of convolution converts the L^p-spaces, $1 \leq p \leq \infty$, into modules over $L^1(\mathbb{R})$ such that the module inequality

$$\|f *_S \xi\|_p \leq \|f\|_p \|\xi\|_1, \qquad f \in L^p\tilde{X}), \xi \in L^1(\mathbb{R})$$

(respectively $\|f *_T \xi\|_p \leq \|f\|_p\|\xi\|_1$, $f \in L^p(X)$, $\xi \in L^1(\mathbb{R})$) is satisfied. We refer the reader to our papers [8-10] or the paper of Forelli [4] for further discussion of this notion of convolution.

As an immediate consequence of Lemma 1, we have

LEMMA 2. *If* $f \in L^1(\tilde{X})$, *and if* $\xi \in L^1(\mathbb{R})$, *then* $\mathbb{P}(f *_S \xi) = (\mathbb{P}f) *_T \xi$.

For $0 < p < \infty$, we define $H^p(\tilde{X})$ to be the closed linear span in $L^p(\tilde{X})$ of those functions which can be written as the product of a function in $L^p(\Omega)$ times a function in $H^p(\mathbb{R})$. It is easy to see that every f in $H^p(\tilde{X})$ has the property that for almost every ω in Ω, where the exceptional null set may depend upon f, $f(\omega,\cdot)$ lies in $H^p(\mathbb{R})$. On the other hand, unless Ω is a standard Borel space or $p \geq 1$, we are unable to show that every function with this property lies in $H^p(\tilde{X})$. Fortunately, however, we need not dwell on this matter of measure theoretic teratology.

LEMMA 3. *When* $0 < p < 1$, $H^p(\tilde{X})$ *has no non-zero, continuous, linear functionals.*

Proof. Suppose ψ is a continuous linear functional on $H^p(\tilde{X})$ and fix f in $H^p(\mathbb{R})$. By Fubini's theorem, the map $g \to \psi(gf)$ is a continuous linear functional on $L^p(\Omega)$ which must vanish by Day's theorem [2]. Since functions of the form gf, $g \in L^p(\Omega)$, $f \in L^p(\mathbb{R})$ span $H^p(\tilde{X})$, ψ must be zero.

LEMMA 4. *For* $0 < p < 1$, $H^1(\tilde{X}) \cap L^p(\tilde{X})$ *is dense in both* $H^1(\tilde{X})$ *and* $H^p(\tilde{X})$.

Proof. Since μ is finite, $L^1(\Omega)$ is dense in $L^p(\Omega)$ and so it suffices to show that $H^1(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $H^1(\mathbb{R})$ and in $H^p(\mathbb{R})$. But $H^1(\mathbb{R}) \cap L^p(\mathbb{R})$ contains the functions $f_a(z) = (z + a)^{-2/p}$, $\mathrm{Im}(a) < 0$, which span both $H^1(\mathbb{R})$ and $H^p(\mathbb{R})$.

Let $H_0^p(X)$ denote the kernel of the linear functional on $H^p(X)$ determined by m and note that because m is finite, $H_0^1(X)$ is contained in $H_0^p(X)$ when $0 < p < 1$.

LEMMA 5. *The closure of* $H_0^1(X)$ *in* $H^p(X)$, $0 < p < 1$, *is* $H_0^p(X)$.

Proof. This is but a special case of Theorem 6.1 on page 131 of [5].

LEMMA 6. *The image under* $\mathbb{P}$ *of* $H^1(\tilde{X})$ *is dense in* $H_0^1(X)$.

Proof. First note that if $f \in L^1(\tilde{X})$ and if ξ is in $H^1(\mathbb{R})$, then $f *_S \xi$ is in $H^1(\tilde{X})$ and that the set of all such convolutions is dense in $H^1(\tilde{X})$. Secondly, note that $H^\infty(X)$ is the set of all functions φ in $L^\infty(X)$ with the property that $\varphi *_T \tilde{\xi} = 0$ for all ξ in $H^1(\mathbb{R})$ where $\tilde{\xi}(t) = \xi(-t)$. This follows from the analysis in section 2 of [10], for example, coupled with the fact that

$$H^1(\mathbb{R}) = \{\xi \in L^1(\mathbb{R}) : \hat{\xi} \text{ is supported in } [0,\infty)\}$$

[3, Theorem 11.10]. Thirdly, note that if φ is in $L^\infty(X)$, if f is in $L^1(X)$, and if ξ is in $L^1(\mathbb{R})$, then $\langle\varphi, f *_T \xi\rangle = \langle\varphi *_T \tilde{\xi}, f\rangle$, where $\langle\varphi, f\rangle = \int_X \varphi f\, dm$ by definition. Finally, note that in this pairing $\langle .,. \rangle$ between $L^\infty(X)$ and $L^1(X)$, $H^\infty(X)$ is the annihilator of $H_0^1(X)$ [12]. Suppose, now, that f is in $L^1(\tilde{X})$ and that ξ is in $H^1(\mathbb{R})$, then using Lemma 2, we find that

$$\langle\varphi, \mathbb{P}(f *_S \xi)\rangle = \langle\varphi, (\mathbb{P}f) *_T \xi\rangle = \langle\varphi *_T \tilde{\xi}, \mathbb{P}f\rangle$$

is zero when φ is in $H^\infty(X)$. This shows that $\mathbb{P}(H^1(\tilde{X})) \subseteq H_0^1(X)$. Next choose φ in $L^\infty(X)$ which annihilates $\mathbb{P}(H^1(\tilde{X}))$. Then this equation yields zero for all $f \in L^1(\tilde{X})$ and $\xi \in H^1(\mathbb{R})$. Since $\mathbb{P}$ maps $L^1(\tilde{X})$ onto $L^1(X)$ by Lemma 1, the fact that the equation yields zero implies that $\varphi * \tilde{\xi} = 0$ for all ξ in $H^1(\mathbb{R})$. As we noted, this means φ is in $H^\infty(X)$ and the proof is completed by appeal to the Hahn-Banach theorem.

LEMMA 7. *When* $0 < p < 1$, $\mathbb{P}(H^p(\tilde{X}))$ *is dense in* $H_0^p(X)$.

Proof. Let $\mathscr{D} = H^1(\tilde{X}) \cap L^p(\tilde{X})$. By Lemma 4, $\mathscr{D}$ is dense in $H^1(\tilde{X})$, so by Lemma 6, $\mathbb{P}(\mathscr{D})$ is dense in $H_0^1(X)$. But then, by Lemma 5, $\mathbb{P}(\mathscr{D})$ is dense in $H_0^p(X)$. By Lemma 4 again, $\mathscr{D}$ is dense in $H^p(\tilde{X})$ and so $\mathbb{P}(H^p(\tilde{X}))$ is dense in $H_0^p(X)$.

It seems reasonable to expect that $\mathbb{P}(H^p(\tilde{X}))$ actually equals $H_0^p(X)$, but we are unable to decide this.

With all the pieces at hand, the proof of the theorem is easily assembled. It suffices to show that each continuous linear functional ψ on $H^p(X)$, $0 < p < 1$, annihilates $H_0^p(X)$. But if ψ is such a functional, then $\psi \circ \mathbb{P}$ is a continuous linear functional on $H^p(\tilde{X})$. By Lemma 3, $\psi \circ \mathbb{P}$ is the zero functional. Thus ψ annihilates the closure of the range of $\mathbb{P}$ restricted to $H^p(\tilde{X})$ and this, by Lemma 7, is H_0^p.

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Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242