COMPACT FAMILIES OF UNIVALENT FUNCTIONS

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Let D be a proper domain in the complex plane $\mathbb{C}$, H(D) the space of holomorphic functions on D, and $H_u(D)$ the subset of univalent functions in H(D). We endow H(D) with the topology of uniform convergence on compact sets. If $L = (\ell_1, \ell_2, \cdots, \ell_n)$ is an n-tuple of continuous, linearly independent, linear functionals on H(D), and $Q = (q_1, q_2, \cdots, q_n) \in \mathbb{C}^n$, define

$$\mathscr{F}(D, L, Q) = \{f \in H_u(D): L(f) = Q\}.$$

In [1], Hengartner and Schober proved

THEOREM A. *If* $\mathscr{F} = \mathscr{F}(D, (\ell_1, \ell_2), (q_1, q_2))$ *is nonempty, and* (ℓ_1, ℓ_2) *satisfies*

$$(*) \qquad \ell_1(1)\,\ell_2(g) \neq \ell_2(1)\,\ell_1(g), \qquad \textit{for every } g \in H_u(D),$$

then $\mathscr{F}$ *is compact. Moreover, if* D *has a "strongly dense boundary" and* $\mathscr{F}$ *is nonempty and compact, then* (*) *holds.*

This paper is concerned with generalizing Theorem A to the case of more than two linear functionals.

Clearly, if (*) held for one pair of the n linear functionals $\ell_1, \ell_2, \cdots, \ell_n$, then $\mathscr{F}(D, L, Q)$ would be compact whenever it were nonempty. On the other hand, as the following example shows, $\mathscr{F}$ may be compact even if (*) fails for each pair of the n linear functionals.

Example. Let D be the unit disk $\Delta = \{z: |z| < 1\}$; let $\ell_1(f) = f''(0) + f'(0)$, $\ell_2(f) = f(0)$, $\ell_3(f) = f''(0)$; and let $q_1 = 1$, $q_2 = q_3 = 0$. If $I(z) = z$, then $I \in \mathscr{F}(\Delta, L, Q)$; so $\mathscr{F}(\Delta, L, Q)$ is nonempty. Clearly,

$$\mathscr{F}(\Delta, L, Q) = \{f \in H_u(\Delta): f(0) = 0,\ f'(0) = 1\} \cap \{f \in H(\Delta): f''(0) = 0\}.$$

The first set on the right-hand side is well known to be compact, and the second is closed. Therefore, $\mathscr{F}(\Delta, L, Q)$ is nonempty and compact. On the other hand, if $h(z) = z - z^2/2$, then $h \in H_u(\Delta)$, and

$$\begin{aligned} 0 &= \ell_1(1)\,\ell_2(h) = \ell_2(1)\,\ell_1(h) \\ &= \ell_1(1)\,\ell_3(I) = \ell_3(1)\,\ell_1(I) \\ &= \ell_2(1)\,\ell_3(I) = \ell_3(1)\,\ell_2(I). \end{aligned}$$

Thus, (*) fails for each pair of the three linear functionals.

The generalization of Theorem A we wish to explore arises from the following observation. Let Ker(L) denote the kernel of L.

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PROPOSITION 1. *If* $L = (\ell_1, \ell_2)$, *then* (*) *is equivalent to*

(**) $$\mathrm{Ker}(L) \cap (H_u(D) \cup \{1\}) = \emptyset.$$

Proof. Clearly, (*) implies (**). Conversely, suppose (*) fails to hold; *i.e.*, $\ell_1(1)\, \ell_2(g) = \ell_2(1)\, \ell_1(g)$ for some g in $H_u(D)$. Then if $1 \notin \mathrm{Ker}(L)$, either

$$g - \ell_1(g)/\ell_1(1) \in \mathrm{Ker}(L) \cap H_u(D) \quad \text{if } \ell_1(1) \neq 0,$$

or

$$g - \ell_2(g)/\ell_2(1) \in \mathrm{Ker}(L) \cap H_u(D) \quad \text{if } \ell_2(1) \neq 0.$$

Hence, (**) fails to hold.

We conjecture that for $\mathscr{F} = \mathscr{F}(D, L, Q)$ nonempty, $\mathscr{F}$ is compact if and only if (**) holds. We prove half of this conjecture.

THEOREM. *If* $\mathscr{F} = \mathscr{F}(D, L, Q)$ *is nonempty and* (**) *is satisfied, then* $\mathscr{F}$ *is compact.*

In order to prove the theorem, we need the following simple lemma.

LEMMA 1. *Suppose* L *satisfies* (**). *Then for each* f *in* $H_u(D)$, $L(f)$ *and* $L(1)$ *are linearly independent.*

Proof. If the lemma were false, there would be a function f in $H_u(D)$ and a complex constant α for which $f - \alpha \in H_u(D) \cap \mathrm{Ker}(L)$.

Proof of theorem. Fix z_0 in D. We will find constants m, M_0, and M_1 such that $\mathscr{F}$ is the intersection of the compact set

$$\{f \in H_u(D)\colon |f(z_0)| \leq M_0,\ m \leq |f'(z_0)| \leq M_1\}$$

and the closed set $\{f \in H(D)\colon L(f) = Q\}$.

Observe first that the set $S = \{f \in H_u(D)\colon f(z_0) = 0,\ f'(z_0) = 1\}$ is compact, and $\ell_1, \ell_2, \cdots, \ell_n$ are continuous. Therefore, for some constant M, and for every h in S,

(1) $$|\ell_j(h)| \leq M, \qquad j = 1, 2, \cdots, n.$$

Now let $f \in \mathscr{F}$. Then $f(z) = a_0 + a_1 h(z)$, where $h \in S$, $a_0 = f(z_0)$, and $a_1 = f'(z_0)$. Applying ℓ_j, we have

(2) $$q_j = \ell_j(f) = a_0\, \ell_j(1) + a_1\, \ell_j(h), \qquad j = 1, 2, \cdots, n.$$

Since $\mathscr{F}$ is nonempty, it follows from Lemma 1 that Q and $L(1)$ are linearly independent. Consequently, there is a polydisk in $\mathbf{C}^n$, centered at Q, disjoint from the one-dimensional subspace spanned by $L(1)$. In other words, there is a positive constant r_0, depending only on Q and $L(1)$, such that

(3) $$\max_j |q_j - \alpha\, \ell_j(1)| \geq r_0, \qquad \text{for every } \alpha \in \mathbf{C}.$$

From (1), (2) and (3), we deduce $r_0/|a_1| \leq M$. Therefore,

$$|a_1| = |f'(z_0)| \geq m = r_0/M.$$

Next suppose for each positive integer k, there is a function f_k in $\mathscr{F}$ such that $|f_k'(z_0)| \geq k$. Then, as before, $f_k(z) = a_0(k) + a_1(k)\, h_k(z)$, where $h_k(z) \in S$, $a_0(k) = f_k(z_0)$, and $a_1(k) = f_k'(z_0)$. Now

$$L(h_k) = \frac{Q}{a_1(k)} - \frac{a_0(k)}{a_1(k)} L(1). \tag{4}$$

Since S is compact, there is a subsequence $h_{k(i)}$ which converges to $h_0 \in S$. On this subsequence, the left-hand side of (4) converges to $L(h_0)$. Consequently, the right-hand side of (4) must converge, and since $|a_1(k)| \to \infty$, the limit must have the form $\alpha\, L(1)$. Hence, $L(h_0)$ and $L(1)$ are linearly dependent, contradicting Lemma 1. Therefore, there is a constant M_1 such that $|f'(z_0)| \leq M_1$ for all f in $\mathscr{F}$.

Finally, suppose for each positive integer k, $|f_k(z_0)| \geq k$, for some f_k in $\mathscr{F}$. Then, as before,

$$f_k(z) = a_0(k) + a_1(k)\, h_k(z),$$

$$q_j = \ell_j(f_k) = a_0(k)\, \ell_j(1) + a_1(k)\, \ell_j(h_k),$$

where $h_k \in S$, $a_0(k) = f_k(z_0)$, and $a_1(k) = f_k'(z_0)$. Since $1 \notin \mathrm{Ker}(L)$, $\ell_j(1) \neq 0$ for some j, $1 \leq j \leq n$. For that fixed j,

$$|q_j| \geq k\,|\ell_j(1)| - |a_1(k)|\,|\ell_j(h_k)| \geq k\,|\ell_j(1)| - MM_1.$$

The right-hand side of the above inequality tends to ∞ as k increases, but the left-hand side remains constant. From this contradiction, we conclude that there exists a constant M_0 such that $|f(z_0)| \leq M_0$, for all f in $\mathscr{F}$.

It is not clear whether (**) is a necessary condition for compactness when $n > 2$. We note that if $1 \in \mathrm{Ker}(L)$, then $\mathscr{F}(D, L, Q)$ is noncompact whenever it is nonempty. Therefore, the necessity of (**) follows from the statement: "If there is a function in $H_u(D) \cap \mathrm{Ker}(L)$, then for every Q in $\mathbb{C}^n$, $\mathscr{F}(D, L, Q)$ is either empty or noncompact". We are able to prove two weaker versions of this statement.

PROPOSITION 2. *Let* D *be simply connected. If there is a function in* $H_u(D) \cap \mathrm{Ker}(L)$ *whose range is not dense in* $\mathbb{C}$, *then, for every* Q *in* $\mathbb{C}^n$, $\mathscr{F}(D, L, Q)$ *is nonempty and noncompact.*

PROPOSITION 3. *Let* D *be simply connected, let* $n = 3$, *and assume* $1 \notin \mathrm{Ker}(L)$. *If there is a function in* $H_u(D) \cap \mathrm{Ker}(L)$ *whose range omits a line segment, then, for every* Q *off some (real) hypersurface in* $\mathbb{R}^6 = \mathbb{C}^3$, $\mathscr{F}(D, L, Q)$ *is nonempty and noncompact.*

The proofs of both propositions are based on the following two observations. First, if $f \in H(D)$, and we define $\tilde{f} \in H(D \times D)$ by $\tilde{f}(z, w) = (f(z) - f(w))/(z - w)$, then $f \in H_u(D)$ if and only if $\tilde{f}(z, w)$ is never 0. In some ways, $\tilde{f}$ behaves like a derivative of f; indeed, $\tilde{f}(z, z) = f'(z)$, f is constant if and only if $\tilde{f} \equiv 0$, $\tilde{I} \equiv 1$, and $\widetilde{f \circ g}(z, w) = \tilde{f}(g(z), g(w))\, \tilde{g}(z, w)$.

The second observation is the following:

LEMMA 2. *Suppose* D *is simply connected, and* Γ *is an arc in* $\mathbb{C} \setminus D$. *Then there are points* $a_1, a_2, \cdots, a_n$ *on* Γ *such that* $\{L(1/(z - a_j))\}_{j=1}^{n}$ *is a basis for* $\mathbb{C}^n$.

Proof. We generalize an argument of Hengartner and Schober. If ℓ is a continuous linear functional on H(D), it can be represented by a measure μ whose support is a compact set $E \subset D$; that is,

$$\ell(f) = \int_E f(s)\, d\mu(s), \quad \text{for } f \in H(D).$$

(See Corollary 4.3 of [2].) One can assume E is simply connected. Let $F_\ell(z) = \int_E (\zeta - z)^{-1}\, d\mu(\zeta)$. Then $F_\ell \in H(\mathbb{C} \setminus E)$, and $F_\ell \equiv 0$ on $\mathbb{C} \setminus E$ if and only if $\ell \equiv 0$ on H(D). (See Corollary 4.4 of [2].) We will say a compact set E is a *support* of ℓ if it supports a measure representing ℓ.

Let E be a simply connected compact subset of D containing supports of $\ell_1, \ell_2, \cdots, \ell_n$. If $F_{\ell_1}(z) \equiv 0$ on Γ, then $F_{\ell_1} \equiv 0$ on $\mathbb{C} \setminus E$. Consequently, $\ell_1 \equiv 0$ on H(D), contradicting the assumption of linear independence of $\ell_1, \ell_2, \cdots, \ell_n$. Therefore, $F_{\ell_1}(a_1) \neq 0$, for some a_1 on Γ. If $f_1(z) = 1/(z - a_1)$, then $f_1 \in H_u(D)$ and $\ell_1(f_1) = F_{\ell_1}(a_1) \neq 0$.

Now suppose we have found $a_1, a_2, \cdots, a_k$ on Γ $(k < n)$ such that if $f_j(z) = 1/(z - a_j)$, then the $k \times k$ matrix $A_k = (\ell_i(f_j))$ is nonsingular. Since the rows of A_k are linearly independent, there are constants $\alpha_1, \cdots, \alpha_k$ such that

$$\ell_{k+1}(f_j) = \sum_{i=1}^{k} \alpha_i\, \ell_i(f_j), \quad \text{for } j = 1, \cdots, k. \tag{5}$$

We claim

$$\sum_{i=1}^{k} \alpha_i F_{\ell_i}(z) - F_{\ell_{k+1}}(z) \not\equiv 0 \quad \text{on } \Gamma. \tag{6}$$

Otherwise, if $\ell = \sum_{i=1}^{k} \alpha_i \ell_i - \ell_{k+1}$, then ℓ would be a continuous linear functional on H(D) with support in E, and F_ℓ would be given by the left-hand side of (6). If $F_\ell \equiv 0$ on Γ, then $F_\ell \equiv 0$ on $\mathbb{C} \setminus E$, and consequently $\ell \equiv 0$ on H(D), again contradicting the linear independence of $\ell_1, \cdots, \ell_n$.

Therefore, there is a point a_{k+1} on Γ such that the function

$$f_{k+1}(z) = 1/(z - a_{k+1})$$

satisfies

$$\sum_{i=1}^{k} \alpha_i F_{\ell_i}(a_{k+1}) - F_{\ell_{k+1}}(a_{k+1}) = \sum_{i=1}^{k} \alpha_i\, \ell_i(f_{k+1}) - \ell_{k+1}(f_{k+1}) \neq 0. \tag{7}$$

If A_{k+1} is the $(k+1) \times (k+1)$ matrix $(\ell_i(f_j))$, the determinant of A_{k+1} is unchanged if each of the first k rows is multiplied by the corresponding α_i and subtracted from the last row. From (5) and (7), it is clear that

$$\operatorname{Det} A_{k+1} = \pm(\operatorname{Det} A_k)\left(\sum_{i=1}^{k} \alpha_i\, \ell_i(f_{k+1}) - \ell_{k+1}(f_{k+1})\right) \neq 0 .$$

By induction, we can choose $a_1, \cdots, a_n$ on Γ such that if $f_j(z) = 1/(z - a_j)$, then the $n \times n$ matrix $A_n = (\ell_i(f_j))$ is nonsingular. This proves the lemma.

Proof of Proposition 2. We are given f in $\operatorname{Ker}(L) \cap H_u(D)$, and $D^* = f(D)$ is not dense in $\mathbb{C}$. If we let $L^*: H(D^*) \to \mathbb{C}^n$ be the linear transformation $L^*(g) = L(g \circ f)$, we see that $\mathscr{F}(D, L, Q)$ is compact and/or nonempty if and only if $\mathscr{F}(D^*, L^*, Q)$ is. Also, D^* is simply connected and not dense in $\mathbb{C}$, and the identity function I is in $\operatorname{Ker}(L^*)$. Let Ω be an open subset of $\mathbb{C} \setminus D^*$, let Γ be a closed arc in Ω, and let $a_1, \cdots, a_n$ be points on Γ obtained by applying the above lemma to L^* and D^*.

Fix Q in $\mathbb{C}^n$. Then $Q = L^*(F)$, where $F(z) = \sum_{j=1}^n b_j/(z - a_j)$, for suitably chosen constants $b_1, \cdots, b_n$. Let σ be the distance from Γ to $\overline{D^*}$, and let $f_j(z) = 1/(z - a_j)$. Then $\sigma > 0$, and

$$|\tilde{f}_j(z, w)| = \frac{1}{|z - a_j|\ |w - a_j|} \leq \frac{1}{\sigma^2}, \qquad j = 1, 2, \cdots, n .$$

Consequently, $|\tilde{F}(z, w)| \leq nM/\sigma^2$ for every (z, w) in $D^* \times D^*$, where $M = \max(|b_1|, \cdots, |b_n|)$. Now if $N > 1$ and $G_N = (NnM/\sigma^2)z + F(z)$, then for every point (z, w) in $D^* \times D^*$, $|\tilde{G}_N(z, w)| \geq nM(N - 1)/\sigma^2$. Thus, $G_N \in H_u(D)$. Since $I \in \operatorname{Ker}(L^*)$, $L^*(G_N) = L^*(F) = Q$. Hence, $\{G_N\}_{N=2}^{\infty}$ is an infinite sequence in $\mathscr{F}(D^*, L^*, Q)$ with no converging subsequence. Therefore $\mathscr{F}(D^*, L^*, Q)$, and consequently $\mathscr{F}(D, L, Q)$, is nonempty and noncompact.

Proof of Proposition 3. By hypothesis, $L(1) \neq 0$. As in the proof of the above proposition, we may assume D omits a line segment Γ, and the identity I is in $\operatorname{Ker}(L)$. Let a_1, a_2, and a_3 be points on Γ obtained by applying Lemma 2 to a proper subinterval of Γ, so that none of the three points is an endpoint of Γ. The vector $L(1)$ and two of the vectors $L(1/(z - a_j))$, say $L(1/(z - a_1))$ and $L(1/(z - a_2))$, form a basis for $\mathbb{C}^3$. Let $\mathscr{R}$ be the real hypersurface in $\mathbb{R}^6 = \mathbb{C}^3$ defined by

$$\mathscr{R} = \left\{\alpha \left(L\left(\frac{1}{z - a_1}\right) - pL\left(\frac{1}{z - a_2}\right)\right) + \gamma L(1):\ \alpha, \gamma \in \mathbb{C},\ p \in \mathbb{R},\ p \geq 0\right\}.$$

We will show that if $Q \notin \mathscr{R}$, then $\mathscr{F}(D, L, Q)$ is nonempty and noncompact.

For $a \in \mathbb{C}$, $r > 0$, and $0 \leq \theta < 2\pi$, let $\Delta(a, r) = \{z \in \mathbb{C}: |z - a| < r\}$, $\Lambda(a, r, \theta) = \{a + se^{i\theta}: -2r \leq s \leq 2r\}$, and $U(a, r, \theta) = z + (re^{i\theta})^2/(z - a)$. Then $U(a, r, \theta)$ maps the complement $(\overline{\Delta(a, r)})^c$ conformally onto $(\Lambda(a, r, \theta))^c$. Now

$$\tilde{U}(a, r, \theta)(z, w) = 1 - \frac{(re^{i\theta})^2}{(z - a)(w - a)},$$

so $\tilde{U}(a, r, \theta)$ maps $(\overline{\Delta(a, r)})^c \times (\overline{\Delta(a, r)})^c$ into $\Delta(1, 1)$. Consequently, if $V(a, r, \theta)(z) = U(a, r, \theta)^{-1}(z)$, then

$$\tilde{V}(a, r, \theta)(z, w) = \left[1 - \frac{(re^{i\theta})^2}{(V(a, r, \theta)(z) - a)(V(a, r, \theta)(w) - a)}\right]^{-1},$$

and $\tilde{V}(a, r, \theta)$ maps $(\Lambda(a, r, \theta))^c \times (\Lambda(a, r, \theta))^c$ into the half-plane $\{z: \Re(z) > 1/2\}$. Also, outside $\Delta(a, 2r)$, the function $V(a, r, \theta)$ has Laurent expansion

$$z - \frac{(re^{i\theta})^2}{z - a} + O(r^3).$$

Return now to the points a_1 and a_2 on Γ. If $Q \notin \mathscr{R}$, then

$$Q = L(\alpha/(z - a_1) + \beta/(z - a_2) + \gamma)$$

for some $\alpha, \beta, \gamma \in \mathbf{C}$, where $\beta \neq 0$ and α/β lies off the negative real axis. If θ is the angle of inclination of the line segment Γ, let $V_1(z) = V(a_1, r, \theta)(z)$ and $V_2(z) = V(a_2, r, \theta)(z)$, where r is chosen very small. More precisely, r should be chosen so small that the Laurent expansions of V_1 and V_2 are valid on the supports of measures representing L, so that

$$L(V_j) = -r^2[e^{2i\theta} L(1/(z - a_j)) + O(r)], \quad j = 1, 2.$$

(Recall that $L(I) = 0$.) Consequently, we can take r so small that $L(V_1)$, $L(V_2)$, and $L(1)$ are linearly independent, and $Q = L(\alpha' V_1 + \beta' V_2 + \gamma')$ for some $\alpha', \beta', \gamma' \in \mathbf{C}$, where $\beta' \neq 0$ and α'/β' is nonnegative. Finally, we choose r so small that $V_1, V_2 \in H(\mathbf{C} \setminus \Gamma)$, and

$$|V_i(z) - a_i| < 2r \implies |V_j(z) - a_j| > 2r, \tag{8}$$

for $i, j = 1, 2$, $i \neq j$. (To see that this is possible, note that the level curves $|V_i(z) - a_i| = rc$ $(c > 1)$ are ellipses centered at a_i, whose major axes coincide with Γ and have length $2r(c + 1/c)$.)

Now let $F(z) = \alpha' V_1(z) + \beta' V_2(z) + \gamma'$. Then $L(F) = Q$, but F need not be univalent. Consider

$$\tilde{F}(z, w) = \alpha' \tilde{V}_1(z, w) + \beta' \tilde{V}_2(z, w),$$

where $\tilde{V}_j(z, w) = \left[1 - \frac{r^2 e^{2i\theta}}{(V_j(z) - a_j)(V_j(w) - a_j)}\right]^{-1}$, $j = 1, 2$.

If z and w are both near a_1; *i.e.*, if $|V_1(z) - a_1| < 2r$ and $|V_1(w) - a_1| < 2r$, then $|V_2(z) - a_2| > 2r$ and $|V_2(w) - a_2| > 2r$. Thus, $|\beta' \tilde{V}_2(z, w)| < (4/3)|\beta'|$, and therefore $\tilde{F}(z, w)$ lies in the half-plane Ω_1 defined by

$$\Omega_1 = \{z: \Re(\overline{\alpha'} z/|\alpha'|) > (1/2)|\alpha'| - (4/3)|\beta'|\}.$$

Similarly, if z and w are both near a_2; *i.e.*, if $|V_2(z) - a_2| < 2r$ and $|V_2(w) - a_2| < 2r$, then $\tilde{F}(z, w)$ lies in the half-plane Ω_2 defined by

$$\Omega_2 = \{z: \Re(\overline{\beta'} z/|\beta'|) > (1/2)|\beta'| - (4/3)|\alpha'|\}.$$

In all other cases, either $|V_j(z) - a_j| > 2r$ or $|V_j(w) - a_j| > 2r$, for each $j = 1, 2$. But $z, w \in D$ implies $|V_j(z) - a_j| > r$ and $|V_j(w) - a_j| > r$ for $j = 1, 2$. Consequently, if z and w are not both near the same a_j, $\tilde{F}(z, w)$ lies in the disk $\Omega_0 = \{z: |z| < 2(|\alpha'| + |\beta'|)\}$. Hence, $\tilde{F}(D \times D) \subset \Omega_0 \cup \Omega_1 \cup \Omega_2$.

Let $\alpha' = |\alpha'| e^{i\phi}$ and $\beta' = |\beta'| e^{i\psi}$, where $0 \le \phi, \psi < 2\pi$. Then $0 \le |\phi - \psi|/2 < \pi$, and, since α'/β' is nonnegative, $|\phi - \psi|/2 \neq \pi/2$. Let

$$N > \left|\sec\left(\frac{|\phi - \psi|}{2}\right)\right| \max\left(2(|\alpha'| + |\beta'|), \left|\frac{4}{3}|\alpha'| - \frac{1}{2}|\beta'|\right|, \left|\frac{4}{3}|\beta'| - \frac{1}{2}|\alpha'|\right|\right).$$

If $0 \le |\phi - \psi|/2 < \pi/2$, let $\lambda_N = -Ne^{i(\phi+\psi)/2}$. Then $\lambda_N \overline{\alpha'}/|\alpha'| = -Ne^{i(\psi-\phi)/2}$, so

$$\Re\left(\lambda_N \frac{\overline{\alpha'}}{|\alpha'|}\right) = -N\cos\left(\frac{\psi - \phi}{2}\right) < \frac{1}{2}|\alpha'| - \frac{4}{3}|\beta'|.$$

Hence, $\lambda_N \notin \Omega_1$. Similarly,

$$\Re\left(\lambda_N \frac{\overline{\beta'}}{|\beta'|}\right) = -N\cos\left(\frac{\phi - \psi}{2}\right) < \frac{1}{2}|\beta'| - \frac{4}{3}|\alpha'|,$$

and $\lambda_N \notin \Omega_2$. The choice of N guarantees that $\lambda_N \notin \Omega_0$. If, on the other hand, $\pi/2 < |\phi - \psi|/2 < \pi$, let $\lambda_N = Ne^{i(\phi+\psi)/2}$, and apply a similar argument to show that $\lambda_N \notin \Omega_0 \cup \Omega_1 \cup \Omega_2$.

Finally, let $F_N(z) = -\lambda_N z + F(z)$. Then $\tilde{F}_N(z, w) = -\lambda_N + \tilde{F}(z, w)$. Since $\tilde{F}(D \times D) \subset \Omega_0 \cup \Omega_1 \cup \Omega_2$ and $\lambda_N \notin \Omega_0 \cup \Omega_1 \cup \Omega_2$, it follows that $0 \notin \tilde{F}_N(D \times D)$, and therefore $F_N \in H_u(D)$. But since $L(I) = 0$, $L(F_N) = L(F) = Q$. Hence, $F_N \in \mathscr{F}(D, L, Q)$. Letting $N \to \infty$, we get a sequence in $\mathscr{F}(D, L, Q)$ with no converging subsequence; hence, $\mathscr{F}(D, L, Q)$ is nonempty and noncompact.

Remarks. (1) It is clear from the proof that the surface $\mathscr{R}$ depends on the choice of a_1 and a_2. Presumably a different choice for a_1 and a_2 might result in a new surface $\mathscr{R}'$, and then $\mathscr{F}(D, L, Q)$ would be nonempty and noncompact for all $Q \notin \mathscr{R} \cap \mathscr{R}'$. It seems plausible that by taking several choices of a_1 and a_2 we could prove that $\mathscr{F}(D, L, Q)$ is nonempty and noncompact for every Q in $\mathbb{C}^3$. Moreover, even if we do not vary a_1 and a_2, the restriction that $Q \notin \mathscr{R}$ is made so that α/β is nonnegative, and consequently, if we choose r sufficiently small, α'/β' is nonnegative. It is possible that even if Q is on $\mathscr{R}$, an appropriate choice of r would still leave α'/β' nonnegative. It is also possible that different choices of r for a_1 and a_2 would keep α'/β' nonnegative. Unfortunately, examples exist where none of these arguments work.

(2) The example of a triple of functionals ℓ_1, ℓ_2, ℓ_3 for which (*) fails for each pair seems somewhat contrived since (*) clearly holds for ℓ_2 and the linear functional $\ell_1 - \ell_3$. This observation leads to another possible generalization of (*):

(***) There are two linearly independent vectors in $\mathbb{C}^n$, $(\alpha_1, \cdots, \alpha_n)$ and $(\beta_1, \cdots, \beta_n)$, such that condition (*) is satisfied by the two linear functionals $\sum_{j=1}^n \alpha_j \ell_j$ and $\sum_{j=1}^n \beta_j \ell_j$.

Both conditions (**) and (***) are statements about the range R of L on $H_u(D) \cup \{\text{nonzero constants}\}$. Condition (**) says that $0 \notin R$, and it is possible to show, using Proposition 1, that condition (***) holds if and only if R does not intersect an (n - 2)-dimensional subspace of $\mathbb{C}^n$.

Condition (***) is clearly stronger than condition (**), so it is a sufficient condition for $\mathscr{F}(D, L, Q)$ to be compact whenever it is nonempty. (This can be proved directly from Theorem A.) We have been unable to find an example in which

$\mathscr{F}$(D, L, Q) is compact and nonempty, and (***) fails. However, we are, of course, unable to show (***) is necessary for $\mathscr{F}$(D, L, Q) to be compact.

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