# INVERSE LIMITS AND THE COMPLETENESS OF QUOTIENT GROUPS

David Wigner

In [5], G. Köthe gives an example of a complete topological vector space and a closed subspace such that the quotient space is not complete. In this paper we consider the question under what conditions the quotient of a complete abelian topological group by a closed subgroup is complete. We give sufficient conditions on the closed subgroup, and in general we define functors $L^i$ from abelian topological groups to abelian groups such that the vanishing of $L^1$ for the closed subgroup implies the completeness of the quotient. The $L^i$ are shown to be closely related to the derived functors of the inverse limit, and we can conclude that the derived functors of the inverse limit of a strongly dense inverse system (in the sense of R. F. Arens [1]) of complete metrizable abelian groups depend only on the natural topology of the inverse limit. All topological groups considered in this paper will be abelian, but not necessarily Hausdorff. We declare a sequence $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ of abelian topological groups and continuous homomorphisms to be exact if it is exact as a sequence of abstract groups, $\sigma$ is a homeomorphism onto its range, and $\tau$ is an open mapping. We use the following facts about exact sequences; the proofs are not difficult, and we omit them.

LEMMA 1. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence, and let* $\alpha\colon A \to A'$ *and* $\gamma\colon C'' \to C$ *be continuous homomorphisms. Then there exist commutative diagrams*

$$\text{(Q)} \qquad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

*and*

$$\text{(Q*)} \qquad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

*with exact rows.*

Here $B''$ is the fiber product of $C''$ and $B$ over $C$. This implies that the category of abelian topological groups is a quasi-abelian category in the sense of N. Yoneda [7].

LEMMA 2. *Let*

Received October 14, 1974.

Michigan Math. J. 22 (1975).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle id_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

*be a commutative diagram with exact rows. Then* $\beta$ *is an isomorphism; that is, there exists a mapping* $\beta_1: B \to B_1$ *such that* $\beta_1 \cdot \beta$ *is the identity.*

If $A$ is an abelian topological group, $\hat{A}$ will denote the Hausdorff completion of $A$ for the group uniformity.

THEOREM 1. *If* $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ *is an exact sequence, then* $0 \to \hat{A} \xrightarrow{\hat{\sigma}} \hat{B} \xrightarrow{\hat{\tau}} \hat{C}$ *is left exact as a sequence of topological groups* ($\hat{\sigma}$ *is a topological inclusion, and* $\hat{\tau}$ *is open onto its range.*) *If* $B$ *is metrizable, then* $\hat{\tau}: \hat{B} \to \hat{C}$ *is surjective.*

For the proof, see N. Bourbaki [2, p. 163].

THEOREM 2. *Let* $0 \to A \to B \xrightarrow{\tau} C \to 0$ *be a short exact sequence of abelian topological groups, and let* $A$ *be metrizable. Then* $\hat{\tau}: \hat{B} \to \hat{C}$ *is surjective.*

*Proof.* There is an invariant pseudometric on $B$ that induces the group topology of $A$. Let $B_1$ denote $B$ with the topology induced by this pseudometric. We get an exact sequence $0 \to A \to B_1 \to C_1 \to 0$ whose Hausdorff completion $0 \to \hat{A} \to \hat{B}_1 \to \hat{C}_1 \to 0$ is exact, by Theorem 1. Consider the diagram (diagram Q*)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{A} & \longrightarrow & E & \longrightarrow & \hat{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_{\hat{A}}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{A} & \longrightarrow & \hat{B}_1 & \longrightarrow & \hat{C}_1 & \longrightarrow & 0, \end{array}$$

where $E$ is the fiber product of $\hat{B}_1$ and $\hat{C}$ over $\hat{C}_1$. Because of Lemma 2, we get a natural isomorphism of $\hat{B}$ onto a dense subgroup of $E$, whence $\hat{B} \cong E$, since $E$ is Hausdorff. Hence $\hat{\tau}$ is surjective.

THEOREM 3. *Let* $A \cong \prod_{\alpha \in I} M_\alpha$ *be a product of metrizable abelian topological groups. Then* $\hat{\tau}: \hat{B} \to \hat{C}$ *is surjective.*

*Proof.* Consider the sequences $0 \to M_\alpha \to E_\alpha \to C \to 0$ induced by the projections $C_\alpha: A \to M_\alpha$. Then, for each $\alpha$, the sequence $0 \to \hat{M}_\alpha \to \hat{E}_\alpha \to \hat{C} \to 0$ is exact. Because the product of exact sequences is exact,

$$0 \to \prod_I \hat{M}_\alpha \to \prod_I \hat{E}_\alpha \to \prod_I \hat{C} \to 0$$

is exact. Consider the diagram (diagram Q*)

$$\begin{tikzcd}
0 \arrow[r] & \hat{A} \arrow[r] \arrow[d, "\mathrm{id}_{\hat{A}}"'] & E \arrow[r] \arrow[d] & \hat{C} \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \prod_I \hat{M}_\alpha \arrow[r] & \prod_I \hat{E}_\alpha \arrow[r] & \prod_I \hat{C} \arrow[r] & 0 ,
\end{tikzcd}$$

where $\delta$ is the diagonal mapping. Again because of Lemma 2, we get a natural isomorphism of $\hat{B}$ with a dense subgroup of E; therefore $\hat{B} \cong E$. Hence $\hat{\tau}$ is surjective.

Because each abelian topological group A is isomorphic to a subgroup of a product $\Pi$ of metrizable groups, we can find a short exact sequence $0 \to A \xrightarrow{\sigma} \Pi \xrightarrow{\tau} C \to 0$. We define $L^1(A)$ as the quotient of $\hat{C}$ by the image of $\hat{\Pi}$ under $\hat{\tau}$. Given another embedding $\sigma_1 : A \to \Pi_1$ of A in a product of metrizable groups, we can form the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\sigma+\sigma_1"] \arrow[d, "\mathrm{id}_A"'] & \Pi \oplus \Pi_1 \arrow[r, "\rho"] \arrow[d, "\alpha"] & D \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & A \arrow[r, "\sigma"] & \Pi \arrow[r, "\tau"] & C \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since $\tau$ and $\alpha$ are open, $\chi$ is also open, and we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & \hat{\Pi}_1 \arrow[r] \arrow[d, "\mathrm{id}_{\hat{\Pi}_1}"'] & \widehat{\Pi \oplus \Pi_1} \arrow[r] \arrow[d, "\hat{p}"] & \hat{\Pi} \arrow[r] \arrow[d, "\hat{\tau}"] & 0 \\
0 \arrow[r] & \hat{\Pi}_1 \arrow[r] & \hat{D} \arrow[r] & \hat{C} \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact, by Theorem 3. This gives an isomorphism between the quotient of $\hat{C}$ by $\hat{\tau}\hat{\Pi}$ and the quotient of $\hat{D}$ by $\hat{p}(\widehat{\Pi \oplus \Pi_1})$. We conclude that $L^1(A)$ is well defined. Note that $L^1(A) \cong L^1(\hat{A})$.

THEOREM 4. *If* $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ *is an exact sequence with* $L^1(A) = 0$, *then* $\hat{\tau}: \hat{B} \to \hat{C}$ *is surjective.*

*Proof.* We embed $\hat{B}$ in a product $\Pi$ of complete metrizable groups and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \hat{A} \arrow[r] \arrow[d, "\mathrm{id}_{\hat{A}}"'] & \hat{B} \arrow[r] \arrow[d] & \hat{\tau}\hat{B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{A} \arrow[r] & \Pi \arrow[r] & D \arrow[r] & 0 .
\end{tikzcd}$$

D is complete, and since $\hat{B}$ is closed in $\Pi$, $\hat{\tau}\hat{B}$ is closed in D, hence complete. Hence $\hat{\tau}$ is surjective.

In the remainder of this paper, we assume familiarity with the theory of the derived functors of the inverse limit (see for instance [3]), which will be shown to be intimately related to the derived functors (in the sense of Yoneda [7]) of the completion functor. Following Arens [1], we call a directed inverse system $[A_\alpha]_{\alpha \in I}$ of abelian topological groups a *strongly dense inverse system* if for each $\beta \in I$ the natural mapping $:\varprojlim A_\alpha \to A_\beta$ has dense range. It is clear that every quotient of a strongly dense inverse system is strongly dense. We need the following lemma.

LEMMA 3. *Let* $0 \to [A_\alpha] \to [B_\alpha] \to [C_\alpha] \to 0$ *be a short exact sequence of strongly dense inverse systems of topological groups; that is, for each* $\alpha$, *let*

$$0 \longrightarrow A_\alpha \xrightarrow{\phi_\alpha} B_\alpha \xrightarrow{\chi_\alpha} C_\alpha \longrightarrow 0$$

*be an exact sequence of topological groups; also, let* $0 \xrightarrow{\phi} A \to B \xrightarrow{\chi} C$ *be the sequence of inverse limits. Then* $\phi$ *is a topological embedding and* $\chi$ *is open onto its range, which is dense in* C (*that is,* $0 \to A \to B \to C$ *is left exact as a sequence of topological groups*).

*Proof.* The only nontrivial assertion is that $\chi$ is open onto its range. It will be enough to show that if U is open in $B_\alpha$, and if

$$\rho_\alpha: B \to B_\alpha, \qquad \Pi_\alpha: C \to C_\alpha, \qquad \sigma_\alpha: A \to A_\alpha$$

are the natural mappings, then $\chi(\rho_\alpha^{-1}(U)) \supset \chi(B) \cap \Pi_\alpha^{-1}(\chi_\alpha(U))$. If

$$x \in \chi(B) \cap \Pi_\alpha^{-1}\, \chi_\alpha(U),$$

we can write $x = \chi(y)$ and $\Pi_\alpha(x) = \chi_\alpha(u)$, with $u \in U$. Then $\chi_\alpha(\rho_\alpha(y) - u) = 0$, and since U is open and $\sigma_\alpha$ has dense range, we can find $n \in A$ such that

$$\phi_\alpha\, \sigma_\alpha(n) + \rho_\alpha(y) = \rho_\alpha\, \phi(n) + \rho_\alpha(y) \in U\,.$$

Then $\gamma = \phi(n) + y \in \rho_\alpha^{-1}(U)$ and $\chi(\gamma) = x$, which proves the lemma.

Now Yoneda [7] has defined the derived functors for each left exact functor from a quasi-abelian category to the category of abelian groups. These functors are unique, subject to the conditions of effaceability and exactness. In particular, we can define the derived functors of the functor L that assigns to each abelian topological group the abstract group of points in its completion.

The $L^i$ are then unique, subject to the conditions

(1) $L^0 = L$,

(2) for each short exact sequence $0 \to A \to B \to C \to 0$ there exists a (functorial) long exact sequence

$$0 \to L^0(A) \to L^0(B) \to L^0(C) \to L^1(A) \to L^1(B) \to L^1(C) \to L^2(A) \to \cdots,$$

(3) if $x \in L\,(A)$ and $i > 0$, there exists a short exact sequence

$$0 \to A \xrightarrow{\alpha} B \to C \to 0$$

such that x is in the kernel of $\alpha_*: L^i(A) \to L^i(B)$.

The discussion above shows that $L^1$ is the first derived functor of L. Let $L^i$ denote the ith derived functor of L.

THEOREM 5. *Let* A *be the inverse limit of a strongly dense directed inverse system* $[A_\alpha]$, *where each* $A_\alpha$ *is a product of complete metrizable abelian topological groups. Let* $\Pi$ *be any product of complete metrizable abelian topological groups. Then, for all* $i > 0$,

(a) $L^i(\Pi) = 0$,

(b) $L^i(A) \cong \lim^{(i)} [A_\alpha]$.

*Here* $\lim^{(i)}$ *denotes the* $i$*th derived functor of the inverse limit* (see [3]).

Note that every complete group A is representable by an inverse limit, as in the theorem.

*Proof* (by induction on i). Suppose $i = 1$ and $x \in L^1(\Pi)$; since $L^1$ is effaceable, there exist a group E and an inclusion $\phi: \Pi \to E$ such that $\phi_*(x) = 0$. But $0 \to \Pi \to \hat{E} \to \widehat{E/\Pi} \to 0$ is exact, by Theorem 3, so that the exactness of the sequence $\hat{E} \to \widehat{E/\Pi} \to L^1(\Pi) \to L^1(E)$ implies $x = 0$. This proves (a) for $i = 1$. For (b), we embed $[A_\alpha]$ in a flasque inverse system $[B_\alpha]$ of products of complete metrizable groups (see [3]). Here $B_\alpha \cong \prod_{\beta \leq \alpha} A_\beta$, for all $\alpha$. We define $B = \varprojlim B_\alpha \cong \prod_{\alpha \in I} A_\alpha$. We get a short exact sequence of inverse systems $0 \to [A_\alpha] \to [B_\alpha] \to [C_\alpha] \to 0$ in which each $C_\alpha$ is complete; this gives rise to an exact sequence

$$0 \to A \to B \to C \to \lim^{(1)} (A_\alpha) \to 0,$$

so that $L^1(A) \cong B/C \cong \lim^1 [A_\alpha]$. This proves (b) for $i = 1$. Now assume the theorem is true for all j $(0 < j \leq i)$. Since $L^{i+1}$ is effaceable, there exists, for each $x \in L^{i+1}(\Pi)$, an exact sequence $0 \to \Pi \xrightarrow{\phi} E$ such that $\phi_*(x) = 0$. Now each coordinate projection of $\Pi = \prod_\eta M_\eta \to M_\eta$ gives rise to an invariant pseudometric $p_\eta$ on $\Pi$ that can be extended to E. We consider the Hausdorff completion $E_\eta$ of E with respect to $p_\eta$. More generally, we consider, for each finite subset $F \subset I$, the completion $E_F$ of E with respect to the invariant pseudometric $\sum_{\eta \in F} p_\eta$. The $E_F$ form an inverse system in a natural way, we have for each F an exact sequence

$$0 \to \prod_{\eta \in F} M_\eta \xrightarrow{\chi} E_F \to E_F / \prod_{\eta \in F} M_\eta \to 0$$

of complete metrizable abelian topological groups, $\Pi \simeq \varprojlim_F \prod_{\eta \in F} M_\eta$, and we define

$$G = \varprojlim_F E_F, \qquad H = \varprojlim_F E_F / \prod_{\eta \in I} M_\eta .$$

The sequence $0 \to \Pi \xrightarrow{\chi} G \to H \to 0$ is exact, and $\chi_*(x) = 0$, since $\phi_*(x) = 0$. But since $\left[ \prod_{\eta \in F} M_\eta \right]$ is a flasque system,

$$\lim_F{}^{(i)} [E_F] \to \lim^{(i)} \left[ E_F / \prod_{\eta \in F} M_\eta \right]$$

is surjective; therefore, by the induction hypothesis, the natural mapping $L^i(G) \to L^i(H)$ is surjective, and the exact sequence

$$L^i(G) \to L^i(H) \to L^{i+1}(\Pi) \xrightarrow{\chi_*} L^{i+1}(G)$$

implies $x = 0$. This proves (a) for $i + 1$. For (b), we embed $[A_\alpha]$ in a flasque system $[B_\alpha]$ whose inverse limit will be a product of complete metrizable groups. If $[C_\alpha]$ denotes the quotient system and C its inverse limit, then we get natural isomorphisms

$$L^i(C) \cong L^{i+1}(A) \quad \text{and} \quad \lim^{(i)} [C_\alpha] \cong \lim^{(i+1)} [A_\alpha].$$

This completes the proof.

COROLLARY. *If* $[A_\alpha]$ *is a strongly dense inverse system of products of complete, metric, abelian topological groups, the functors* $\lim^{(i)} [A_\alpha]$ *depend only on the natural group topology of the inverse limit group* A.

## REFERENCES

1. R. F. Arens, *Dense inverse limit rings*. Michigan Math. J. 5 (1958), 169-182.
2. N. Bourbaki, *Elements of mathematics. General topology*. Part II. Hermann, Paris; Addison-Wesley Publishing Co., Reading, Mass. - London - Don Mills, Ontario; 1966.
3. C. U. Jensen, *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*. Lecture Notes in Mathematics No. 254, Springer-Verlag, Berlin - Heidelberg - New York, 1972.
4. S. Kakutani, *Über die Metrisation der topologischen Gruppen*. Proc. Imperial Acad. Tokyo 12 (1936), 82-84.
5. G. Köthe, *Die Quotientenräume eines linearen vollkommenen Raumes*. Math. Z. 51 (1947), 17-35.
6. L. S. Pontrjagin, *Continuous groups*. GITTL, Moscow, 1938.
7. N. Yoneda, *On* Ext *and exact sequences*. J. Fac. Sci. Univ. Tokyo Sect. I 8 (1960), 507-576.

University of Michigan
Ann Arbor, Michigan 48104