THE STRUCTURE OF JORDAN H-ALGEBRAS

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1. INTRODUCTION

In [3], I. N. Herstein considered an associative ring R with the property that for each x in R there exists a polynomial $p_x(X)$ with integral coefficients, in an indeterminant X, such that $x^2 p_x(x) - x$ lies in the center of R, and he proved that every associative ring with this property is commutative. Further, Herstein and S. Montgomery [6] have extended Herstein's result by proving that if an associative division ring with involution has the property that to each symmetric element $x = x^*$ there corresponds an integer $n > 1$ such that $x^n - x$ is central, then the ring is commutative or four-dimensional over its center. In [1], W. Burgess and M. Chacron have demonstrated that in an associative ring R with involution, the property that $x^2 p_x(x^2) - x$ is central for each symmetric element implies that R is an integral extension of its center of degree not larger than 2. It is our aim to determine the structure of a Jordan algebra over a field of characteristic not 2 that satisfies the condition of Herstein's theorem [3]. In particular, we shall call a Jordan algebra J a *Jordan* H-*algebra* if for every x in J there exists a polynomial $p_x(X)$, in an indeterminant X with integral coefficients such that $x^2 p_x(x) - x$ lies in the center of J. The following result is our main theorem.

THEOREM 1. *Let* J *be a Jordan* H-*algebra over a field of characteristic not two; then* J *is isomorphic to a subdirect sum of Jordan algebras* $\{J_i\}_{i\in\Lambda}$, *where each* J_i *is either an associative algebra or a simple periodic Jordan algebra of capacity two.*

We remark that in [7], an associative ring R is called an H-ring if for every x in R there exists an integer $n(x) > 1$ such that $x^{n(x)} - x$ is in the center of R. However, from results in [3] and [2, p. 220] it follows immediately that this condition and the centrality of $x^2 p_x(x) - x$ are equivalent. Though we have not proved this equivalence for Jordan algebras, we shall use here the more general condition to denote a Jordan H-algebra.

As corollaries to Theorem 1 we shall prove that if J is as in Theorem 1, then J is associative if either its idempotents are central, or it contains no idempotents different from zero or one, or it contains a unique nonzero idempotent. Further, we shall prove analogues, for Jordan algebras over a field of characteristic not 2, of results given in [4] and [9].

The proof of Theorem 1 will proceed as follows. After some preliminary lemmas, we shall reduce the problem to one of determining the structure of a subdirectly irreducible Jordan algebra satisfying the hypothesis of Theorem 1. Next, the subdirectly irreducible case will be divided into two cases, depending on whether the algebra contains an idempotent different from zero and the identity, that is, whether the algebra contains a nontrivial idempotent. When the subdirectly irreducible algebra contains nontrivial idempotents, the algebra will be a simple periodic Jordan algebra of capacity 2. The proof of this will proceed similarly to the proof

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of the corresponding result for periodic Jordan rings [14]. We shall achieve the proof of the remaining case by first imbedding the algebra in an algebra of continuous sections, to prove that it is associative provided it has no nonzero nilpotent elements. Then we shall show that it is associative, by employing an argument for associators similar to the one used in [3] for commutators. At several stages in the proof of Theorem 1, we shall have occasion to use arguments similar to those employed in [3] and [14]. For the reader's convenience we shall include these arguments.

Finally, we remark that as in [3], a more general condition is sufficient in the division ring case. In particular, a Jordan division algebra J of characteristic different from 2 is associative provided for each x in J there exist an integer $n(x) = n > 1$ and a polynomial $p_x(X)$ with integer coefficients such that $x^{n+1}p_x(x) - x^n$ lies in the center of J. Because we shall have no occasion to refer to this result, we shall not prove it here; we merely note that it can be proved in an analogous manner to the corresponding result in [3], by means of Lemma 1 of [10].

2. PRELIMINARY LEMMAS

Throughout the remainder of this paper, all Jordan algebras will be over a field of characteristic different from 2, Z(J) will denote the center of the Jordan algebra J, and N will denote the set of nilpotent elements. We begin by proving some lemmas that will be useful in both subdirectly irreducible cases. The first one and its proof coincide essentially with the result of Herstein's [3].

LEMMA 1. *If* J *is a Jordan* H-*algebra, then* N *is an ideal contained in the center of* J.

Proof. Let $x \in N$, and suppose that $x^{2^n} = 0$. Then, since $x^2 p_x(x) - x \in Z(J)$, we see that $x^4 q(x) - x^2 p_x(x) \in Z(J)$, where

$$q(x) = [p_x(x)]^2 p_y(x^2 p_x(x)) \qquad \text{and} \qquad y = x^2 p_x(x).$$

Hence $x^4 q(x) - x \in Z(J)$. Continuing this process, we deduce after $n - 1$ steps that

$$x = x - x^{2^n} r(x) \in Z(J),$$

where $r(x)$ is a polynomial in x with integral coefficients. Thus $N \subseteq Z(J)$, and it follows immediately that N is an ideal.

For a Jordan ring J with an idempotent e, we shall use the Pierce decomposition $J = J_1(e) + J_{1/2}(e) + J_0(e)$ of J with respect to e. Also, in accordance with [14], we say that x in J is *anti-integral* provided there exists a polynomial $q_x(X)$ with integral coefficients and such that $x^2 q_x(x) - x = 0$. With this terminology we can prove the next lemma, which is useful in the application of the techniques of [14] for periodic Jordan rings to Jordan H-algebras.

LEMMA 2. *Let* J *be a Jordan* H-*algebra and* e *an idempotent in* J. *If* $J_{1/2}(e) \neq (0)$, *then* J *has prime characteristic and every element of* $J_{1/2}(e)$ *is periodic.*

Proof. Let x be a nonzero element of $J_{1/2}(e)$. We begin by showing that x is anti-integral. Without loss of generality, we can suppose that the element $y = x^2 p_x(x) - x \in Z(J)$ is not zero. Now let $y = y_1 + y_{1/2} + y_0$, with $y_i \in J_i(e)$ for $i = 0, 1/2, 1$. Then, since $y \in Z(J)$, we see that $ye = (ye)e$ and hence

$$y_1 + \frac{1}{2} y_{1/2} = ye = \left(y_1 + \frac{1}{2} y_{1/2} \right) e = y_1 + \frac{1}{4} y_{1/2}.$$

Thus, $0 = y_{1/2} = (x^2 p_x(x) - x)_{1/2}$, or equivalently,

$$(x^2 p_x(x))_{1/2} = x_{1/2} = x.$$

Now, if $p_x(x) = \sum_{i=0}^n t_i x^i$ for integers t_i, then $x^2 p_x(x) = \sum_{i=0}^n t_i x^{i+2}$, and

$$(x^2 p_x(x))_{1/2} = \sum_{i=0}^{n} t_i (x^{i+2})_{1/2} = \sum t_i x^{i+2},$$

where the last sum is taken over all odd indices i. Hence, if $q_x(X) = \sum t_i X^i$ (summed on the odd indices i), then

$$x^2 q_x(x) = [x^2 p_x(x)]_{1/2} = x,$$

and $x^2 q_x(x) - x = 0$.

If F is the base field of the algebra J, and if F has characteristic zero, it follows that the subring of J generated by $x \neq 0$ in $J_{1/2}(e)$ is an associative ring in which $0 \neq \alpha x$ is anti-integral for each integer α. But then Proposition 13.1 of [14] implies that the ordinary integers form a field, which is absurd. Hence, F has prime characteristic p.

Let Z_p denote the Galois field of p elements. Then, if x is a nonzero element of $J_{1/2}(e)$, it follows that $Z_p[x]$ is a finite, associative, commutative algebra. But then there exist positive integers m and n with $(x^{p^n} - x)^m = 0$. Moreover, since p is odd, $x^{p^n} - x \in J_{1/2}(e) \cap N$. But $J_{1/2}(e) \cap Z(J) = 0$, since each $z \in J_{1/2}(e) \cap Z(J)$ satisfies the equations $\frac{1}{2} z = ez = e(ez) = \frac{1}{4} z$. Thus $x^{p^n} = x$, and the proof of Lemma 2 is complete.

The next two lemmas are analogous to results of J. M. Osborn in [14]. First we need some terminology. In accordance with [14] and [8], a pair of orthogonal idempotents e and f of a Jordan algebra J is called *connected* if $J_{1/2}(e) \cap J_{1/2}(f)$ contains a unit in JU_{e+f}, and it is called *weakly connected* if $J_{1/2}(e) \cap J_{1/2}(f)$ is not zero.

LEMMA 3. *Let* J *be a Jordan* H-*algebra; then* J *does not contain three connected idempotents.*

Proof. Let e_1, e_2, e_3 be three connected idempotents in J. By the coordinization theorem [8, p. 137], there is an alternative algebra with involution (A, *) whose symmetric elements S(A, *) are contained in the nucleus of A, and with the property that $J' = JU_{e_1+e_2+e_3} = S(A_3, J_a)$. In other words,

$$J' = \left\{ \begin{pmatrix} \alpha & \beta^* \mu & \delta^* \nu \\ \beta & \gamma & \mu^{-1} \varepsilon^* \nu \\ \delta & \varepsilon & \eta \end{pmatrix} : \alpha, \nu, \mu \in S(A, *), \ \gamma^* = \mu \gamma \mu^{-1}, \ \eta^* = \nu \eta \nu^{-1} \right\}.$$

Since the nucleus of A is invariant under $*$, we can suppose without loss of generality that A is associative. For convenience, we let D denote A_3.

Let b_{12} be a unit in $J_{1/2}(e_1) \cap J_{1/2}(e_2)$, and let b_{13} be a unit in $J_{1/2}(e_1) \cap J_{1/2}(e_3)$, each chosen so that its entries lie in the nucleus of A. Next, let $D = \sum D_{ij}$ for i, j = 1, 2, 3 be the Pierce decomposition of D by e_1, e_2, e_3. Then

$$b_{12} = e_1 b_{12} + e_2 b_{12},$$

where $e_1 b_{12} \in D_{12}$, $e_2 b_{12} \in D_{21}$, and juxtaposition denotes the associative product in D. If $e_1 b_{12} = h + s$, where h is symmetric and s is skew-symmetric, then

$$0 = (e_1 b_{12})^2 = (h+s)^2 = (h^2 + s^2) + (hs + sh),$$

and it follows that $hs = -sh$ and $h^2 = -s^2$.

Let B be the subring of D generated by h and s, and let ψ be the subring of B generated by h^2. Then ψ is contained in the center of B. Moreover, since $h = \frac{1}{2}(e_1 b_{12} + b_{12} e_1) \in J_{1/2}(e_1)$, we see that h is periodic, and hence h^2 is periodic. Therefore ψ is finite with identity e'. If x in ψ is a nonzero nilpotent, then $x \in J_1(e_1) + J_0(e_1)$, and therefore xh (the product taken in J) is in $J_{1/2}(e_1)$. Thus $xh = 0$, since the elements of $J_{1/2}(e_1)$ are periodic and J is power-associative. But then $U_h = U_{b_{12}}$ has nonzero kernel in $JU_{e_1+e_2}$, which is impossible, since b_{12} is a unit in $JU_{e_1+e_2}$. Hence, ψ contains no nonzero nilpotent elements, and therefore ψ is a direct sum of a finite number of Galois fields. Let e" be a primative idempotent in ψ; then e" is in the center of B. Now, replacing B by e" B and ψ by e" ψ, we find that e" B is an algebra over e" ψ spanned by {e", e" h, e" sh, e" s} with

$$(e''h)(e''s) = -(e''s)(e''h) \quad \text{and} \quad (e''h)^2 = -(e''s)^2.$$

For simplicity of notation, it will suffice for our purposes to suppose that ψ is a field and consider B an algebra over ψ with spanning set {e', h, sh, s}.

Our next task is to prove that B is a simple algebra. Let C be an ideal of B, and suppose that $a = \alpha_1 e' + \alpha_2 h + \alpha_3 sh + \alpha_4 s$, with $\alpha_i \in \psi$, is a nonzero element of C. Then

$$(ah - ha)s = (-2\alpha_3 h^4)e' + (2\alpha_4 h^2)h$$

is an element of C. Hence, whether or not $\alpha_3 = \alpha_4 = 0$, there exist β_1, $\beta_2 \in \psi$, not both zero, with

$$0 \neq a' = \beta_1 e' + \beta_2 h \in C.$$

If $\beta_2 \neq 0$, then

$$(a's - sa')sh = -2\beta_2 h^4 e',$$

and it follows that in any case $e' \in C$, so that $C = B$ and B is a simple associative algebra of dimension at most 4 over ψ.

Now $(e_1 b_{12})(e_2 b_{12}) = (e_1 b_{12})(e_1 b_{12})^* = (h+s)(h-s) \in S$. Therefore B contains the component of b_{12}^2 in $D_{11} = D_1(e_1)$. However, b_{12} connects e_1 and e_2 and is periodic; hence, e_1, $e_2 \in B$, and it follows that B is isomorphic to ψ_2. If we let e_{12}, $e_{21} \in B$ be the usual matrix units, then $\{e_1, e_{12}, e_{21}, e_2\}$ is a basis for B with $e_{12}^* = e_{21}$, since $B^* = B$. Similarly, we can do the same for e_1 and e_3 to get e_{13}, e_{31}. Next, let $e_{23} = e_{21}e_{13}$ and $e_{32} = e_{31}e_{12}$; then, over the prime field Z_p of ψ, we get a Jordan H-ring K of capacity 3 contained in $S(A_3, J_a)$.

By Lemma 15.2 of [14], for each pair of elements μ and ν in Z_p, there exist element ρ, σ, $\tau \in Z_p$, not all zero with $\rho^2 + \sigma^2\mu + \tau^2\nu = 0$. But then the element

$$x = \begin{pmatrix} \rho^2 & \rho\sigma\mu & \rho\tau\nu \\ \rho^2\sigma & \sigma^2\mu & \sigma\tau\nu \\ \rho\tau & \sigma\tau\mu & \tau^2\nu \end{pmatrix}$$

is in K and squares to zero. Hence $x \in N \in Z(K)$. However, this implies that two of the elements ρ, σ, τ are zero, which in turn implies that all three must be zero; this gives a contradiction and completes the proof of Lemma 3.

Using Lemma 3, we can prove the final result of this section.

LEMMA 4. *If* J *is a Jordan* H-*algebra with orthogonal idempotents* e, f, g *such that* e *and* f *are connected, then* g *is not weakly connected to* e.

Proof. We begin by supposing that $J_{1/2}(e) \cap J_{1/2}(g)$ is not zero and show that this implies that J contains three connected idempotents. An appeal to Lemma 3 will then complete the proof of Lemma 4.

Let $c \neq 0$ be an element of $J_{1/2}(e) \cap J_{1/2}(g)$; then c is periodic, so that c^n is idempotent for some n. Let $c^n = u + v$, with $u \in J_1(e)$ and $v \in J_1(g)$. If $v = 0$, then $u \neq 0$ and $c = cu \in J_1(u)$. But

$$J_1(u) = JU_u = JU_{uU_e} = JU_e U_u U_e \subseteq J_1(e),$$

which is impossible. Similarly, $u \neq 0$, and it follows that u and v are connected orthogonal idempotents in J.

Next, let b connect e and f, so that b is periodic and $b^m = e + f$ for some positive integer m. Now we consider $2bu$. Since $J_{1/2}(e) \subseteq J_{1/2}(u) + J_0(u)$, we see that

$$2bu \in J_{1/2}(u)J_1(u) \subseteq J_{1/2}(u) \quad \text{and} \quad 2bu \in J_{1/2}(f)J_0(f) \subseteq J_{1/2}(f).$$

If $bu = 0$, then $b \in J_0(u)$. But b is a unit in JU_{e+f}, and $u \in JU_{e+f}$, so that $uU_b \neq 0$. Hence $u(2R_b^2 - R_{b^2}) \neq 0$, and it follows that $0 \neq uR_{b^2}$. But $b^2 \in J_0(u)$, so that $uR_{b^2} = 0$, and this contradiction implies that $bu \neq 0$. Thus, $2bu$ weakly connects e and f. By Lemma 2, $2bu$ is periodic, in other words, there exists a positive integer s with $(2bu)^s = u_1 + w$, where $u_1 \in J_1(u)$ and $w \in J_1(f)$ and both are idempotent. Moreover, as above, both u_1 and w are nonzero. Further, $(2bu)^2 = h_1 + h_2$, where $h_1 \in J_1(u)$ and $h_2 \in J_1(f)$, is also periodic and hence $h_1 = (2bu)^2u$ is periodic.

We claim that $((2bu)^2u)^\ell = u$ for some positive integer ℓ. By restricting our attention to the subalgebra generated by b and u, we can assume, by virtue of the Shirshov-Cohn Theorem, that J is a special Jordan algebra [8]. In particular, $J \subseteq A^+$, where A is an associative algebra and where for each pair $x, y \in J$, xy

denotes the product in A and x . y the product in A^+. We want to show that in this notation,

$$[(ab\,.\,u)^2\ .\ u]^{\ell} = u\,.$$

Let $A = A_{11} + A_{10} + A_{01} + A_{00}$ be the Pierce decomposition of A with respect to u. Then $b \in A_{10} + A_{01} + A_{00}$ and $ubu = 0$. Therefore

$$(2b\,.\,u)^2\ .\ u = (bu + ub)^2\ .\ u = (bub + ub^2u)\,.\,.\ u = ub^2u\,.$$

Therefore, we must show that in turn $(ub^2u)^{\ell} = u$. Using the fact that $(ub^2u)^{\ell}$ is idempotent, we obtain the relations

$$t = 2(u - (ub^2u)^{\ell})\ .\ b = ub + bu - (ub^2u)^{\ell}b - b(ub^2u)^{\ell}\,.$$

Hence,

$$t^2 = bub - b(ub^2u)b\ .$$

Therefore,

$$t^4 = bub^2ub - b(ub^2u)^{\ell+1}b - b(ub^2u)^{\ell+1}b + b(ub^2u)^{2\ell+1}b = 0\ .$$

But $2b\,.\,u \in J_{1/2}(u)$, so that $(2b\,.\,u)^2\ .\ u \in J_1(u)$; also, $b \in J_{1/2}(u)$, so that $t \in J_{1/2}(u)$ and hence t cannot be nilpotent. Thus $t = 0$, and since $b^m = e + f$, we have the relations

$$0 = tb^{m-1} = ub^m + bub^{m-1} - (ub^2u)^{\ell}b^m - b(ub^2u)^{\ell}b^{m-1}$$

$$= u - (ub^2u)^{\ell} + t^2b^{m-2} = u - (ub^2u)^{\ell}\,.$$

Hence $u = (ub^2u)^{\ell}$, as we claimed.

But now

$$u_1 = u_1^{\ell} = [(2b\,.\,u)^2\ .\ u]^{s\ell} = [[(2b\,.\,u)^2\ .\ u]^{\ell}]^s = u^s = u\ ,$$

and it follows that u, v, and w are connected. Therefore, it follows from Lemma 3 that g is indeed not weakly connected to e.

Next we note that each Jordan H-algebra can be considered as a subdirect sum of subdirectly irreducible Jordan H-algebras. Hence we shall restrict our attention to the case where J is a subdirectly irreducible Jordan H-algebra with unique nonzero minimal ideal S. Thus, if $N \neq (0)$, then $S \subseteq N$. In order to determine the structure of subdirectly irreducible Jordan H-algebras, we shall consider separately algebras with idempotents, different from zero or the identity, that is, with nontrivial idempotents, and algebras without nontrivial idempotents. We begin with the former case.

3. THE CASE WITH NONTRIVIAL IDEMPOTENTS

In this section, J will always denote a subdirectly irreducible Jordan H-algebra with a nontrivial idempotent e and Pierce decomposition $J_1(e) \oplus J_{1/2}(e) \oplus J_0(e)$ with respect to e. We shall prove that J is then a simple, periodic Jordan algebra of capacity 2.

LEMMA 5. *If* $N \neq 0$, *then* $S \subseteq J_1(e)$ *or* $S \subseteq J_0(e)$, *and* $S \subseteq J_i(e)$ *implies* $N \subseteq J_i(e)$ *for* $i = 0, 1$.

Proof. The first statement in the lemma follows from the fact that N is an ideal and the sets Se and $\{se - s: s \in S\}$ are ideals, with $Se \in J_1(e)$ and $\{se - s: s \in S\} \subseteq J\ (e)$. Next, $x \in N$, $x = x_1 + x_{1/2} + x_0$, with respect to e, implies

$$x_1 + \frac{1}{2}x_{1/2} = xe = (xe)e = x_1 + \frac{1}{4}x_{1/2}.$$

Hence $x_{1/2} = 0$. Now $N = N \cap J_1(e) + N \cap J_0(e)$ and $N \cap J_i(e)$ $(i = 0, 1)$ are disjoint ideals of J. Since J is subdirectly irreducible, the lemma follows.

Using Lemma 5, we can prove our next result.

LEMMA 6. *If* $N \neq (0)$, *then* J *does not contain* n *primitive, orthogonal idempotents whose sum is the identity.*

Proof. If $n = 2$ and $1 = e + f$, where e and f are primitive orthogonal idempotents, then by Lemma 5 we can suppose that $N \subseteq J_0(e) = J_1(f)$. Moreover, since J is subdirectly irreducible, $J_{1/2}(e) \neq (0)$. Take $y \neq 0$ in $J_{1/2}(e)$; then there exists a positive integer m such that $y^m = u + v$, where $u^2 = u \in J_1(e)$ and $v^2 = v \in J_0(e)$. Further, $u \neq 0 \neq v$; for if $v = 0$, then as in Lemma 4, $u \neq 0$ and $y = yu \in J_1(e)$. But

$$J_1(e) = JU_u = J_u U_e \subseteq J_1(e),$$

which contradicts the choice of y. Since e and f are primitive,

$$u = e, \qquad v = f, \qquad \text{and } y^m = 1.$$

Now if we take $x \neq 0$ in N, then $xy \in J_{1/2}(e) \cap N = (0)$, so that

$$x = xy^m = (xy)y^{m-1} = 0.$$

However, because $x \neq 0$, we must conclude that if 1 is the sum of n primitive orthogonal idempotents, then $n > 2$.

Suppose that $\sum_{i=1}^{n} e_i$, where the e_i form a set of primitive, orthogonal idempotents, and suppose that $J = \sum_{i \leq j} J_{ij}$ is the Pierce decomposition of J with respect to the e_i. Since J is subdirectly irreducible, there exist integers k and ℓ such that $J_{k\ell}$ is different from zero. But then, as above, e_k and e_ℓ are connected idempotents in J, and hence, by Lemma 4, J_{ki} and $J_{\ell i}$ are zero for each i different from ℓ and k. Since this implies that

$$JU_{e_k + e_\ell} = J_1(e_k) \oplus J_{k\ell} \oplus J_1(e_\ell) \quad \text{and} \quad \sum J_{ij} \quad (i, j \neq k \text{ and } i, j \neq \ell)$$

are disjoint ideals of J, we have proved the lemma.

LEMMA 7. J *has capacity* 2, *and it has no nonzero nilpotent elements.*

Proof. Let e be a nontrivial idempotent in J; then by hypothesis $J_{1/2}(e)$ is nonzero, and by Lemma 2, $x \in J_{1/2}(e)$ $(x \neq 0)$ is periodic. Therefore, J contains connected idempotents u and v. We claim that u and v are primitive idempotents and that $J = J_1(u+v)$. Suppose that there exists $x \neq 0$ in $J_{1/2}(u+v)$, such that $x = x_1 + x_2$, where $x_1 \in J_{1/2}(u) \cap J_0(v)$ and $x_2 \in J_{1/2}(v) \cap J_0(u)$. Hence

$$x_1^2 \in [J_1(u) + J_0(u)] \cap J_0(v) \subseteq J_1(u) + J_0(u) \cap J_0(v).$$

Therefore, if $x_1 \neq 0$ and n is taken with $x_1^{2n} = g + h$, where g is idempotent in $J_1(u)$ and h is idempotent in $J_0(u) \cap J_0(v)$, then u, v, and h are orthogonal, and u and h are weakly connected. But this contradicts Lemma 4; therefore $J_{1/2}(u+v) = (0)$. Since J is subdirectly irreducible, $J = J_1(u+v)$. Moreover, if u is the sum of orthogonal idempotents e_1 and e_2 and if $v = e_3$, then one of the spaces J_{12}, J_{13}, J_{23} must be nonzero. If $J_{12} \neq (0)$, then there exist connected idempotents f and g with $f \in J_{11}$ and $g \in J_{22}$. However, Lemma 4 then implies that $J = J_1(f+g) \subseteq J_1(e_1 + e_2)$, and therefore $J_1(v) = (0)$, which is impossible. Similarly, the assumption that $J_{13} \neq (0)$ or $J_{23} \neq (0)$ leads to a contradiction. Thus u and v must be primitive idempotents, and Lemma 8 implies that $N = (0)$.

It remains to show that $J_1(u)$ is a Jordan division ring. In fact, we shall prove that it is a periodic field. If x is a nonzero element of $J_1(u)$ and x is anti-integral over the integers modulo p, then x is periodic. If x is not anti-integral, then $J_1(u)$ contains a nonzero element z in the center of J. Each nonzero y in $J_{1/2}(u)$ is a unit in J, so that $0 \neq zy^2 = (zy)y$, and $zy \neq 0$ and $zy \in J_{1/2}(u)$. Now, if $y^n = u + v$ and $(zy)^m = u + v$, then

$$u + v = (zy)^{(m-1)(n-1)+1} = z^{(m-1)(n-1)+1}.$$

Since $J_1(u)^2 = J_1(u)$, the algebra $J_1(u)$ is periodic. Therefore, $J_1(u)$ is a periodic Jordan algebra with a unique idempotent. Thus, by Theorem 16.2 of [14], $J_1(u)$ is a periodic field.

For our final result of this section we determine the structure of a subdirectly irreducible Jordan H-algebra with a nontrivial idempotent.

LEMMA 8. *Let* J *be a subdirectly irreducible Jordan* H*-algebra with a nontrivial idempotent; then* J *is a simple periodic Jordan ring of capacity* 2. *In particular there exist a periodic field* Φ *and an element* $\mu \in \Phi$ *such that* $-\mu$ *is a nonsquare and*

$$J = \left\{ \begin{pmatrix} \alpha & \beta\mu \\ \beta & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \Phi \right\}.$$

Proof. We begin by noting that it follows immediately from Lemma 7 that J is a simple Jordan algebra of capacity 2. By [8], J is a Jordan algebra of a nondegenerate symmetric bilinear form f. We write $J = \Phi \oplus V$ and take $e = \alpha + v$ for $\alpha \in \Phi$ and $v \in V$. Since $\alpha + v = e = e^2 = (\alpha^2 + v^2) + 2\alpha v$, we see that $\alpha = 1/2$ and $v^2 = 1/4$. Now take $w \in V$, with $f(v, w) = 0$ and $w \neq 0$. Note that such an element w exists, since J is subdirectly irreducible. If $f(w, w) = \mu$, then it is clear that the subalgebra K of J generated by 1, v, and w is isomorphic to the 2-by-2 Hermitean matrices over Φ under the correspondence

$$\beta_0 + \beta_1 v + \beta_2 w \longleftrightarrow \begin{pmatrix} \beta_0 + \frac{1}{2}\beta_1 & \beta_2\mu \\ \beta_2 & \beta_0 - \frac{1}{2}\beta_1 \end{pmatrix}.$$

Next we claim that Φ is periodic. Since

$$\begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix} \in J_{1/2}(e),$$

μ is periodic. If $\gamma \in \Phi$, then with $\beta_2 = \gamma$, we have the relation

$$\begin{pmatrix} 0 & \gamma\mu \\ \gamma & 0 \end{pmatrix} \in J_{1/2}(e),$$

and it follows that γ is periodic; hence Φ is a periodic field and K is a periodic Jordan ring of capacity 2.

It remains to prove that $K = J$. If $K \neq J$, then there exists $x \in J$ $(x \neq 0)$ such that v, w, x are pairwise orthogonal. Let $a = \alpha_0 v + \alpha_1 w + \alpha_2 x$, with $\alpha_0, \alpha_1, \alpha_2 \in \Phi$; then

$$a^2 = \alpha_0^2 f(v, v) + \alpha_1^2 f(w, w) + \alpha_2^2 f(x, x).$$

Now, by Lemma 15.2 of [14], $\alpha_0, \alpha_1, \alpha_2$ can be chosen so that $a^2 = 0$, which is impossible. Hence $J = K$.

4. THE CASE WITHOUT NONTRIVIAL IDEMPOTENTS

Our aim in this section is to complete the proof of Theorem 1 by showing that every subdirectly irreducible Jordan H-algebra without nontrivial idempotents must be associative. We shall achieve this by first showing that every such algebra without nonzero nilpotent elements must be associative, or equivalently, every associator in a Jordan H-algebra without nontrivial idempotents is contained in N.

By a procedure analogous to the one employed in [5], each Jordan algebra without nonzero nil ideals is isomorphic to a subdirect sum of prime Jordan algebras, each with a nonnilpotent element that is nilpotent modulo every nonzero ideal (by a prime Jordan algebra we mean an algebra in which the product of nonzero ideals is never zero). By virtue of this, we can determine the structure of a Jordan H-algebra without nonzero nilpotent elements by restricting our attention to a prime Jordan H-algebra J with a nonnilpotent element a, nilpotent modulo every nonzero ideal.

We begin by supposing that J has no nontrivial idempotents. Note that the center Z(J) of J is an integral domain, so that we can imbed J in Q, a Jordan ring with identity, by forming as in [5] the quotient ring of Z(J) in J. We claim that Q has no nontrivial idempotents. If $z/z \in Q$ is idempotent, then $zx^2 = z^2x$, that is, $z(x^2 - zx) = 0$. Hence $x^2 - zx = 0$ in J. Now $x^2 p_x(x) - x = z_1 \in Z(J)$, so that

$$zx^2 p_x(x) - zx + x^3 p_x(x) - x^2 = zz_1 \in Z(J),$$

and hence $xz_1 = z_2 \in Z(J)$. Therefore $(zx)/z = z_2/z_1$ in Q, so that $x \in Z(J)$. But now $x(x - z) = 0$ implies that $x = 0$ or $x = z$ and $x/z = 1$, as we claimed.

Let x be a nonzero element of J. If $x^2 p_x(x) - x = 0$, then $x p_x(x)$ is a nonzero idempotent, and therefore $xp_x(x) = 1$. Hence x is a unit in $J \subseteq Q$. If $x^2 p_x(x) - x \neq 0$, then $x^2 p_x(x) - x$ is a unit in Q and

$$x[x p_x(x) - 1)(x^2 p_x(x) - x)^{-1}] = 1$$

in Q. Therefore Q is a Jordan division ring.

We claim that Q is in fact a field. To prove this, we shall use an argument analogous to the one for Theorem 2 of [3]. Let Z be the center of Q. If Z has prime characteristic p and Z is algebraic over its prime field, then Q is a field, since it is algebraic over Z and hence periodic [4, Proposition 15.1]. Hence, suppose that Z is not algebraic over its prime field if it has characteristic p, and that Q is not associative. Then, by [10], there must exist an element $x \in Q$ ($x \notin Z$) that is separable over Z. Since zx is in turn separable over Z for each $z \in Z$, we can suppose that $x \in J$.

Let $L = Z[x]$; then by [13] there exists a pair of distinct, nonarchimedean exponential valuations ρ_1 and ρ_2 on L such that $\rho_1(z) = \rho_2(z)$ for all $z \in Z$. In particular, $\rho_1(x) \neq \rho_2(x)$. Now, if $\rho_1(x) > 0$, then Q is associative, by an argument in [3]. Hence we may assume that $\rho_1(x) < 0$. If Z has characteristic zero, then for each i the valuation ρ_i is p-adic on the rationals, and hence, if λ is sufficiently large, then

$$\rho_i(p^\lambda x) = \lambda + \rho_i(x) > 0.$$

Thus, by replacing x by $p^\lambda x$, we can suppose that Z has prime characteristic and ρ_i is trivial on Z_p, the set of integers modulo p. If $x^2 p_x(x) - x = \alpha \in Z$ and $p_x(x) = \sum_{i=0}^n t_i x^i$, then

$$\begin{aligned}\rho_1(\alpha) &= \min_{i=0,1,\cdots,n} \{\rho_1(t_i x^{i+2}), \rho_1(x)\}\\ &= \min_{i-0,1,\cdots,n} \{\rho_1(t_i) + (i+2)\rho_1(x), \rho_1(x)\} = (n+2)\rho_1(x).\end{aligned}$$

Therefore, if $\rho_2(x) < 0$, then $\rho_1(\alpha) = \rho_2(\alpha) = (n+2)\rho_2(x) < 0$, so that $\rho_1(x) = \rho_2(x)$, which is impossible. Hence $\rho_2(x) > 0$; but then

$$0 > \rho_1(\alpha) = \rho_2(\alpha) = \rho_2(x) > 0,$$

which again is impossible. Hence Q must be a field, and J is an associative integral domain.

Next we suppose that J is a prime Jordan H-algebra with a nontrivial idempotent e. Since J is prime, we see that $J_{1/2}(e) \neq (0)$ and J has prime characteristic and a nonzero periodic element. Moreover, since the nilpotent elements of J are central, N = (0). Finally, as above, J contains a pair of connected orthogonal idempotents u and v, and as in Lemma 7, $J = J_1(u + v)$ and u and v are primitive.

Next we want to show that $J_1(u)$ is a field. Let $x \in J_1(u)$, $z = x^2 p_x(x) - x$, and $0 \neq y \in J_{1/2}(u)$. Again, if $z = 0$, then x is a unit; therefore, without loss of generality, we can suppose that $z \neq 0$. Now $0 \neq zy \in J_{1/2}(u)$; and hence zy is periodic. Therefore, $(zy)^2 = z^2 y^2 \in J_1(u)$ is periodic. Consequently, $Z_p[uy^2]$ is a finite, commutative, associative ring without nonzero nilpotent elements and with a unique nonzero idempotent. Hence, $Z_p[uy^2]$ is a field. The element z is algebraic over this field, so that there exists a positive integer m with $z^m = u$. But then

$$u = (x^2 p_x(x) - x)^m = x[x^{m-1}(x p_x(x) - u)^m],$$

and x is a unit in $J_1(u)$. Similarly, $J_1(v)$ is a field, and we have proved that J has capacity 2. As in the proof of Lemma 8, it also follows that J is a simple, periodic Jordan algebra. Combining these facts we have the following lemma.

LEMMA 9. *Let* J *be a Jordan* H-*algebra without nonzero nilpotent elements; then* J *is isomorphic to a subdirect sum of Jordan algebras* $\{J_i\}_{i\in\Lambda}$ *such that for each* $i \in \Lambda$, J_i *is either an associative integral domain or a simple periodic Jordan algebra of capacity* 2.

Note that in the proof of Lemma 9 we never used the fact that the prime homomorphic images of J could be chosen so that each contains a nonnilpotent element that is nilpotent modulo each nonzero ideal. Hence we have in fact proved that every prime homomorphic image of a Jordan H-algebra is either an associative, commutative integral domain or a simple periodic Jordan algebra of capacity 2.

As we stated in the introduction to this section, our present aim is to prove that a Jordan H-algebra without nontrivial idempotents and without nonzero nilpotent elements must be associative. It is for this purpose that we shall next show that if J is an H-algebra and N = (0), then each element of the ideal generated by all elements of the form $(xy)z - x(yz)$ $(x, y, z \in J)$ is periodic. We shall accomplish this by simultaneously considering the structure of J as given in Lemma 9 and considering J as a subring of a ring of continuous sections. We now develop this second representation.

We know that J is isomorphic to a subdirect sum of Jordan algebras $\{J_i\}_{i\in\Lambda}$, where each J_i is as given in Lemma 9. Let $\bar{J}$ be the complete direct sum of rings $\{K_i\}_{i\in\Lambda}$, where $K_i = J_i$ if J_i is not associative, and where K_i is the quotient field of J_i otherwise. Further, let ϕ_i denote the projection of $\bar{J}$ onto K_i, and let E be the set of all central idempotents of $\bar{J}$. It is immediately clear that E forms a Boolean algebra.

Let $X(\bar{J})$ be the set of all maximal ideals of E topologized with the hull-kernel topology. Hence, as described in [15], $X(\bar{J})$ is a totally disconnected, compact, Hausdorff space in which the set

$$\mathscr{N}(e) = \{M \in X(\bar{J}): e \notin M\}$$

is open and closed for each $e \in E$. In fact, the sets $\mathscr{N}(e)$ form a basis for the topology on $X(\bar{J})$. Next, let $\bar{J}_M = \bar{J}/\bar{J}M$ for each $M \in X(\bar{J})$, and let $\mathscr{J}$ be the disjoint union of the rings $\bar{J}_M$. Further, for each $a \in \bar{J}$, define $\sigma_a: X(\bar{J}) \to \mathscr{J}$ so that $\sigma_a(M)$ is the image of a in $\bar{J}_M$, and topologize $\mathscr{J}$ so that the sets

$$\{\sigma_a(\mathscr{N}(e)): a \in \bar{J} \text{ and } e \in E\}$$

form a neighborhood basis for each point in $\mathscr{J}$. Then each σ_a is continuous, and if $\pi: \mathscr{J} \to X(\bar{J})$ is defined by $\pi^{-1}(M) = \bar{J}_M$, then, as in [11], $(\mathscr{J}, \pi)$ is a sheaf of Jordan rings and $\bar{J}$ is isomorphic to the ring $\Gamma(X(\bar{J}), \mathscr{J})$ of all continuous functions of $X(\bar{J})$ to $\mathscr{J}$ with respect to the mapping $\xi(a) = \sigma_a$. Moreover, $\bar{J}$ is a subdirect sum of the rings $\bar{J}_M$. Let ψ_M be the projection of $\bar{J}$ onto $\bar{J}_M$.

We claim that for each M, the ring $\bar{J}_M$ has no nonzero nilpotent elements. To show this, suppose that $x \in \psi_M(\bar{J})$ is nilpotent. In particular, take a positive integer n such that x^n is the zero 0_M in $\bar{J}_M$. Next, let $U = \{N \in X(\bar{J}): \sigma_{y^n}(N) = 0\}$, where y is a preimage of x in $\bar{J}$. Because U is a neighborhood of M in $X(\bar{J})$, there exists $e \in E$ with $M \in \mathcal{N}(e) \subseteq U$. Define $\sigma \in X(\bar{J}) \to \mathcal{J}$ by

$$\sigma(N) = \begin{cases} \sigma_y(N) & \text{if } N \in \mathcal{N}(e), \\ 0_N & \text{if } N \notin \mathcal{N}(e). \end{cases}$$

Since $\mathcal{N}(e)$ is open and closed, $\sigma \in \Gamma(X(\bar{J}), \mathcal{J})$ and $\sigma = \sigma_z$ for some $z \in \bar{J}$. It follows immediately that $z^n = 0$, so that $z = 0$, since $\bar{J}$ has no nonzero nilpotents, and hence x must be 0_M.

As in the previous paragraph we can prove that for each idempotent e in $\bar{J}_M$, there exists an idempotent $f \in \bar{J}$ with $\psi_M(f) = e$. This follows from the fact that if y is any preimage of e, then

$$U = \{N \in X(\bar{J}) : \sigma_{y^2-y}(N) = 0_N\}$$

is a neighborhood of M, so that $M \in \mathcal{N}(g) \subseteq U$ for some $g \in E$. Thus the mapping $\sigma_f \in \Gamma(X(\bar{J}), \mathcal{J})$ defined by

$$\sigma_f = \sigma_y \text{ on } \mathcal{N}(g) \qquad \text{and} \qquad \sigma_f = 0 \text{ otherwise}$$

will have the desired properties.

Let A be the associator ideal of J; that is, let A be the ideal of J generated by all elements of J of the form $(xy)z = x(yz)$. We can now prove that every element of A is periodic.

LEMMA 10. *Let* J *be a Jordan* H-*algebra without nonzero nilpotent elements; then for every* $a \in A$, *there exists an integer* $n = n(x) > 1$ *with* $a^n = a$.

Proof. We begin by showing that if for some $M \in X(\bar{J})$, the projection $\psi_M(J)$ is not associative, then $\psi_M(J)$ is periodic. Let I be the kernel of ψ_M under the restruction of ψ_M to J. If I is a prime ideal of J, then by the remark following Lemma 9, $\psi_M(J)$ is periodic. Hence it suffices to prove that I is prime.

If I is not prime, there exist ideals P and Q of J containing I and such that $PQ \subseteq I$. If there are elements $a \in P$ and $b \in Q$ neither of which is anti-integral modulo I, then there are elements α and β in the center of J with $\alpha\beta \in I$ and $\alpha \notin I$, $\beta \notin I$. Moreover, since $\alpha, \beta \in Z(J)$, there exist elements $\gamma, \delta \in Z(\bar{J})$ with $\alpha\gamma = f \in E$ and $\beta\delta = g \in E$, and with $\alpha f = \alpha$ and $\beta g = \beta$. Since $\alpha\beta \in I$, it follows that $fg \in \bar{J}M$, more precisely that $fg = xh$ with $x \in \bar{J}$ and $h \in M$. However, then $fgh = fg$, so that $fg \in M$, and since M is maximal it follows that $f \in M$ or $g \in M$. But $f \in M$ implies that $\alpha = f\alpha \in \bar{J}M \cap J = I$. Hence the assumption cannot hold; therefore P is anti-integral modulo I, and hence periodic modulo I, since $\bar{J}_M$ has no nonzero nilpotent elements.

Let P' be the image of P in $J' = \psi_M(J)$. If P' contains an idempotent e' that is not in the center of J', then $J'_1(e') + J'_{1/2}(e') \subseteq P'$. Therefore, if a' is a nonzero element of $J'_{1/2}(e')$, then a' connects a pair of nonzero orthogonal idempotents u' and v' in P'. Now there exists an idempotent u of $\bar{J}$ with $\psi_M(u) = u'$. Moreover, if $u \in E$, then $1 - u \in \bar{J}M$, which implies that $1_M = \psi_M(u) = u' \in P'$, where 1_M denotes the identity of $\bar{J}_M$. Because this is impossible, $u \notin E$. Now take g idempotent in $\bar{J}$, with $\phi_i(g) = 0$ if $\phi_i(u)$ is central and $\phi_i(g) = \phi_i(u)$ if $\phi_i(u)$ is not central.

Next, let a be a pre-image of a' in $\bar{J}$, and consider $2aU_{g,1-g}$. There exists $h \in E$ with $g + h = u$; if $\psi_M(h) \neq 0_M$, then $\psi_M(h) = 1_M$ and $u' = 1_M + \psi_M(g)$. But this implies that $2\psi_M(g) = 0$, and in turn we conclude that $u' = 1_M$, which cannot happen. Therefore, $\psi_M(g) = u'$, and $\psi_M(2aU_{g,1-g}) = a'$. However, if $0 \neq 2aU_{g,1-g}$, then there exist idempotents $k \in \bar{J}_1(g)$ and $\ell \in \bar{J}_1(g)$ and $\ell \in \bar{J}_0(g)$ such that $2aU_{g,1-g}$ connects k and ℓ, and $k + \ell \in E$. Since $a' \notin J_0'(u')$, it follows that $\psi_M(k) \neq 0_M$. Therefore $\psi_M(k + \ell) = 1_M \in P'$, which is again impossible. Thus, our assumption is incorrect, and we see that every idempotent of P' is central in J'.

If e' is idempotent in P' and $e^2 = e$ in $\bar{J}$, with $\psi_M(e) = e'$, then $e \notin E$. Hence, if $a \in \bar{J}_{1/2}(e)$, then $e'\psi_M(a) = \frac{1}{2}\psi_M(a)$. Moreover, if $\psi_M(a) = 0_M$, then $\psi_M(ea^2) = 0_M$, which is impossible since e and a can be chosen in $\bar{J}$ so that ea^2 is a unit in $\bar{J}_1(e)$. Hence P' must contain idempotents that are not central in $\bar{J}_M$. But then an application of the argument given in the previous paragraph shows that we can find connected orthogonal idempotents u and v in $\bar{J}_M$ with

$$\bar{J}_M = \psi_M(\bar{J})_1(u) \oplus \psi_M(\bar{J})_{1/2}(u) \oplus \psi_M(\bar{J})_1(v),$$

and $u \in P'$. Since $u \in Z(J')$, we can assert that

$$J' = (\psi_M(\bar{J})_1(u) \cap J') + (\psi_M(\bar{J})_1(v) \cap J').$$

However, $\psi_M(\bar{J})_1(u)$ and $\psi_M(\bar{J})_1(v)$ are both homomorphic images of the subdirect sum of fields, and therefore each must be associative. But this implies that J' is associative, contrary to the hypothesis. Therefore I is a prime ideal of J and $\psi_M(J)$ is periodic, as we claimed.

Now take $a \in A$, the associator ideal of J. For each positive integer n, let

$$U_n = \{M \in X(\bar{J}): \sigma_{a^n - a}(M) = 0_M\}.$$

Clearly, the sets U_n form an open covering of $X(\bar{J})$. Since $X(\bar{J})$ is compact, there are integers $n_1, \cdots, n_m$ with $X(\bar{J}) = \bigcup_{i=1}^m U_{n_i}$. Hence, if we let

$$t = \prod_{i=1}^{m} (n_i - 1) + 1,$$

then $\sigma_{a^t - a}(M) = 0$ for all $M \in X(\bar{J})$. Therefore $a^t = a$, and Lemma 10 is proved.

We can now prove the desired result for Jordan H-algebras having no nontrivial idempotents and no nonzero nilpotent elements. It will be useful to state this theorem in a slightly more general setting.

LEMMA 11. *If* J *is a Jordan* H-*algebra without nontrivial idempotents, then* J/N *is associative.*

Proof. Our first task is to prove that J/N contains no nontrivial idempotents. Suppose that $e \in J/N$ is idempotent; then, if a is any pre-image of e in J, $a^2 - a \in N$. Thus, a is algebraic over F. Let J* denote the algebra obtained by adjoining an identity to J, if J does not have identity. Now, applying Lemma 1 on page 149 of [8] to F[a], we see that $g(X) \in F[X]X$, where g(a) is a nonzero idempotent. Moreover, since $g(0) = 0$, then $g(a) \in J$ and hence g(a) is the identity 1 in

J. Therefore, a is a unit in J, and therefore e is a unit in J/N, or equivalently, e is a trivial idempotent.

Next take a in the associator ideal A of J/N. Then a is periodic. If $a \neq 0$, then a is a unit in J/N, and it follows that J/N is isomorphic to a subdirect sum of simple, nonassociative periodic Jordan algebras $\{J_i\}_{i \in \Lambda}$. Thus, if $A \neq (0)$, then A is also a subdirect sum of these algebras. But A is periodic, and [10, Corollary 1] implies that A contains a nontrivial idempotent, which is impossible. Therefore $A = (0)$, and J/N is associative, as we claimed.

Returning to the subdirectly irreducible case, let J be a subdirectly irreducible Jordan H-algebra without nontrivial idempotents. By Lemma 11, $N \neq (0)$; hence, if S is the minimal ideal of J, then $S \subseteq N \subseteq Z(J)$. Therefore the set

$$A(S) = \{x \in J\colon xs = 0 \text{ for all } s \in S\}$$

is an ideal of J. Moreover, if $x \in N$ and $xS \neq (0)$, then for each positive integer n,

$$x^n S = x^{n-1}(xS) = x^{n-1} S = \cdots = S,$$

which is impossible. Hence $N \subseteq A(S)$. Next we shall prove that $A(S) \subseteq Z(J)$.

LEMMA 12. *Let* J *be a subdirectly irreducible Jordan* H-*algebra without nontrivial idempotents; then* $A(S) \subseteq Z(J)$.

Proof. We begin by considering $(a, b, c) = (ab)c - a(bc)$ for $a, c \in J$ and $b \in A(S)$. If $(a, b, c) \neq 0$, then $(a, b, c) \in Z(J)$, and the ring J^* obtained by adjoining an identity to J contains an element r such that $0 \neq r(a, b, c) = s \in S$. If $p_b(b) = \sum_{i=0}^n t_i b^i$ with integral coefficients t_i, and if $b^2 p_b(b) - b \in Z(J)$, then, using the identity

$$(w, xy, z) = x(w, y, z) + (w, x, z)y$$

(which holds in every linear Jordan algebra [8, p. 34]), we see that

$$0 \neq r(a, b, c) = r\left(a, b^2\left(\sum_{i=0}^n t_i b^i\right), c\right) = \sum_{i=0}^n t_i r(A, b^{i+2}, c)$$

$$= \left[\sum_{i=0}^n t_i(i+2) b^{i+1}\right][r(a, b, c)] = \left[\sum_{i=0}^n t_i(i+2) b^{i+1}\right][s] = 0.$$

Hence (a, b, c) must be zero for each $b \in A(S)$ and $a, c \in J$.

Next we consider the identity

$$(x, y, z) - (x, z, y) + (z, x, y) = 0,$$

which can easily be seen to hold in every commutative ring. If we take x and z in A(S) and y in J, then $(x, y, z) = 0$; combining this with the result of the previous paragraph, we see that every associator involving two elements from A(S) must vanish.

The Teichmüller identity, which is again easily seen to hold in every ring, states that for all $w, x, y, z \in J$,

$$0 = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z.$$

If we take $w, x \in A(S)$, then, since $A(S)$ is an ideal, we have the equation

$$0 = (wx, y, z) - w(x, y, z),$$

or equivalently, $(wx, y, z) = w(x, y, z)$.

If $a \in A(S)$ and $b, c \in J$ with $(a, b, c) \neq 0$, then again there exists $r \in J^*$ such that $0 \neq r(a, b, c) \in S$. If we take $a^2 p_a(a) - a \in Z(J)$, then, since (a, b, c) is contained in an ideal in the center of J, we see that

$$0 \neq r(a, b, c) = r(a^2 p_a(a), b, c) = r[a\, p_a(a)\,(a, b, c)]$$

$$= [p_a(a)r]\,[a(a, b, c)] = p_a(a)\,[ar(a, b, c)] = 0.$$

Hence, $(a, b, c) = 0$. Finally, since J is flexible, $(c, b, a) = 0$; it follows that $A(S) \subseteq Z(J)$.

We are now in a position to complete the proof of Theorem 1.

LEMMA 13. *If* J *is a subdirectly irreducible Jordan* H*-algebra without nontrivial idempotent elements, then* J *is associative.*

Proof. By virtue of Lemma 12, we can begin by assuming that $A(S) \neq J$. Our first task is to prove that $J/A(S)$ is a field. For this purpose we shall show that $A(S)$ is identical with the set of nonzero divisors of zero of J. Take $x \in J$ and $y \neq 0$, with $xy = 0$. If $xS \neq (0)$, then $xS = S$ and $yS = y(xS) = (yx)S = (0)$. Hence $y \in A(S) \subseteq Z(J)$. Let T be the principal ideal of J generated by y. Then $S \subseteq T$, and since $y \in Z(J)$, we see that $xS \subseteq xT = (0)$. Thus, it follows that $x \in A(S)$, as we claimed.

Let $x, y \in J$, with $x, y \notin A(S)$. To prove that $J/A(S)$ is a field, it suffices to prove that there exists $a \in J$ with $\bar{x}\bar{a} = \bar{y}$ in $J/A(S)$. Take $s \in S$; then $xJs = S$. Because $yS = S$, it follows that there exists an $a \in J$ with

$$0 \neq xas = ys.$$

Hence $xa - y \in A(S)$, or equivalently, $\bar{x}\bar{a} = \bar{y}$.

Next we consider the set $Z = \{\bar{a} \in J/A(S): a \in Z(J)\}$. Take $e \in J$, where $\bar{e}$ is the identity in $J/A(S)$. If $Z \neq (0)$, take $a \in Z(J)$, with $\bar{0} \neq \bar{a} \in Z$. Then $ae - a \in A(S) \subseteq Z(J)$, so that $ae \in Z(J)$. But then all $x, y \in J$ satisfy the equation $0 = (ae, x, y) = a(e, x, y)$, and a is not a zero divisor. Therefore, $(e, x, y) = 0$. Similarly, $(x, e, y) = (x, y, e) = 0$, and it follows that $\bar{e} \in Z$. Next, if $\bar{x} \in Z$ and $y \in J$, with $\bar{x}\bar{y} = \bar{e}$, then the same argument proves that $\bar{y} \in Z$, so that Z is a subfield of $J/A(S)$. Moreover, we can suppose that $Z \neq J/A(S)$. Finally, we know that $\bar{x}^2 p_x(\bar{x}) - \bar{x} \in Z$ for every $x \in J$. By a theorem due to I. N. Herstein [3], $J/A(S)$ has characteristic $p \neq 0$, and either $J/A(S)$ is purely inseparable over Z, or $J/A(S)$ is algebraic over the Galois field of p elements, and hence it is periodic. If $Z = (0)$, then $\bar{x}^2 p_x(\bar{x}) - \bar{x} = \bar{0}$ for all $x \in J$, and again $J/A(S)$ is periodic.

We claim that if $Z \neq J/A(S)$, then all elements of $J/A(S)$ not in Z are periodic. Take $a \in J$, with $\bar{a} \notin Z$ and a not periodic. If we let $e \in J$, so that $\bar{e}$ is the identity in $J/A(S)$ as above, then for every $b, c \in J$

$$e(b, a, c) = (b, ae, c) = (b, a, c) = (b, a^2 p_a(a), c)$$

$$= \left(\sum_{i=0}^{n} t_i a^{i+2}, c \right) = \sum_{i=0}^{n} t_i a^{i+1}(b, a, c),$$

where $p_a(X) = \sum_{i=0}^{n-2} t_{i+2} X^i$. But this implies that $\sum_{i=0}^{n} t_i \bar{a}^{i+1} - \bar{e} = \bar{0}$ or $(b, a, c) = 0$. However, $\bar{a}$ is not periodic; therefore $(b, a, c) = 0$. Next, by the Teichmüller identity, if $x = a$ and w is a polynomial in a, then for all $y, z \in J$,

$$0 = (wa, y, z) - (w, ay, z) - w(a, y, z).$$

If $(w, ay, z) \neq 0$, then $\overline{ay} \notin Z$, and there exists a positive integer m with $\overline{ay}^{p^m} \in Z$. But as above, $\overline{ay}$ is periodic; therefore there exists a positive integer u with $\overline{ay}^{p^u} = \overline{ay}$. Hence, if v is the least common multiple of m and u, then $mk = v$ and

$$\overline{ay} = \overline{ay}^{p^v} = (\overline{ay}^{p^m})^{p^{(k-1)m}} \in Z.$$

But this is impossible, and it follows that $(w, ay, z) = 0$. Hence, for $w = ap_a(a)$,

$$e(a, y, z) = (a, y, z) = (a^2 p_a(a), y, z) = a p_a(a)(a, y, z).$$

As before, this implies that $(a, y, z) = 0$. Therefore, $a \in Z(J)$, since J is flexible; hence $\bar{a} \in Z$. Because this is contrary to the assumption, all elements of $J/A(S)$ that are not in Z are periodic.

Finally, let $x, y, z \in J$, and suppose that $(x, y, z) \neq 0$. Then $\bar{x}, \bar{y}, \bar{z} \notin Z$, and therefore the field K obtained by attaching $\bar{x}, \bar{y}, \bar{z}$ to the prime field of Z is finite. Hence there exists $a \in J$ with the property that $\bar{a}$ generates the nonzero multiplicative group of K, and in particular there exist integers t_1, t_2, t_3 and elements $x', y', z' \in A(S)$ with

$$x = a^{t_1} + x', \qquad y = a^{t_2} + y', \qquad z = a^{t_3} + z'.$$

Since $A(S) \subseteq Z(J)$, we have the relations

$$(x, y, z) = (a^{t_1} + x', a^{t_2} + y', a^{t_3} + z') = (a^{t_1}, a^{t_2}, a^{t_3}) = 0.$$

It follows that J is associative, and the proof of Theorem 1 is complete.

Before proceeding, we note that the structure of commutative, associative, subdirectly irreducible rings is given in [12].

5. COROLLARIES

We shall now consider several corollaries to Theorem 1.

COROLLARY 1. *If* J *is a Jordan* H-*algebra and all idempotents of* J *are central, then* J *is associative.*

Proof. As in the proof of Lemma 11, the idempotents in J/N are central, and hence, by Lemma 10 and the representation of J/N given in the proof of Lemma 10,

J/N must be associative. Let I be an ideal of J with the property that J/I is subdirectly irreducible. If J/I is not associative, then J/I must be periodic, and it follows that $N \subseteq I$. But then J/I is a homomorphic image of J/N. Therefore, each subdirectly irreducible homomorphic image of J is associative, and the corollary follows from Theorem 1.

COROLLARY 2. *If* J *is a Jordan* H-*algebra with a unique nonzero idempotent, then* J *is associative.*

Proof. Let e be the unique nonzero idempotent of J. By Corollary 1 it suffices to show that e is in the center of J. If it is not, then $J_{1/2}(e) \neq (0)$. But this implies the existence of another nonzero idempotent, which is impossible.

COROLLARY 3. *If* J *is a Jordan* H-*algebra without nontrivial idempotents, then* J *is associative.*

Proof. By Corollary 2, we can suppose that J has no nonzero idempotents, and hence, as in Corollary 1, we can assert that J/N has no nonzero idempotents. Moreover, by the proof of Corollary 1 it suffices to prove that J/N is associative. But Lemma 10 implies that J/N is associative or has nonzero idempotents; therefore J/N is associative, as desired.

In Corollaries 1 and 3 we used the following result, which we state without proof.

COROLLARY 4. *Let* J *be a Jordan* H-*algebra; then* J *is associative if and only if* J/N *is associative.*

Note that Corollary 1 is the analogue to Corollary 1 of [10] and Corollary 2 is analogous to Theorem 16.2 of [14]. The next two corollaries are analogous to the results of [4].

COROLLARY 5. *Let* J *be a Jordan algebra over a field of characteristic not* 2, *and such that for every element* x *in* J *there exist an integer* $n = n(x) > 0$ *and a polynomial* $p(t) = p_x(t)$ *with integer coefficients such that* $x^{n-1}p(x) = x^n$. *If further all the nilpotent elements of* J *are in the center of* J, *then* J *has the structure given in Theorem* 1.

Proof. Since $x^{n+1}p(x) = x^n$, we see that $(x^2p(x) - x)x^{n-1} = 0$. Note that without loss of generality we can suppose that $n > 1$. Then

$$(x^2p(x) - x)^n = (x^2p(x) - x)(x^2p(x) - x)^{n-1} = 0,$$

and it follows that J is a Jordan H-algebra.

COROLLARY 6. *Let* J *be a Jordan algebra of characteristic not* 2 *such that every element of* J *generates a finite subring. If the nilpotent elements of* J *are all in the center, then* J *has the structure given in Theorem* 3.

Proof. This result follows immediately from Corollary 5 and the fact that $x^m = x^n$ for all x in J and some pair of integers m and n $(m \neq n)$.

As our final corollary, we prove the following result, which is analogous to the Jordan-ring case of [9].

COROLLARY 7. *Let* J *be a Jordan ring, and let* p *be a prime integer different from* 2. *Suppose that for every* $x \in J$, $px = 0$ *and* $x^p - x \in Z(J)$. *Then* J *is associative.*

Proof. By Theorem 3, it suffices to prove that if J is a simple Jordan algebra of a symmetric bilinear form f that is three-dimensional over its center, then J cannot satisfy the hypothesis of the corollary. Now $J = F1 + V$, where $\dim V = 2$ over F, and if $\alpha + x, \beta + y \in J$, then $(\alpha + x)(\beta + y) = \alpha\beta + f(x, y) + \alpha y + \beta x$, where $\alpha, \beta \in F$ and $x, y \in V$. Moreover, there exists $w \in V$ with $w^2 = f(w, w) = 1$. Now take $x \in V$, with $f(x, w) = 0$, and let $\beta = f(x, x) \neq 0$. Consider $\alpha w + x$ for $\alpha \in F$. By direct computation,

$$(\alpha w + x)^p = (\alpha^2 + \beta)^{(p-1)/2}(\alpha w + x).$$

Since $(\alpha w + x)^p - (\alpha w + x) \in Z(J) = F$, and since $\alpha w + x \notin Z(J)$, it follows that $(\alpha^2 + \beta)^{(p-1)/2} = 1$. Now take $\alpha \in Z_p$; then, since $\alpha^2 + \beta \in Z_p$, it follows that $\beta \in Z_p$. If $\alpha = 0$, then clearly β is a square, and therefore there exists $\gamma \in Z_p$ with $\gamma^2 = \beta$. Next take $z = x/\gamma$. Then $f(z, z) = 1$ and $f(w, z) = 0$. Replacing x by z in the above, we see that for all $\sigma, \lambda \in Z_p$

$$(\lambda w + \sigma z)^p = (\lambda^2 + \sigma^2)^{(p-1)/2}(\lambda w + \sigma z).$$

As before, this implies that

$$(\lambda^2 + \sigma^2)^{(p-1)/2} = 1.$$

Hence the sum of each pair of squares in Z_p is again a square in Z_p. But 1 is a square; therefore all elements of Z_p are squares; this is plainly impossible, and the corollary is proved.

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