FIBRATIONS OF COMPACTLY GENERATED SPACES

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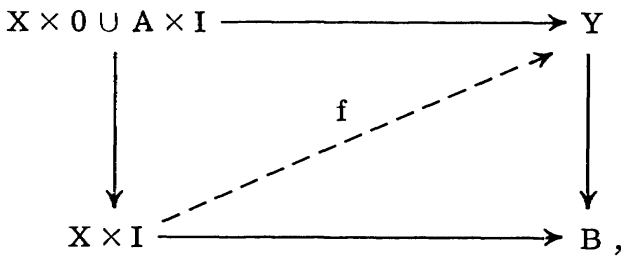
1. INTRODUCTION

N. E. Steenrod showed that the category CG of *compactly generated Hausdorff* spaces is a convenient category for algebraic topology [8]. In particular, he showed that cofibrations and colimits have various good properties. See Theorems 3.2, 4.2, and 4.3, below. We shall show that in CG, fibrations (maps having the covering-homotopy property in CG) and limits have similar good properties.

Consequently, CG, together with the usual classes of cofibrations, fibrations, and homotopy equivalences, is a *closed model category* (D. G. Quillen, [5, Definitions I.1.1, I.5.1]). (Note that cofibrations in CG are automatically closed; see [7, p. 57], for example.) A. Strøm has shown that all spaces, together with the usual classes of cofibrations, fibrations, and homotopy equivalences is also a closed model category [9].

Our main tool is the following covering-homotopy-extension property for fibrations in CG (see Section 2). Given a cofibration $A \to X$, a fibration $Y \to B$, and a solid-arrow diagram



we can find a filler f. Under additional hypotheses, G. Allaud and E. Fadell gave an earlier proof of this result for regular fibrations in the category of all spaces [1, Theorem 2.4]. The theorem also holds in the category of simplicial sets; see J. P. May [4, Corollary 7.17], for example.

From now on, unless otherwise stated, all spaces and maps are in CG, and we make all constructions there. In Section 3, we prove a "Polish" Theorem for fibrations and the *function-space* functor Map in CG [8, Section 5]. Towers and towers of fibrations will be discussed in Sections 4 and 5.

In a subsequent paper (extending [3]), these results will be used to discuss the relation between M. Rothenberg and N. E. Steenrod's characterization of the classifying space of a topological group [6, Definition 1.1] and A. K. Bousfield and D. M. Kan's realization of a cosimplicial space [2].

Sections 2, 3, and 5 are contained in the author's dissertation [3]. This dissertation was written under the direction of Professors N. E. Steenrod and J. C. Moore, to whom the author is grateful for their help and guidance.

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2. A COVERING-HOMOTOPY-EXTENSION THEOREM

THEOREM 2.1. *Let* i: $A \to X$ *be a cofibration, let* p: $Y \to B$ *be a fibration, and suppose that the diagram* (*of solid arrows*)

$$
\text{(2.1)} \qquad \begin{array}{ccc}
X \times 0 \cup A \times I & \xrightarrow{\ f\ } & Y \\
{\scriptstyle j}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle p} \\
X \times I & \xrightarrow{\ h\ } & B
\end{array}
$$

commutes. Then there exists a map $H\colon X \times I \to Y$ *such that* $Hj = f$ *and* $pH = h$.

Proof. Since i is a cofibration, it follows by [8, Theorem 7.1] that $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$. Let $r\colon X \times I \to X \times 0 \cup A \times I$ be the retraction, and let $\Gamma\colon X \times I \times I' \to X \times I$ be the homotopy from jr to $1_{X \times I}$ relative to $X \times 0 \cup A \times I$.

There also exists a halo function $u\colon X \times I \to [0, 1]$ with $u^{-1}(0) = X \times 0 \cup A \times I$ [8, Theorem 7.1]. Following a suggestion of Dold and Steenrod (private communication), we shall first replace Γ with another homotopy $\Gamma'\colon X \times I \times I' \to X \times I$ such that $\Gamma'(x, t, t') = (x, t)$ for $t' \geq u(x, t)$.

Let Z be the quotient of $X \times I \times I'$ obtained by collapsing $(X \times 0 \cup A \times I) \times I'$ to $(X \times 0 \cup A \times I) \times 0$. Factor Γ through Z to obtain a map $\Gamma''\colon Z \to X \times I$.

Also, let

$$W = \{(x, t, t') \mid 0 \leq t' \leq u(x, t)\} \subset X \times I \times I'.$$

Define a mapping $g\colon X \times I \times I' \to W$ by $g(x, t, t') = (x, t, u(x, t)t')$. Then g induces a homeomorphism $g'\colon Z \to W$. Define the required homotopy Γ' by

$$\Gamma'(x, t, t') = \begin{cases} \Gamma'' g^{-1}(x, t, t') & \text{for } t' \leq u(x, t), \\ (x, t) & \text{for } t' \geq u(x, t). \end{cases}$$

Now consider the diagram

$$
\begin{array}{ccccc}
X \times I & \xrightarrow{\ r\ } & X \times 0 \cup A \times I & \xrightarrow{\ f\ } & Y \\
\| {\scriptstyle \cong} & & & & \\
X \times I \times 0 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle p} \\
\downarrow & & & & \\
X \times I \times I' & \xrightarrow{\ \Gamma'\ } & X \times I & \xrightarrow{\ h\ } & B
\end{array}
$$

Since p is a fibration, there exists a map $H'\colon X \times I \times I' \to Y$ that extends fr and satisfies the condition $pH' = h\Gamma'$. Define $H\colon X \times I \to Y$ by the equation

$$H(x, t) = H'(x, t, u(x, t)).$$

Since $\Gamma'(x, t, u(x, t)) = (x, t)$, the mapping H is the required filler in diagram (2.1). ■

Remark 2.2. There is an analogous theorem for simplicial sets. See May [4, Corollary 7.17], for example.

3. A POLISH THEOREM FOR FIBRATIONS

Let $i: A \to X$ be a cofibration, and let $p: Y \to B$ be a fibration. In Figure 1 below, P is the pullback, and the mapping $q: \mathrm{Map}(X, Y) \to P$ is induced by $\mathrm{Map}(1_X, p)$ and $\mathrm{Map}(i, 1_Y)$.

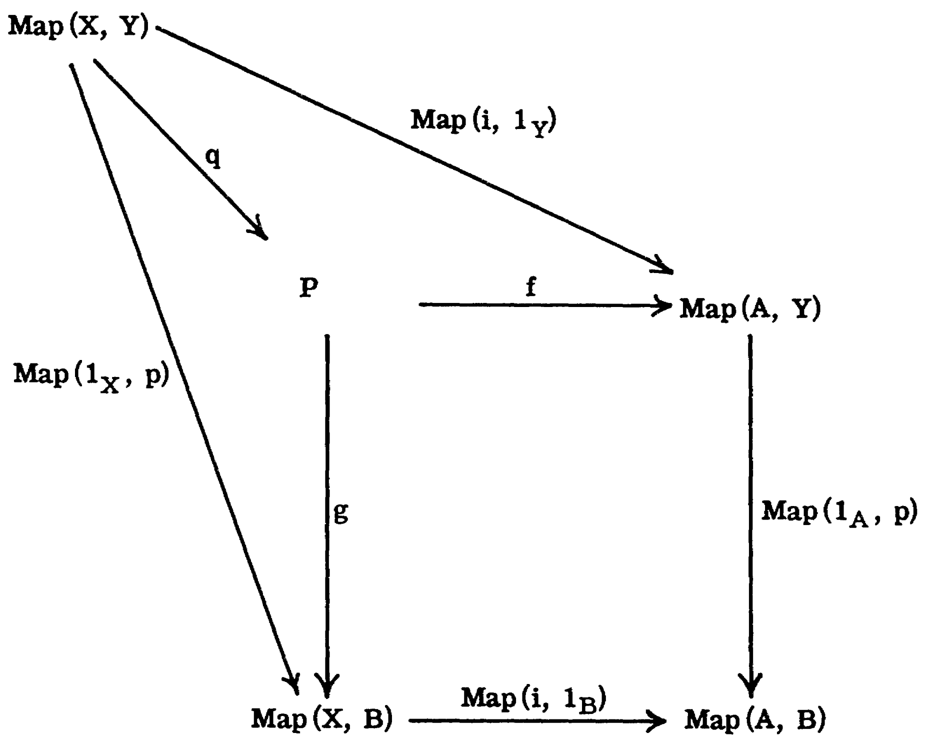


Figure 1.

THEOREM 3.1. *The mapping* q *is a fibration.*

This is roughly dual to the following result of Steenrod [8, Theorem 6.3].

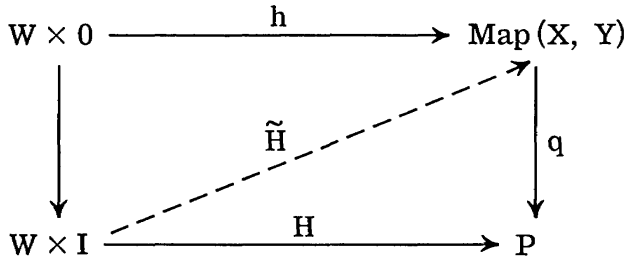
THEOREM 3.2. *Let* $A \to X$ *and* $B \to Y$ *be cofibrations. Then the induced map*

$$A \times Y \cup X \times B \to X \times Y$$

is also a cofibration.

Proof of Theorem 3.1. We shall show that the map q in the solid-arrow diagram

(3.1)



has the covering-homotopy property.

An application of the exponential law [8, Theorem 5.6] to the mappings h, fH, and gH yields mappings

$$h': W \times X \times 0 \to Y,$$

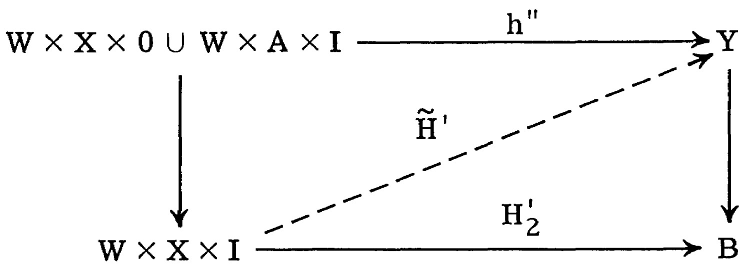
$$H_1': W \times A \times I \to Y,$$

$$H_2': W \times X \times I \to B,$$

respectively. Since h' and H_1' agree on their intersection $W \times A \times 0$ (by diagram (3.1)) and their domains are closed in their union, h' and H_1' induce a mapping

$$h'': W \times X \times 0 \cup W \times A \times I \to Y.$$

Further, since $\mathrm{Map}(1_A, p)fH = \mathrm{Map}(i, 1_B)gH$, the mapping H_2' extends pH_1'. Hence there is a commutative solid-arrow diagram



Theorem 2.1 yields the filler $\tilde{H}'$. An application of the exponential law to the composite of $\tilde{H}'$ with the isomorphism $W \times I \times X \to W \times X \times I$ yields the required map $\tilde{H}$ in diagram (3.1). ∎

COROLLARY 3.3. *Suppose that a basepoint* $*$ *is chosen in* B, *that* F *is the fibre of* p, *and that a basepoint* $*$ *is chosen in* $F \subset Y$. *Then, with respect to the basepoint* (Map(A, $*$), Map(X, $*$)) *in* P, *the fibre of* q *is* Map((X, A), (F, $*$)).

Note that Theorem 3.1 only requires the covering-homotopy-extension property (Theorem 2.1), and a function-space construction adjoint to the product (exponential law). For example, Theorem 3.1 holds in the category of simplicial sets.

4. TOWERS

We shall now dualize some results of Steenrod [8, Section 10] on filtered spaces.

Definition 4.1. A *filtered space* is a diagram

$$X_0 \to X_1 \to \cdots \to X_i \to \cdots \to X = \operatorname{colim} \{X_i\};$$

this is usually denoted simply by X.

Unlike in [8], the maps $X_i \to X_{i+1}$ need not be inclusions. See Remark 4.7, below. A space X is said to be *filtered by cofibrations* if all the maps $X_i \to X_{i+1}$ are cofibrations. Then (see [8]) the maps $X_i \to X$ are also cofibrations.

Let X and Y be filtered spaces. Define $(X \times Y)_n$ by the coequalizer (identification space) diagram

$$(4.1) \qquad \coprod_{i+j=n-1} X_i \times Y_j \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{i+j=n} X_i \times Y_j \to (X \times Y)_n,$$

where f and g are induced by the maps $X_i \to X_{i+1}$ and $Y_j \to Y_{j+1}$, respectively. If these maps are inclusions, then

$$(X \times Y)_n = \bigcup_{i+j=n} X_i \times Y_j,$$

as usual.

There are induced maps $(X \times Y)_n \to (X \times Y)_{n+1}$ that yield the following.

THEOREM 4.2 (compare [8, Theorem 10.3]). $X \times Y$ *is a filtered space.*

Proof. To show that $X \times Y = \operatorname{colim} \{(X \times Y)_n\}$, observe that

$$X \times Y = \operatorname{colim} \{X_i\} \times \operatorname{colim} \{Y_j\} = \operatorname{colim} \{X_i \times Y_j\} = \operatorname{colim} \{X_n \times Y_n\}.$$

Since $\{(X \times Y)_n\}$ is cofinal in $\{X_n \times Y_n\}$, that is, since there exist suitable natural mappings

$$X_n \times Y_n \to (X \times Y)_{2n} \to X_{2n} \times Y_{2n},$$

the conclusion follows. ■

THEOREM 4.3 [8, Theorem 10.5]. *If* X *and* Y *are filtered by cofibrations, then so is* $X \times Y$.

Definition 4.4. A *cofiltered space* (*tower*) is a diagram

$$\lim Y^j = Y \to \cdots \to Y^j \to \cdots \to Y^1 \to Y^0;$$

this is usually denoted simply by Y. If in addition each map $Y^{j+1} \to Y^j$ is a fibration, Y is said to be *cofiltered by fibrations*. In this case the maps $Y \to Y^j$ are also fibrations.

Definition 4.5. Suppose that X is a filtered space and Y is a cofiltered space. Define $\operatorname{Map}(X, Y)^n$ by the equalizer diagram

$$\text{(4.2)} \qquad \operatorname{Map}(X, Y)^n \to \prod_{i+j=n-1} \operatorname{Map}(X_i, Y^j) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{i+j=n} \operatorname{Map}(X_i, Y^j).$$

Here f is induced by the maps $X_i \to X_{i+1}$ and g by the maps $Y^{j+1} \to Y^j$. Compare diagram (4.1).

There exist natural mappings $\operatorname{Map}(X, Y)^{n+1} \to \operatorname{Map}(X, Y)^n$.

THEOREM 4.6 (compare Theorem 4.2). Map(X, Y) *is a cofiltered space.*

Proof. As in the proof of Theorem 4.2, observe that $\{\operatorname{Map}(X, Y)^n\}$ is cofinal in $\{\operatorname{Map}(X_n, Y^n)\}$, whose limit is Map(X, Y). We omit the details. ■

Remark 4.7. Even if the maps $X_i \to X_{i+1}$ are inclusions and the maps $Y^j \to Y^{j+1}$ are projections, the maps $\operatorname{Map}(X, Y)^{n+1} \to \operatorname{Map}(X, Y)^n$ need not be projections. To see this, let $X_0 = \{0, 1\}$, $X_i = [0, 1]$ for $i \geq 1$, and $Y^j = \{0, 1\}$ for all j.

5. FIBRATION TOWERS

THEOREM 5.1 (compare Theorem 4.3). *Let* X *be filtered by cofibrations, and let* Y *be cofiltered by fibrations. Then* Map(X, Y) *is cofiltered by fibrations.*

Proof. Given any solid-arrow diagram

$$\text{(5.1)} \qquad \begin{array}{ccc} W \times 0 & \xrightarrow{\quad h \quad} & \operatorname{Map}(X, Y)^{n+1} \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow \\ W \times I & \xrightarrow{\quad H \quad} & \operatorname{Map}(X, Y)^n, \end{array}$$

we shall construct the filler H.

First represent $\operatorname{Map}(X, Y)^n$ as the space of diagrams in Figure 2.

$$\begin{array}{ccc} X_0 & \longrightarrow & Y^n \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & Y^{n-1} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & Y^0 \end{array}$$

Figure 2.

By the exponential law [8, Theorem 5.4], h and H correspond to the respective diagrams in Figure 3.

$$
\begin{array}{ccc}
W \times 0 \times X_0 & \xrightarrow{h_0'} & Y^{n+1} \\
\downarrow & & \downarrow \\
W \times 0 \times X_1 & \xrightarrow{h_1'} & Y^{n} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
W \times 0 \times X_n & \xrightarrow{h_n'} & Y^{1} \\
\downarrow & & \downarrow \\
W \times 0 \times X_{n+1} & \xrightarrow{h_{n+1}'} & Y^{0}
\end{array}
\qquad
\begin{array}{ccc}
W \times I \times X_0 & \xrightarrow{H_0'} & Y^{n} \\
\downarrow & & \downarrow \\
W \times I \times X_1 & \xrightarrow{H_1'} & Y^{n-1} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
W \times I \times X_n & \xrightarrow{H_n'} & Y^{0} \\
\downarrow & & \downarrow \\
W \times I \times X_{n+1} & \xrightarrow{H_{n+1}'} & Y^{-1} = *
\end{array}
$$

Figure 3.

By diagram (5.1), the two diagrams

$$
\begin{array}{ccc}
W \times 0 \times X_j & \xrightarrow{h_j'} & Y^{n-j+1} \\
\downarrow & & \downarrow \\
W \times I \times X_j & \xrightarrow{H_j'} & Y^{n-j}
\end{array}
$$

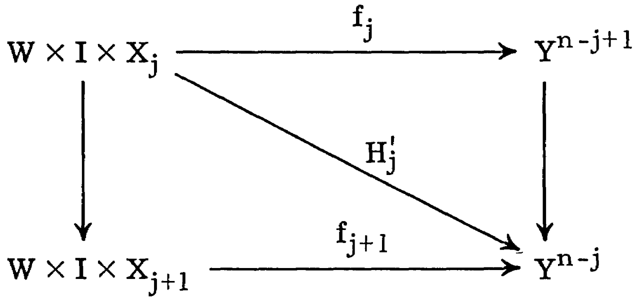
and

$$
\begin{array}{ccc}
W \times 0 \times X_{j-1} & \longrightarrow & W \times 0 \times X_j \\
\downarrow & & \downarrow h_j' \\
W \times I \times X_{j-1} & \xrightarrow{H_{j-1}'} & Y^{n-j+1}
\end{array}
$$

commute. They yield solid-arrow diagrams

$$
\tag{5.2}
\begin{array}{ccc}
W \times 0 \times X_j \cup W \times I \times X_{j-1} & \xrightarrow{h_j' \cup H_{j-1}'} & Y^{n-j+1} \\
\downarrow & \overset{f_j}{\nearrow} & \downarrow \\
W \times I \times X_j & \xrightarrow{H_j'} & Y^{n-j}
\end{array}
$$

$(X_{-1} = \emptyset)$. The covering-homotopy-extension theorem (2.1) yields the fillers f_j. Since diagram (5.2) and the diagrams



commute, an application of the exponential law to $\{f_j\}$ yields the required map $H: W \times I \to \mathrm{Map}(X, Y)^{n+1}$ in diagram (5.1). ■

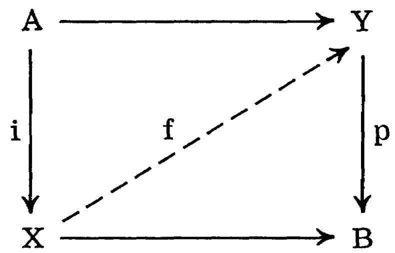
Remark 5.2. We shall call a tower Y of simplicial sets *cofiltered by fibrations* if every map $Y^{j+1} \to Y^j$ is a fibration (as in Definition 4.4) and, additionally, Y^0 is a Kan complex. The results of Sections 4 and 5 then hold for simplicial sets.

6. CG IS A MODEL CATEGORY

We shall show that the covering-homotopy-extension theorem (2.1) implies the lifting property for a model category (Quillen, [5, Definition I.1.1]). The remaining axioms for a *closed model category* [5, Definitions I.1.1 and I.5.1] may easily be verified for CG, together with the usual classes of cofibrations, fibrations, and (homotopy) equivalences.

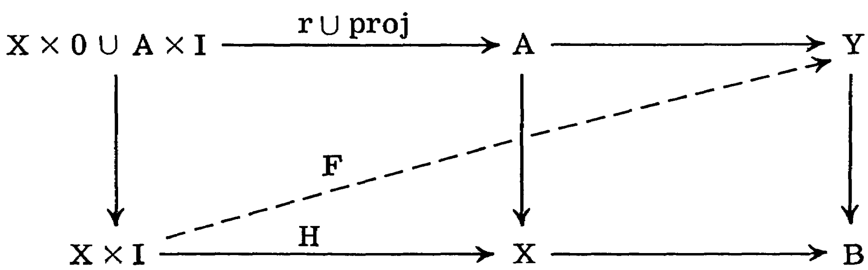
THEOREM 6.1. *For every solid-arrow diagram*

(6.1)



where i *is a cofibration,* p *is a fibration, and either* i *or* p *is a homotopy equivalence, the filler* f *exists.*

Proof. Suppose that i is a homotopy equivalence. Then (see for example [8, Section 1.4]) the space A is a strong deformation retract of X. Let $r: X \to A$ be the retraction, and let $H: X \times I \to X$ be a homotopy relative to A from ri to 1_X. We obtain the commutative solid-arrow diagram



By Theorem 2.1, the filler exists. Let $f(x) = F(x, 1)$; then f is the required map in diagram (6.1).

If instead p is a homotopy equivalence, the filler f may be constructed in a similar way. ■

Remark 6.2. More generally, in any category where homotopy is defined with a cylinder functor, the lifting property of model categories (diagram 6.1) and the covering-homotopy-extension property (diagram 6.2) are equivalent. We omit the details.

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