A FIXED-POINT THEOREM FOR HOMOGENEOUS CONTINUA

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A space is *homogeneous* if for each pair x, y of its points there exists a homeomorphism of the space onto itself that takes x to y. The union S of a finite collection $\mathscr{F}$ of arcs is called a *star* provided that one point of S is the common part of each pair of elements of $\mathscr{F}$. A continuum X is *star-like* if for each $\varepsilon > 0$ there exists an ε-mapping of X onto a star. Here we prove that every homogeneous star-like continuum has the fixed-point property for homeomorphisms.

A *topological transformation group* (G, X) is a topological group G together with a topological space X and a continuous mapping $(g, x) \to gx$ of $G \times X$ into X such that $ex = x$ for all $x \in X$ (e denotes the identity of G) and $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$.

For each $x \in X$, let G_x be the stabilizer subgroup of x in G (that is, the set of all $g \in G$ such that $gx = x$). If we let G/G_x denote the left-coset space with the usual topology, the mapping of G/G_x onto Gx sending gG_x to gx is one-to-one and continuous. We call the set Gx the *orbit* of x.

Henceforth, X is a continuum (that is, a nondegenerate, compact, connected metric space) and G is the topological group of homeomorphisms of X onto itself with the compact open topology. It follows from a theorem of E. G. Effros [2, Theorem 2.1] that each orbit is a G_δ-set in X if and only if for each $x \in X$, the mapping $gG_x \to g(x)$ of G/G_x onto Gx is a homeomorphism. In [5], G. S. Ungar pointed out that if X is homogeneous, then $Gx = X$ for each $x \in X$, and therefore T_x: $g \to g(x)$, being the composition of the natural open mapping of G onto G/G_x and a homeomorphism of G/G_x onto X, is an open mapping of G onto X.

A continuous function f of X onto a space Y is called an *ε-mapping* if for each $y \in Y$, the diameter of $f^{-1}(y)$ in X is less than ε. A finite sequence $\{L_i\}_{i=1}^n$ of open sets in X is a *chain* provided that $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

THEOREM. *Suppose that* X *is a homogeneous star-like continuum. Then for each homeomorphism* h *of* X *onto itself, there exists a point* x *of* X *such that* $h(x) = x$.

Proof. Assume there is a homeomorphism h of X onto itself that moves each point of X. There exist a positive number ε and an open set U in G containing h such that for each $f \in U$ and each $x \in X$, the distance from x to f(x) in X is greater than ε.

Since X is star-like, there is a sequence $\{\mathscr{C}_i\}_{i=1}^\infty$ of open covers of X such that for each i, (1) each element of $\mathscr{C}_i$ has diameter less than i^{-1} and (2) there is an element Y_i of $\mathscr{C}_i$ such that $\mathscr{C}_i - \{Y_i\}$ consists of finitely many mutually disjoint chains, each having only one element, an end-link, that meets Y_i. For each i, define y_i to be a point of Y_i.

Let y be a limit point of $\{y_i\}_{i=1}^\infty$, and let $x = h^{-1}(y)$. Note that $T_x[U]$ is an open set in X that contains y. Let j be an integer such that $j^{-1} < \varepsilon/2$ and the closure of Y_j is a subset of $T_x[U]$.

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Define K to be the x-component of X - Y_j. There is a point z of K that belongs to the boundary of Y_j [4, Theorem 50, p. 18]. Let $\{L_i\}_{i=1}^n$ be the longest chain that is a subcollection of $\mathscr{C}_j - \{Y_j\}$ and covers K. We assume without loss of generality that $z \in L_1$. Let $\{L_i\}_{i=1}^m$ be the shortest subchain of $\{L_i\}_{i=1}^n$ that covers K.

Since $z \in T_x[U]$, there is a homeomorphism f in U such that $f^{-1}(z) = x$. For each k $(1 \leq k \leq m)$, define

$$A_k = \left\{ p \in K \cap L_k \colon f^{-1}(p) \in \bigcup_{i=k}^{n} L_i \right\}$$

and

$$B_k = \left\{ p \in K \cap L_k \colon f^{-1}(p) \in X - \bigcup_{i=k}^{n} L_i \right\}.$$

Note that $A = \bigcup_{k=1}^m A_k$ and $B = \bigcup_{k=1}^m B_k$ are disjoint closed sets and $A \cup B = K$. The point z belongs to A. Since K is connected and A is not empty, B is empty. It follows that $f^{-1}[K] \cap Y_j = \emptyset$. But since $x \in K \cap f^{-1}[K]$, this implies that $f^{-1}[K]$ lies in the component K. Hence B contains $K \cap L_m$, and this is a contradiction. This completes the proof.

Our only example of a homogeneous star-like continuum is the pseudo-arc [1]. Showing that there are no others would make our result a corollary to O. H. Hamilton's theorem [3].

REFERENCES

1. R. H. Bing, *A homogeneous indecomposable plane continuum*. Duke Math. J. 15 (1948), 729-742.

2. E. G. Effros, *Transformation groups and* C^**-algebras*. Ann. of Math. (2) 81 (1965), 38-55.

3. O. H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua*. Proc. Amer. Math. Soc. 2 (1951), 173-174.

4. R. L. Moore, *Foundations of point set theory*. Revised edition, Amer. Math. Soc. Colloquium Publications, Vol. 13. Amer. Math. Soc., Providence, R.I., 1962.

5. G. S. Ungar, *On all kinds of homogeneous spaces* (preprint).

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