FINITELY GENERATED FUCHSIAN GROUPS AND CHARACTER-AUTOMORPHIC NORMAL FUNCTIONS

E. Röding

Ch. Pommerenke [4] (Corollary 2) has shown that for every infinitely generated Fuchsian group there exists a character-automorphic function f(z) in $D = \{|z| < 1\}$ with

$$1 \leq \sup_{z \in D} (1 - |z|^2) f^{\#}(z) \leq K_0 < \infty,$$

where K_0 is an absolute constant. Here we use the notation

$$f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$$

for the spherical derivative. We prove the following supplementary result.

THEOREM. *For every finitely generated Fuchsian group* Γ *there exists a nonconstant character-automorphic function* g(z) *with*

$$\sup_{z \in D} (1 - |z|^2) g^{\#}(z) \leq K_0 < \infty, \tag{1}$$

where K_0 *is an absolute constant.*

Proof. The case where Γ is finitely generated and of the second kind has been treated by Pommerenke in Section 3 of his paper [4]. Clearly, it suffices to consider the case where D/Γ is a compact Riemann surface. According to A. Marden [2], one can choose a conjugate group $\Gamma^* = \psi \circ \Gamma \circ \psi^{-1}$ such that there exists a fundamental region of Γ^* in D whose interior contains a circle K around 0 with radius $\rho > 0$ independent of Γ^*. There exists a single-valued potential function u on $R = D/\Gamma^*$ that has the singular behavior of $\log|z/(z - z_0)|$ near the points on R corresponding to 0 and some fixed point $z_0 \in K$ ($z_0 \neq 0$), and is harmonic elsewhere. If $\tilde{u}$ denotes a conjugate harmonic of u, then the function $f = \exp(u + i\tilde{u})$ is a nonconstant character-automorphic function in D with respect to Γ^*.

Let ρ_1 denote a positive number such that $|z_0| < \rho_1 < \rho$, let

$$B = \{|z| < \rho\}, \quad \beta = \{|z| = \rho\}, \quad B_1 = \{|z| < \rho_1\}, \quad \alpha = \{|z| = \rho_1\},$$

let A denote the complement of B_1 on R, and let $u_0 = \log|z/(z - z_0)|$. The alternating method of Schwarz (see R. Nevanlinna [3, pp. 151-153]) requires the construction of functions u_n and v_n, harmonic in A and B, respectively, and with the boundary values

$$u_n = v_{n-1} + u_0 \text{ on } \alpha \ (v_0 \equiv 0), \qquad v_n = u_n - u_0 \text{ on } \beta. \tag{2}$$

Received July 31, 1974.

Michigan Math. J. 21 (1974).

Let K_i denote positive constants independent of Γ^*. With $q = 2\rho_1/(\rho_1 + \rho)$, we obtain from (2) (see Nevanlinna [3, p. 153]) the inequalities

$$(3) \qquad |v_{n+1} - v_n| \leq 2K_1 q^n \quad \text{in } B, \qquad |u_{n+1} - u_n| \leq 2K_1 q^n \quad \text{in } A.$$

From the relations $v = \sum_{\nu=1}^{\infty}(v_\nu - v_{\nu-1})$ in B and $u = u_0 + \sum_{\nu=1}^{\infty}(u_\nu - u_{\nu-1})$ in A and (3) we obtain bounds

$$(4) \qquad |v| \leq K_2 \text{ in } B, \qquad |u| \leq K_3 \text{ in } A; \qquad \text{therefore } |f| \leq K_4 \text{ in } A.$$

On the other hand, (4) implies that

$$|f(z)| = \exp(u_0 + v) \geq \exp u_0 \exp(-K_2) \geq K_5 \exp u_0 \quad \text{in } B_1.$$

Thus, writing $F = \exp(u_0 + i\tilde{u}_0)$ and $G = \exp(v + i\tilde{v})$ in B_1, we get the inequalities

$$(5) \qquad \begin{aligned} f^{\#}(z) &\leq \frac{|F'|\,|G| + |G|\,|F'|}{1 + K_5^2 \exp 2u_0} \leq \frac{|z_0|\,|G|}{|z - z_0|^2\left(1 + \left|\frac{z}{z - z_0}\right|^2 K_5^2\right)} + \frac{|G'|\,|F|}{1 + |F|^2 K_5^2} \\ &\leq |z_0|\,|G|\,K_6 + |G'|\,K_7 \leq K_8 \quad \text{in } B_1, \end{aligned}$$

where $K_6^{-1} = |z_0/2|^2 K_5^2$; to see this, observe that $|G| \leq K_9$ in B_1, by (4), and that $|G'| \leq K_{10}$ in B_1, by (4) and the Cauchy integral formula. Now let $E = \{|w| > K_4 + 1\}$. Then $f^{-1}(E) \subset \bigcup_{\phi \in \Gamma^*} \phi(B_1)$, by (4). From (5) and the equation

$$(1 - |\phi(z)|^2)\,|f'(\phi(z))| = (1 - |z|^2)\,|f'(z)|$$

for $\phi \in \Gamma^*$ we obtain a bound

$$(1 - |z|^2) f^{\#}(z) \leq K_{11} \quad \text{for all } z \in f^{-1}(E).$$

From this we conclude by Theorem 2 in a paper by A. J. Lohwater and Ch. Pommerenke [1] that the nonconstant character-automorphic function $g(z) = f(\psi(z))$ with respect to $\Gamma = \psi^{-1} \circ \Gamma^* \circ \psi$ satisfies (1), because this supremum is invariant under ψ.

REFERENCES

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Technische Universität Berlin, Germany, Fachbereich Mathematik