MULTIPLIERS, SPECTRAL THEORY, AND THE INTERPOLATION OF CLOSED OPERATORS

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Let (B^0, B^1) be an interpolation pair of Banach spaces, and let B_s, or $[B^0, B^1]_s$, denote the analytic interpolation space of A. P. Calderón [1]. In a sense to be made precise below, we consider the problem of interpolation of a closed operator T on B^0 and B^1. Our principal results concern the equality of $[\mathscr{D}_0(T), \mathscr{D}_1(T)]_s$ and $\mathscr{D}_s(T)$, where $\mathscr{D}_j(T)$ is the natural domain of T in B^j, and where $\mathscr{D}_s(T)$ is the domain of T as a closed operator on B_s (see Theorem 3.3, Corollary 4.2, and Theorem 4.9). These theorems depend partly on the spectral properties of the closed operators involved. Thus we also obtain some results concerning the way in which the spectrum of a closed operator on B_s can change with the parameter s $(0 < s < 1)$. Our examples depend on the theory of multipliers of Fourier series.

1. *Notation and Definitions.* Corresponding to each pair of Banach spaces X and Y, we denote by $O(X, Y)$ the space of bounded linear operators taking X into Y, by $O(X)$ the algebra $O(X, X)$, and by $\|T\|_{(X,Y)}$ the norm of an operator $T \in O(X, Y)$.

If $T \in O(X)$, or more generally, if T is a closed operator with domain and range in X, we denote the *spectrum* of T in X by $sp(T, X)$. By $R(\lambda, T)$ we denote the *resolvent* of T, and by $\rho(T, X)$ the *resolvent set* of T in X. If T is bounded, and if f is a function analytic in a neighborhood of $sp(T, X)$, we let $f(T)$ be the element

$$\frac{1}{2\pi i}\int_C f(\lambda)\, R(\lambda, T)\, d\lambda,$$

where C is an envelope of $sp(T, X)$, contained in the domain of f. If A is a commutative Banach algebra, $\Delta(A)$ will denote the maximal-ideal space of A.

It is well known and simple to show that if T is a closed operator with domain $\mathscr{D}(T)$ and with its range contained in the Banach space X, then $\mathscr{D}(T)$ becomes a Banach space under the norm

$$\|x\|_{\mathscr{D}(T)} = \|x\|_X + \|T(x)\|_X .$$

Let (B^0, B^1) be a pair of complex Banach spaces continuously embedded in a topological linear space. We define the Banach spaces $B^0 \cap B^1$, $B^0 + B^1$, and $[B^0, B^1]_s = B_s$ as in [1, Sections 1 to 3]. The basic properties of the analytic interpolation spaces B_s can be found in [1], and we shall use them freely.

Finally, we introduce some notation concerning multipliers of Fourier series. Let G be a locally compact Abelian group (in short, an LCA group) with dual group

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Γ. For $1 \leq p \leq \infty$, we denote by $L_p(G)$ (or L_p) the usual L_p-space with respect to the Haar measure of G. The symbol $f^\wedge$ will denote the Fourier transform of a function $f \in L_p$ $(1 \leq p \leq 2)$. We denote by $M_p(G)$ the subalgebra of $O(L_p)$ consisting of the operators that commute with all translations on L_p. The elements of $M_p(G)$ are called *multipliers*. It is well known that if $1 \leq p < \infty$ and $T \in M_p(G)$, or if $T \in M_\infty(G)$ and T is continuous with respect to the weak* topology of $L_\infty(G)$, then there exists a unique function $T^\wedge \in L_\infty(\Gamma)$ such that $T(f)^\wedge = T^\wedge f^\wedge$ a.e. on Γ, for all integrable simple functions on G (see [8, Chapters 3 and 4]). $T^\wedge$ will be called the *transform* of T.

We denote by $CM_p(G)$ those multipliers T in $M_p(G)$ for which $T^\wedge$ is continuous on Γ, and we write

$$C_0 M_p(G) = \{T \in CM_p(G) \mid T^\wedge \text{ vanishes at } \infty\}.$$

If $f \in L_1(G)$, we define the operator T_f by the equation

$$T_f(g) = f * g,$$

for all $g \in L_p(G)$, where the symbol $*$ denotes convolution. Then $T_f \in C_0 M_p(G)$. We denote the closure of $\{T_f \mid f \in L_1(G)\}$ in the norm of $O(L_p)$ by $m_p(G)$. A. Figà-Talamanca and G. Gaudry [5] have shown that if G is the n-torus T^n or Euclidean n-space R^n, then $m_p(G)$ is a proper subspace of $C_0 M_p(G)$ $(1 < p < \infty,\ p \neq 2)$. This result will play an important role later in this note (see Sections 2 and 3). The basic properties of $M_p(G)$, $C_0 M_p(G)$, and $m_p(G)$ can be found in [7] and [8]; we shall use them throughout the paper.

2. In this section, we obtain some results concerning the spectra of multipliers as closed operators. Let G be a compact LCA group. Then, if $1 \leq p \leq \infty$ and $\phi \in L_\infty(\Gamma)$, the operator T defined by the equation

$$T(g)^\wedge = \phi g^\wedge$$

for $g \in \mathscr{D} = \{f \in L_p(G) \mid \phi f^\wedge \in L_p^\wedge\}$ is a closed operator on $L_p(G)$ with natural domain $\mathscr{D}_p(T) = \mathscr{D}$. Moreover, $\mathscr{D}_p(T)$ is a dense subspace of $L_p(G)$ for $p < \infty$, since $\mathscr{D}_p(T)$ contains all trigonometric polynomials and G is compact. Thus each multiplier $T \in M_2(G)$ may be viewed as a closed operator on $L_p(G)$ for $1 \leq p \leq \infty$, with natural domain

$$\mathscr{D}_p(T) = \{f \in L_p(G) \mid T^\wedge f^\wedge \in L_p^\wedge\}.$$

It is clear that if $T \in M_p(G)$, then $L_p(G) = \mathscr{D}_p(T)$. In order to find the spectra of multipliers as closed operators, we require the following known lemmas.

LEMMA 2.1 (see [3, Section 16.6.2] or [7, Theorem 1.16]). *Let* G *be a compact* LCA *group, and let* $1 \leq p, q < \infty$ *with* $\left|\frac{1}{q} - \frac{1}{2}\right| < \left|\frac{1}{p} - \frac{1}{2}\right|$. *Then* $C_0 M_p(G) \subseteq m_q(G)$.

LEMMA 2.2. *Let* G *be a compact* LCA *group, let* $1 \leq p, q < \infty$, *with* $\left|\frac{1}{q} - \frac{1}{2}\right| < \left|\frac{1}{p} - \frac{1}{2}\right|$, *and suppose* $T \in C_0 M_p(G)$. *Then* $sp(T, L_q) = T^\wedge(\Gamma) \cup \{0\}$.

This result follows immediately from Lemma 2.1 and the following two elementary facts:

(1) The maximal-ideal space of the Banach algebra $m_q(G)$ may be identified with Γ.

(2) The spectrum $sp(T, L_q)$ equals the spectrum of T as an element of $m_q(G)$.

Lemmas 2.1 and 2.2 remain valid for general LCA groups. The proofs for the case where $G = R^n$ are essentially given in [7]. The proofs for arbitrary G require no new ideas.

THEOREM 2.3. *Let* G *be a compact* LCA *group, let* $1 < q < \infty$, *let* $T \in C_0 M_q(G)$, *and suppose that* $T \notin C_0 M_r(G)$, *for each* r *satisfying the inequality* $\left|\frac{1}{q} - \frac{1}{2}\right| < \left|\frac{1}{r} - \frac{1}{2}\right|$. *Then* $sp(T, L_r)$ *is the complex plane, whenever* $\left|\frac{1}{q} - \frac{1}{2}\right| < \left|\frac{1}{r} - \frac{1}{2}\right|$ *and* $r < \infty$.

Proof. Suppose, to the contrary, that there exists $\lambda \in \rho(T, L_r)$, for some r with $\left|\frac{1}{q} - \frac{1}{2}\right| < \left|\frac{1}{r} - \frac{1}{2}\right|$ and $r < \infty$. Let S denote the operator $(\lambda I - T)^{-1}$ taking L_r onto

$$\mathscr{D}_r(G) = \{f \in L_r \mid T\hat{}\, f\hat{} \in L_r\hat{}\}.$$

Since $S \circ (\lambda I - T)(f) = f$ for all trigonometric polynomials on G, it is simple to verify that

(1) $\quad S = (\lambda I - T)^{-1}$ commutes with all translations on G,

(2) $\quad S(f)\hat{} = \dfrac{1}{\lambda - T\hat{}} f\hat{}$ for all $f \in L_r$.

Choose p with $\left|\frac{1}{q} - \frac{1}{2}\right| < \left|\frac{1}{p} - \frac{1}{2}\right| < \left|\frac{1}{r} - \frac{1}{2}\right|$. Since $S\hat{}$ has the limit $1/\lambda$ at ∞, $S - \frac{1}{\lambda} I \in C_0 M_r(G)$, and therefore by Lemmas 2.1 and 2.2, $S - \frac{1}{\lambda} I \in m_p(G)$ and

$$sp\left(S - \frac{1}{\lambda} I, L_p\right) = \left(S - \frac{1}{\lambda} I\right)\hat{}\,(\Gamma) \cup \{0\}.$$

It follows easily that

(3) $$sp(S, L_p) = \left\{ \frac{1}{\lambda - T\hat{}(\gamma)} \;\middle|\; \gamma \in \Gamma \right\} \cup \left\{ \frac{1}{\lambda} \right\}.$$

By (3), $0 \notin sp(S, L_p)$, so that the function $h(z) = 1/z$ is holomorphic in a neighborhood of $sp(S, L_p)$. Thus $\lambda I - T = h(S) \in O(L_p)$, and therefore $T \in O(L_p)$. This contradicts the choice of p and proves the theorem.

The following lemma will be useful in studying the spectrum of the multiplier of Figà-Talamanca and Gaudry referred to in Section 1.

LEMMA 2.4. *Let* G *be a compact* LCA *group, let* $1 < p < q \leq 2$, *let* $T \in C_0 M_p(G)$, *and suppose that* T *is also a bounded operator from* $L_p(G)$ *into* $L_q(G)$. *Then*

(a) $T^2 \in m_p(G)$,

(b) $sp(T, L_p) = T\hat{}(\Gamma) \cup \{0\}$.

Proof. We first prove (a). Let $\frac{1}{p}+\frac{1}{p'} = 1$ and $\frac{1}{q}+\frac{1}{q'} = 1$. Then, by Theorem 5.2.1 of [8], T is also a bounded operator from $L_{q'}(G)$ into $L_{p'}(G)$. Moreover, Lemma 2.1 implies that $T \in m_{q'}(G)$. Hence there exists a sequence of trigonometric polynomials f_n on G such that

(1) $$\|T - T_{f_n}\|_{(L_{q'},L_{q'})} \to 0 \quad \text{as } n \to \infty,$$

where $T_{f_n}(g) = f_n * g$ for all $g \in L_{q'}(G)$.

It is clear that $T \circ T_{f_n}$ is the multiplier corresponding to convolution by the trigonometric polynomial whose Fourier transform is $T^\wedge f_n^\wedge$. For all $f \in L_{p'}(G)$, we obtain the relations

$$\begin{aligned}\|T^2(f) - T \circ T_{f_n}(f)\|_{L_{p'}} &\leq \|T\|_{(L_{q'},L_{p'})} \|T - T_{f_n}\|_{(L_{q'},L_{q'})} \|f\|_{L_{q'}} \\ &\leq \|T\|_{(L_{q'},L_{p'})} \|T - T_{f_n}\|_{(L_{q'},L_{q'})} \|f\|_{L_{p'}},\end{aligned}$$

the last inequality following since G is compact and $q' < p'$. Hence, by (1), $\|T^2 - T \circ T_{f_n}\|_{(L_{p'},L_{p'})} \to 0$ as $n \to \infty$, and therefore $T^2 \in m_{p'}(G)$. Part (a) now follows immediately, by Theorem 4.1.2 of [8].

In order to show (b), we use the previously stated fact that $\Delta(m_p(G)) = \Gamma$. Since $T^2 \in m_p(G)$, we see that $h(T^2) = 0$ for all homomorphisms $h \in \Delta M_p(G) \setminus \Gamma$. Thus $h(T) = 0$ for all $h \in \Delta M_p(G) \setminus \Gamma$, and since $sp(T, L_p)$ is the spectrum of T as an element of the Banach algebra $M_p(G)$, we see that $sp(T, L_p) = T^\wedge(\Gamma) \cup \{0\}$, as desired. The proof is complete.

We now discuss the important multiplier of Figà-Talamanca and Gaudry [5]. Let G be the circle group, fix p $(1 < p < 2)$, and define $r = 2p(2 - p)^{-1}$. It is shown in [5] that then there exists a multiplier $T \in C_0 M_p(G)$ such that

(a) $T \notin m_p(G)$,

(b) $T \notin C_0 M_q(G)$ for $1 \leq q < p$,

(c) $T^\wedge(m) = \pm 1/2^{n/r}$ for $2^n \leq m \leq 2^{n+1} - 1$ $(n = 0, 1, 2, \cdots)$ and $T^\wedge(m) = 0$ for $m \leq 0$,

(d) T is also a bounded operator from L_p into L_2.

The fundamental property of T is (a). Properties (b), (c), and (d) of T follow by an examination of the construction given in the proofs of Lemma 1 and Theorem B of [5], as well as the remarks following Theorem B.

Using the previous results of this section, together with the properties of T, we obtain the following theorem.

THEOREM 2.5. *Let* G *be the circle group, and let* Z *denote the group of integers. Fix* p $(1 < p < 2)$, *and let* T *be the multiplier of Figà-Talamanca and Gaudry corresponding to* p. *Then*

(a) $sp(T, L_q)$ *is the complex plane, for* $1 \leq q < p$,

(b) $sp(T, L_q)$ *is the countable set* $T^\wedge(Z) \cup \{0\}$, *for* $p \leq q \leq 2$.

Proof. Part (a) is an immediate consequence of Theorem 2.3 and property (b) of the operator T. Part (b) of the theorem follows by Lemma 2.4(b), property (d) of T, and the fact that $T \in m_q(G)$ for $p < q \leq 2$.

Remark 2.6. Theorem 2.5 is also valid if $2 < p < \infty$. In this case, we let T correspond to p', where $\frac{1}{p} + \frac{1}{p'} = 1$. Then $sp(T, L_q)$ is the complex plane, for $p < q < \infty$, and $sp(T, L_q) = T\hat{}(Z) \cup \{0\}$, for $2 \leq q \leq p$. This result may be obtained by an argument similar to that above, or by a duality argument.

Remark 2.7. We note, by property (c) of the multiplier T, that $T^{2\wedge}(m) = 1/2^{2n/r}$ for $2^n \leq m \leq 2^{n+1} - 1$ $(n = 0, 1, 2, \cdots)$, and that $T^{2\wedge}(m) = 0$ for $m \leq 0$. Hence $T^{2\wedge}$ is of uniformly bounded variation on all dyadic "intervals" of the integers. By the classical multiplier theorem of J. Marcinkiewicz [12, Chapter 15, Theorem 4.14], $T^2 \in C_0 M_q(G)$ if $1 < q < \infty$. Thus Lemma 2.1 asserts that $T^2 \in m_q(G)$ for $1 < q < \infty$. Hence we could obtain Theorem 2.5 by using these observations, without Lemma 2.4. We have given our alternate approach, since it seems to apply in more general situations.

3. In this section, we consider the problem of the interpolation of the domains of closed operators. The multiplier of Figà-Talamanca and Gaudry considered in the preceding section plays a fundamental role in the principal result, Theorem 3.3.

Notation. Let (B^0, B^1) be an interpolation pair of Banach spaces continuously embedded in a topological linear space. Let T_j be a closed operator with domain $\mathscr{D}_j(T_j)$ in B^j and with range contained in B^j $(j = 0, 1)$. Suppose $T_0(x) = T_1(x)$ for all $x \in \mathscr{D}_0(T_0) \cap \mathscr{D}_1(T_1)$. We define $T(x) = T_0(x_0) + T_1(x_1)$ for all $x = x_0 + x_1$ with $x_j \in \mathscr{D}_j(T_j)$ $(j = 0, 1)$. It is well known and easy to prove that T is the unique linear extension of T_j to $\mathscr{D}_0(T_0) + \mathscr{D}_1(T_1) \subseteq B^0 + B^1$. Hereafter, we identify T_j and the restriction of T to $\mathscr{D}_j(T_j)$ $(j = 0, 1)$, and we drop the subscripts on the T_j $(j = 0, 1)$. For simplicity, write $\mathscr{D}(T) = \mathscr{D}_0(T) + \mathscr{D}_1(T)$. Define

$$\mathscr{D}_s(T) = \{x \in \mathscr{D}(T) \cap B_s \mid T(x) \in B_s\}.$$

For $x \in \mathscr{D}_s(T)$, we define the norm $\|x\|_{\mathscr{D}_s(T)} = \|x\|_{B_s} + \|T(x)\|_{B_s}$.

With the notation of the preceding paragraph, we can state a theorem of P. Grisvard [6, pp. 168-169] as follows.

THEOREM 3.1. *Suppose there exists* $\lambda \notin sp(T, B^0) \cup sp(T, B^1)$ *such that if* S_j *is the inverse of* $\lambda I - T$ *on* B^j, *then* $S_0 = S_1$ *on* $B^0 \cap B^1$. *Then* T *is a closed operator on* B_s *with domain* $\mathscr{D}_s(T)$, *and* $[\mathscr{D}_0(T), \mathscr{D}_1(T)]_s = \mathscr{D}_s(T)$ $(0 < s < 1)$, *with equivalence of norms.*

The condition stated in Theorem 3.1, asserting that the inverses S_0 and S_1 agree on $B^0 \cap B^1$, deserves some further comment. It is shown in example 5.28 of [11] that S_0 and S_1 need not coincide on $B^0 \cap B^1$, even if the operator T is bounded on both B^0 and B^1. However, with the same notations as in Theorem 3.1, we have the following result.

LEMMA 3.2. *If* $\mathscr{D}_0(T) \subseteq \mathscr{D}_1(T)$, *if there exists* $\lambda \notin sp(T, B^0) \cup sp(T, B^1)$, *and if* S_j *denotes the inverse of* $\lambda I - T$ *on* B^j, *then* $S_0 = S_1$ *on* $B^0 \cap B^1$.

Proof. Let $x \in B^0 \cap B^1$, and let $y = S_0(x)$. Then $y \in \mathscr{D}_0(T) \subseteq \mathscr{D}_1(T)$, and $(\lambda I - T)(y) = x$. Hence $S_1(x) = S_1((\lambda I - T)(y)) = y = S_0(x)$; this proves the lemma.

We now turn to our principal example concerning the interpolation of closed operators. For notational convenience, we write $\mathscr{D}_r(T)$ as the domain of the closed operator T on an L_r-space. Thus, if our interpolation pair is (L_{r_0}, L_{r_1}), we write $\mathscr{D}_{r_j}(T) = \mathscr{D}_j(T)$ $(j = 0, 1)$. We recall that if G is a compact LCA group and $T \in M_2(G)$, then the natural domain of the closed operator T on L_r is $\mathscr{D}_r(T) = \{f \in L_r \mid T^\wedge f^\wedge \in L_r^\wedge\}$. (See the beginning of Section 2.)

THEOREM 3.3. *Let* $1 < p < 2$, *and let* T *denote the multiplier of Figà-Talamanca and Gaudry on the circle group* G, *corresponding to* p (see Section 2). *Let* $1 \leq q < p$, *and choose* s *so that* $\frac{1}{p} = \frac{1-s}{q} + \frac{s}{2}$. *Then* $[\mathscr{D}_q(T), \mathscr{D}_2(T)]_s \neq \mathscr{D}_p(T)$.

Proof. We begin by noting that

$$L_p = \mathscr{D}_p(T) = \{f \in [\mathscr{D}_q(T) + \mathscr{D}_2(T)] \cap L_p \mid T(f) \in L_p\} = \mathscr{D}_s(T).$$

The first equality holds because T is bounded on L_p, the second, because $L_p \subseteq \mathscr{D}_q(T)$, which in turn is a simple consequence of the fact that $L_p \subseteq L_q$; the last is a matter of definition. Hence this theorem does indeed show that the natural extension of Theorem 3.1 fails.

Let $\{f_n\}$ be an approximate identity of trigonometric polynomials on the circle group G, with $\|f_n\|_{L_1} = 1$ for all n, and define $T_{f_n}(g) = f_n * g$ for all $g \in L_1$. (For example, the operator T_{f_n} may be taken as the "n^{th}-Cesaro-sum operator".) Let $S_n = T \circ T_{f_n}$. Then S_n is the multiplier corresponding to convolution by the trigonometric polynomial whose Fourier transform is $T^\wedge f_n^\wedge$. Moreover, since $T^\wedge$ vanishes at ∞ on the integers, it follows that

$$\|T - S_n\|_{(L_2, L_2)} \to 0 \quad \text{as } n \to \infty. \tag{1}$$

Now let $g \in \mathscr{D}_q(T)$. Then $S_n(g) = T_{f_n} \circ T(g)$, and we obtain the relations

$$\|S_n(g)\|_{L_q} = \|T_{f_n} \circ T(g)\|_{L_q} \leq \|T_{f_n}\|_{(L_q, L_q)} \|T(g)\|_{L_q} \leq \|g\|_{\mathscr{D}_q(T)}.$$

Hence

$$S_n \in O(\mathscr{D}_q(T), L_q) \quad \text{and} \quad \|S_n\|_{(\mathscr{D}_q(T), L_q)} \leq 1. \tag{2}$$

Now assume, contrary to the theorem, that $[\mathscr{D}_q(T), \mathscr{D}_2(T)]_s = \mathscr{D}_p(T)$. By the closed-graph theorem, the norms of these spaces are equivalent. Also, by the convexity theorem in [1, Section 4], applied to the operator $T - S_n$, we obtain the relations

$$\begin{aligned} \|T - S_n\|_{([\mathscr{D}_q(T), \mathscr{D}_2(T)]_s, L_p)} &\leq \|T - S_n\|^{1-s}_{(\mathscr{D}_q(T), L_q)} \|T - S_n\|^{s}_{(\mathscr{D}_2(T), L_2)} \\ &\leq 2^{1-s} \|T - S_n\|^{s}_{(\mathscr{D}_2(T), L_2)}, \end{aligned} \tag{3}$$

the last inequality following by (2). However, $\mathscr{D}_p(T) = L_p$ and $\mathscr{D}_2(T) = L_2$, with equivalence of norms; therefore, by (1) and (3),

$$\|T - S_n\|_{(L_p, L_p)} \to 0 \quad \text{as } n \to \infty .$$

But then $T \in m_p(G)$, contrary to the choice of the operator T. This concludes the proof.

The analogue of this result is also valid for $2 < p < \infty$ (see Remark 2.6). The proof is similar.

4. In view of Theorems 2.5, 3.1 and 3.3, it is natural to ask whether $[\mathscr{D}_0(T), \mathscr{D}_1(T)]_s \neq \mathscr{D}_s(T)$ if $sp(T, B^0)$ is the complex plane. We shall show (see Theorem 4.9) that the answer is negative. Our example is based on the properties of the conjugate function operator.

THEOREM 4.1. *Let* X *be a Banach space, and let* T *be a closed, unbounded operator with domain* $\mathscr{D}(T) \subseteq X$ *and with range in* X. *Define*

$$\mathscr{D}(T^2) = \{ x \in \mathscr{D}(T) \mid T(x) \in \mathscr{D}(T) \},$$

and suppose $\mathscr{D}(T^2)$ *is dense in* X.

(a) *If* $T^2(x) = x$ *for all* $x \in \mathscr{D}(T^2)$, *then* $sp(T, X)$ *is the complex plane.*

(b) *If* $T^2(x) = T(x)$ *for all* $x \in \mathscr{D}(T^2)$, *then* $sp(T, X)$ *is the complex plane.*

Proof. We first prove (a). Suppose, to the contrary, there exists $\lambda \notin sp(T, X)$. We begin by showing that $\lambda \neq 1$ and $\lambda \neq -1$. If $\lambda = 1$, then $(I - T)^2(x) = 2(I - T)(x)$ for all $x \in \mathscr{D}(T^2)$. Thus

$$R(1, T) \circ (I - T)^2(x) = 2R(1, T) \circ (I - T)(x),$$

and therefore $T(x) = -x$, for all x in the dense subspace $\mathscr{D}(T^2)$ of X. It follows that T can be extended to a bounded operator on X, a contradiction. Thus $\lambda \neq 1$. Similarly, $\lambda \neq -1$.

Now define the closed, unbounded operator $S_\lambda = \lambda(\lambda^2 - 1)^{-1} I + (\lambda^2 - 1)^{-1} T$. Using the fact that $T^2(x) = x$ for all $x \in \mathscr{D}(T^2)$, we can easily verify that

$$(\lambda I - T) \circ S_\lambda(x) = x \tag{1}$$

for all $x \in \mathscr{D}(T^2)$.

Since $R(\lambda, T)$ exists, we see by (1) that $S_\lambda(x) = R(\lambda, T)(x)$ for all x in the dense subspace $\mathscr{D}(T^2)$ of X. But then the unbounded operator S_λ can be extended to a bounded operator on X. This contradiction completes the proof of (a).

The proof of (b) is similar. We now replace the operator S_λ by $U_\lambda = \lambda^{-1} I + (\lambda^2 - \lambda)^{-1} T$ and repeat the preceding argument. The proof of the theorem is complete.

It is interesting to compare Theorem 4.1 with Theorem 10 in Section 9, Chapter 7, of [2]. In particular, Theorem 4.1 provides an example showing that the latter result fails without the assumption that $\rho(T, X)$ is nonempty.

We now discuss some applications of Theorem 4.1. Let T be the multiplier on the circle group G defined by the equation $T(f)\hat{}(n) = \operatorname{sgn} n\, f\hat{}(n)$, for all $f \in L_2(G)$ (we define $\operatorname{sgn} 0 = 1$, and we adhere to this definition of sgn throughout this paper). Then T is a slight modification of the usual conjugate operator C for which

$C(f)^\wedge(n) = -i \operatorname{sgn} n\, f^\wedge(n)$ for $n \neq 0$, and $C(f)^\wedge(0) = 0$. It is well known that $T \in M_p(G)$ for $1 < p < \infty$, and that T is an unbounded operator on L_1 [12, Chapter 7]. As a corollary to Theorem 4.1, we have the following result.

COROLLARY 4.2. *Let* G *be the circle group, and define the multiplier* T *by* $T(f)^\wedge(n) = \operatorname{sgn} n\, f^\wedge(n)$, *for all integers* n. *Then* $\operatorname{sp}(T, L_1)$ *is the complex plane.*

Proof. It is evident that the class of trigonometric polynomials on G is contained in $\mathscr{D}(T^2) = \{f \in L_1 \mid T(f) \in \mathscr{D}(T)\}$. The result now follows easily by Theorem 4.1(a).

The analogue of Corollary 4.2 also holds for the Hilbert transform H on the real line, even though the natural domain of H on L_1 is not dense in L_1, so that Theorem 4.1 does not apply.

THEOREM 4.3. *Let* R *denote the real line, and define* $T(f)^\wedge(x) = \operatorname{sgn} x\, f^\wedge(x)$ *for all* $x \in R$ *and* $f \in L_2(R)$. *Then* $\operatorname{sp}(T, L_1)$ *is the complex plane.*

Proof. Suppose, to the contrary, that there exists $\lambda \notin \operatorname{sp}(T, L_1)$. Since $R(\lambda, T) \circ (\lambda I - T)(f) = f$ for all $f \in \mathscr{D}(T) = \{g \in L_1 \mid \operatorname{sgn} x\, g^\wedge(x) \in L_1^\wedge\}$, it is not difficult to verify that $R(\lambda, T)$ commutes with all translations on L_1. Thus, for all $f \in L_1$, $(\lambda I - T) \circ R(\lambda, T)(f)^\wedge = f^\wedge$ and

$$(\lambda - \operatorname{sgn} x) R(\lambda, T)^\wedge(x) f^\wedge(x) = f^\wedge(x)$$

for all $x \in R$. It follows that $R(\lambda, T)^\wedge(x) = \dfrac{1}{\lambda - \operatorname{sgn} x}$ for $x \in R$. However, by Theorem 0.1.1 of [8], $R(\lambda, T)^\wedge$ must coincide with the Fourier-Stieltjes transform of a regular Borel measure on R. In particular, the function $\dfrac{1}{\lambda - \operatorname{sgn} x} = R(\lambda, T)^\wedge(x)$ must be continuous. This contradiction proves the theorem.

We had originally obtained Theorem 4.3 by an argument somewhat similar to that in the proof of Theorem 4.1. The elegant proof given here was suggested to the author by Professor J. D. Stafney.

We now proceed to give another application of Theorem 4.1. Let G be the circle group, and let Z denote the group of integers. Let $0 < q < \infty$. Following W. Rudin [9], we say that a set $E \subseteq Z$ is of *type* $\Lambda(q)$ if and only if there exist an r $(0 < r < q)$ and a constant $B > 0$ such that

$$\|f\|_{L_q(G)} \leq B \|f\|_{L_r(G)}$$

whenever f is a trigonometric polynomial satisfying $f^\wedge(n) = 0$ for each $n \notin E$. It is shown in [9] that this definition is independent of r. Moreover, it is clear that if E is a $\Lambda(q)$-set, then E is a $\Lambda(q - \varepsilon)$-set for all $\varepsilon > 0$ with $\varepsilon < q$. We state the following result of Rudin (see [9, Theorem 4.8]).

LEMMA 4.4. *Let* q *be an even integer* $(q \geq 4)$. *Then there exists a set* E *of type* $\Lambda(q)$ *such that* E *is not of type* $\Lambda(q + \varepsilon)$ *for any* $\varepsilon > 0$.

We also require the following lemma.

LEMMA 4.5. *Let* G *denote the circle group, and let* $2 \leq q < \infty$. *Let* E *be a set of type* $\Lambda(q)$. *Define* $T(f)^\wedge = \chi_E f^\wedge$ *for* $f \in L_2(G)$, *where* χ_E *is the characteristic function of* E. *Then*

(a) $T \in M_q(G)$; *in fact,* T *takes* $L_2(G)$ *into* $L_q(G)$;

(b) *if* E *is not of type* $\Lambda(r)$ *for some* $r > q$, *then* $T \notin M_r(G)$.

Proof. For (a), we refer the reader to Theorem 5.4 of [9], or to the argument used in Section 5.7.8 of [10], combined with Theorem 5.2.1 of [8].

To show (b), we suppose that, to the contrary, $T \in M_r(G)$. Choose $p < 2$ with $\frac{1}{p} + \frac{1}{r} = 1$. Then $T \in M_p(G)$. Hence, if $f \in L_p$, then $T(f) \in L_p$ and $T(f)\hat{\ }(n) = 0$ for $n \notin E$. Since E is of type $\Lambda(q)$ with $q \geq 2$, it follows that $T(f) \in L_2$, and T is a bounded operator from L_p into L_2. By Theorem 5.2.1 of [8], T is also a bounded operator from L_2 into L_r. It follows easily that E is of type $\Lambda(r)$, and this contradiction completes the proof of the lemma.

COROLLARY 4.6. *Let* G *denote the circle group, and let* q *be an even integer* ($q \geq 4$). *Let* E *be the* $\Lambda(q)$*-set of Lemma* 4.4. *Define* $T(f)\hat{\ } = \chi_E f\hat{\ }$, *for* $f \in L_2(G)$. *Then, for all* $r > q$, *the spectrum* $sp(T, L_r)$ *is the complex plane. Also, if* $\frac{1}{p} + \frac{1}{q} = 1$, *then* $sp(T, L_r)$ *is the complex plane for* $1 \leq r < p$.

The first assertion is an immediate consequence of Theorem 4.1(b) and Lemmas 4.4 and 4.5. The second part follows easily from the equality of $M_r(G)$ and $M_{r'}(G)$ whenever $\frac{1}{r} + \frac{1}{r'} = 1$.

Remark 4.7. Corollaries 4.2 and 4.6 differ from Theorems 2.3 and 2.5 in the following respects:

(a) In Corollaries 4.2 and 4.6, the operators involved do not have transforms vanishing at ∞.

(b) In the case of Corollary 4.2, the modified conjugate function operator T is unbounded on L_1, and it is bounded on L_r for all r in the half-open interval $1 < r \leq 2$. In Theorems 2.3 and 2.5, the operators involved are unbounded on L_r for $1 \leq r < p$, and they are bounded on L_r for all r in the closed interval $p \leq r \leq 2$.

Remark 4.8. Theorem 2.5 provides an example of a closed operator T such that $sp(T, L_r)$ is the complex plane for $1 \leq r < p$, and $sp(T, L_r)$ is a countable set for $p \leq r \leq 2$.

Corollary 4.2 gives an example of a closed operator T for which $sp(T, L_r)$ consists of two points, for $1 < r \leq 2$, but for which $sp(T, L_1)$ is the complex plane.

Corollary 4.6 asserts the existence of a closed operator T for which $sp(T, L_r)$ consists of two points for $p \leq r \leq 2$, but for which $sp(T, L_r)$ is the complex plane for $1 \leq r < p$.

We now return to the question raised at the beginning of this section.

THEOREM 4.9. *Let* G *denote the circle group, and define the multiplier* T *by the condition* $T(f)\hat{\ }(n) = \operatorname{sgn} n\, f\hat{\ }(n)$, *for all integers* n. *Let*

$$\mathscr{D}_1(T) = \{f \in L_1 \mid \operatorname{sgn} n\, f\hat{\ }(n) \in L_1\hat{\ }\}.$$

Let $1 < p < r < \infty$, *and let* $0 < s < 1$, *with* $\frac{1}{p} = \frac{1-s}{1} + \frac{s}{r}$. *Then* $[\mathscr{D}_1(T), L_r]_s = L_p$, *with equivalence of norms.*

The theorem is essentially known. The analogue of the result for the natural domain of the Riesz transforms on R^n is stated, although not explicitly proved, by C. Fefferman and E. M. Stein in [4]. Also, all the basic ideas for our particular result can be found in [12, Chapter 12, Theorem 3.9]. Since, however, our theorem does not seem to appear explicitly in the literature, we sketch a proof, based on the argument of [12, Chapter 12, Theorem 3.9].

Proof of Theorem 4.9. Let H^q denote the Hardy space as defined in [12, Chapter 7], with $0 < q < \infty$.

We first show that $[H^1, H^r]_s = H^p$. It is almost trivial to prove that $[H^1, H^r]_s \subseteq H^p$. We thus prove the reverse inclusion. For any integrable simple functions g_1 and g_2, we define the bilinear mapping ι by $\iota(g_1, g_2) = F_1 F_2$, where

$$F_j(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, g_j(t)\, dt \qquad (|z| < 1,\ j = 0, 1). \tag{1}$$

The argument of Theorem 3.9, Chapter 12, of [12] shows that there exists a constant $K > 0$ such that

$$\begin{cases} \|\iota(g_1, g_2)\|_{H^1} \leq K\, \|g_1\|_{L_2}\, \|g_2\|_{L_2}, \\ \|\iota(g_1, g_2)\|_{H^r} \leq K\, \|g_1\|_{L_{2r}}\, \|g_2\|_{L_{2r}}. \end{cases} \tag{2}$$

Moreover, we may extend ι to $L_{2r} \oplus L_{2r}$ so that we still have the relation $\iota(g_1, g_2) = F_1 F_2$, where F_j is defined by (1), and so that (2) remains valid.

By the multilinear-interpolation theorem in [1, Section 10.1], we see that

$$\|\iota(g_1, g_2)\|_{[H^1, H^r]_s} \leq K\, \|g_1\|_{L_{2p}}\, \|g_2\|_{L_{2p}} \tag{3}$$

for all $g_1, g_2 \in L_{2r}$. Now, if P is a complex polynomial, we write $P = BF^2$, where B is the Blaschke product formed with the zeros of P lying in the set $\{z \mid |z| < 1\}$, F is a bounded, holomorphic function in $\{z \mid |z| < 1\}$, and $F(0) > 0$ (see [12, pp. 274-275]). Multiplying P by a number of absolute value one, we may suppose that $P(0)$ is real. Then there exist real-valued functions $g_1, g_2 \in L_{2r}$ such that BF corresponds to g_1 and F corresponds to g_2 as in (1). Hence, by (3),

$$\|P\|_{[H^1, H^r]_s} = \|\iota(g_1, g_2)\|_{[H^1, H^r]_s} \leq K\, \|g_1\|_{L_{2p}}\, \|g_2\|_{L_{2p}} \leq K\, \|P\|_{H^p}, \tag{4}$$

the last inequality following as in the argument of [12, Chapter 12, Theorem 3.9]; here K is an absolute constant. By (4) and the fact that the complex polynomials are dense in $[H^1, H^r]_s$ and in H^p, we see that $H^p \subseteq [H^1, H^r]_s$. Hence $H^p = [H^1, H^r]_s$.

Define the mapping $\omega(f)(x) = f(-x)$ for $x \in G$ and measurable functions f. Define

$$H^q_- = \{f \in L_q \mid f\hat{}(n) = 0 \text{ for } n \geq 0\} \qquad (1 \leq q < \infty).$$

Since H^q may be identified with $\{f \in L_q \mid f\hat{}(n) = 0 \text{ for } n < 0\}$, and since ω is an isometric mapping of H^q onto $\{f \in L_q \mid f\hat{}(n) = 0 \text{ for } n > 0\}$, it is easy to see that $[H^1_-, H^r_-]_s = H^p_-$. Hence

$$[\mathscr{D}_1(T), L_r]_s = [H^1 \oplus H^1_-, H^r \oplus H^r_-]_s = [H^1, H^r]_s \oplus [H^1_-, H^r_-]_s = H^p \oplus H^p_- = L_p,$$

with equivalence of norms. This concludes the proof of the theorem.

Remark 4.10. Let G be an infinite, compact, connected LCA group, with ordered dual group Γ (see [10, Chapter 8]).

Let $1 < p < r < \infty$, and let $0 < s < 1$ with $\frac{1}{p} = \frac{1-s}{1} + \frac{s}{r}$. Then, in the notation of [10, Chapter 8], we have the relation $[H^1(G), H^r(G)]_s = H^p(G)$. The proof of this is similar to that of Theorem 4.9. We now define the bilinear mapping ι by the equation $\iota(g_1, g_2) = \Phi(g_1)\Phi(g_2)$ for all trigonometric polynomials on G, where Φ is as in Section 8.7.1 of [10]. Arguing as in the proof of Theorem 4.9, we see that

$$\| \iota(g_1, g_2)\|_{[H^1(G), H^r(G)]_s} \leq K \|g_1\|_{L_{2p}} \|g_2\|_{L_{2p}}$$

for all bounded functions g_1 and g_2. Now, if f is any trigonometric polynomial in $H^p(G)$ with $f^\wedge(0) \neq 0$, the proof of Theorem 8.4.4 of [10] shows that we may write

$$f = \alpha\beta,$$

where $|f| = |\alpha|^2 = |\beta|^2$, and where α and β are of analytic type. Hence

$$\|f\|_{[H^1(G), H^r(G)]_s} = \|\iota(\alpha, \beta)\|_{[H^1(G), H^r(G)]_s} \leq K \|\alpha\|_{L_{2p}} \|\beta\|_{L_{2p}} = K \|f\|_{H^p(G)} .$$

Since this is valid for all trigonometric polynomials f of analytic type with $f^\wedge(0) \neq 0$, it is simple to verify that

$$\|g\|_{[H^1(G), H^r(G)]_s} \leq K \|g\|_{H^p(G)}$$

for all trigonometric polynomials g of analytic type. It follows that $H^p(G) \subseteq [H^1(G), H^r(G)]_s$. The reverse inclusion is elementary.

We also note that if C is the operator defined by the equation

$$C(f) = -i \sum_{\gamma > 0} a_\gamma \gamma + i \sum_{\gamma < 0} a_\gamma \gamma$$

for all trigonometric polynomials $f = \sum_{\gamma \in \Gamma} a_\gamma \gamma$ on G, then

$$[\mathscr{D}_1(C), L_r(G)]_s = L_p(G),$$

where $\mathscr{D}_1(C)$ is the domain of C on L_1. The proof is similar to that of Theorem 4.9. Moreover, the analogue of Corollary 4.2 is valid for C, by Section 8.7.5 of [10]. This concludes the remark.

By combining Corollary 4.2 and Theorem 4.9, or by Remark 4.10, we immediately see that the answer to the question posed at the beginning of this section is negative. Note also that Theorem 4.3, combined with the previously stated interpolation theorem of Fefferman and Stein [4], shows that the Hilbert transform on R provides a counterexample to this question. However, in a sense, this is a less decisive example, since the domain of the Hilbert transform on $L_1(R)$ is not dense in $L_1(R)$.

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