CONVEXITY PROPERTIES OF OPERATOR RADII ASSOCIATED WITH UNITARY ρ-DILATIONS

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1. INTRODUCTION

For $\rho > 0$, let $\mathscr{C}_\rho$ denote the class of bounded linear operators T on a Hilbert space $\mathscr{H}$ whose powers admit a representation

$$T^n h = \rho P U^n h \qquad (h \in \mathscr{H};\ n = 1, 2, \cdots),$$

where U is a unitary operator (called a *unitary ρ-dilation*) on some Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ as a subspace, and where P is the projection from $\mathscr{K}$ to $\mathscr{H}$. Intrinsic characterizations of operators of class $\mathscr{C}_\rho$ were given by B. Sz.-Nagy and C. Foiaş [6]. Later, J. A. R. Holbrook [3] and J. P. Williams [7] introduced the concept of the operator radius $w(\rho)$ of an operator T, relative to $\mathscr{C}_\rho$. The operator radius is defined by the formula

$$w(\rho) = w(\rho; T) = \inf\{\gamma: \gamma > 0,\ \gamma^{-1} T \in \mathscr{C}_\rho\}.$$

It is known that $w(1)$ coincides with the norm $\|T\|$ while $w(2)$ is simply the numerical radius

$$w(2) = \sup\{|(Th, h)|: \|h\| = 1\}.$$

Holbrook [3], [4] investigated basic properties of $w(\rho)$. Among other things, he showed that $w(\rho)$ is a nonincreasing function of ρ, that

$$w(1) \leq \rho \cdot w(\rho) \leq (2\rho' - \rho) \cdot w(\rho') \qquad (\rho \leq \rho'),$$

and that $w(\infty) = \lim_{\rho \to \infty} w(\rho)$ coincides with the spectral radius of T.

Further, Holbrook [4] proved the convexity of $w(\rho)$ on $(0, 1)$, and he asked whether $w(\rho)$ is convex on the whole interval $(0, \infty)$. Our main purpose in this paper is to prove that $\log w(\rho)$ is convex on $(0, \infty)$.

In Section 2, using function-theoretic methods, we shall show that $\log w(\rho)$ and $\log\{(e^\xi + 1) w(e^\xi + 1)\}$ are convex on $(0, \infty)$ and $(-\infty, \infty)$, respectively. Incidentally, we point out the reciprocity law

$$\rho \cdot w(\rho) = (2 - \rho) \cdot w(2 - \rho) \qquad (0 < \rho < 2),$$

which has hitherto been overlooked. As a consequence of convexity, we show that $\rho \cdot w(\rho)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

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An explicit algebraic form of $w(\rho)$ has until now been known only when $T^2 = 0$ or T is a *normaloid*, that is, $w(1) = w(\infty)$. In Section 3, we shall calculate $w(\rho)$ for the cases $T^2 = T$ and $T^2 = I$.

2. LOGARITHMIC CONVEXITY

Let $w(\rho)$ denote the operator radius for a fixed operator T. The criterion of Sz.-Nagy and Foiaş [6] says that $w(\rho) \leq 1$ if and only if the operator-valued function

$$F(z) = I + 2 \cdot \rho^{-1} z T(I - zT)^{-1} \qquad (|z| < 1)$$

has positive real part.

THEOREM 1. *The function* $\log w(\rho)$ *is convex on* $(0, \infty)$.

Proof. Consider the operator-valued function

$$G(z) = -2zT(I - zT)^{-1}.$$

Here the condition $w(\rho) \leq \alpha$ is equivalent to the inequality

$$(*) \qquad \Re G(z) \leq \rho I \qquad (|z| < \alpha^{-1}).$$

An application of a modified form of Hadamard's three-circle theorem [5, p. 250] to the operator-valued analytic function $G(z)$ shows that if $\rho_1, \rho_2 > 0$, $1 > \lambda > 0$, and $\Re G(z) \leq \rho_i I$ $(|z| < w(\rho_i)^{-1};\ i = 1, 2)$, then

$$\Re G(z) \leq (\lambda\rho_1 + (1 - \lambda)\rho_2) I \qquad (|z| = w(\rho_1)^{-\lambda} w(\rho_2)^{-1+\lambda}).$$

By the maximum-modulus principle, the last inequality holds on the disc $|z| < w(\rho_1)^{-\lambda} w(\rho_2)^{-1+\lambda}$. Now (*) yields the relation

$$w(\lambda\rho_1 + (1 - \lambda)\rho_2) \leq w(\rho_1)^{\lambda} w(\rho_2)^{1-\lambda},$$

and this completes the proof.

COROLLARY 2. *The operator radius* $w(\rho)$ *is a convex function on* $(0, \infty)$.

Another useful variant (see [1]) of the criterion of Sz.-Nagy and Foiaş says that the inequality $w(\rho) \leq \alpha$ is equivalent to the condition

$$(**) \qquad \| T\{(\rho - 1)T - \rho\alpha z I\}^{-1} \| \leq 1 \qquad (|z| \geq 1).$$

Since $|\rho - 1| = |(2 - \rho) - 1|$, the following useful reciprocity law follows immediately from (**).

THEOREM 3. $\rho \cdot w(\rho) = (2 - \rho) \cdot w(2 - \rho) \quad (0 < \rho < 2)$, *and*

$$\lim_{\rho \to 0} 2^{-1} \rho \cdot w(\rho) = w(2).$$

THEOREM 4. *The function* $\log\{(e^x + 1) \cdot w(e^x + 1)\}$ *is convex on* $(-\infty, \infty)$, *while* $\log\{(1 - e^x) \cdot w(1 - e^x)\}$ *is convex on* $(-\infty, 0)$.

Proof. Let $1 < \rho_1 < \rho_2$ and $0 < \lambda < 1$. Define ρ and α by the equations

$$\rho - 1 = (\rho_1 - 1)^\lambda \cdot (\rho_2 - 1)^{1-\lambda}$$

and

$$\rho \cdot \alpha = (\rho_1 \cdot w(\rho_1))^\lambda (\rho_2 \cdot w(\rho_2))^{1-\lambda}.$$

Since $T(T - zI)^{-1}$ is an operator-valued analytic function on

$$|z| \geq \rho_2(\rho_2 - 1)^{-1} \cdot w(\rho_2),$$

one sees from (**) that

$$\|T(T - zI)^{-1}\| \leq \rho_j - 1 \qquad (|z| = \rho_j(\rho_j - 1)^{-1} \cdot w(\rho_j);\ j = 1, 2).$$

Thus, it follows from Hadamard's three-circle theorem [5, p. 250] that

$$\|T(T - zI)^{-1}\| \leq (\rho_1 - 1)^\lambda \cdot (\rho_2 - 1)^{1-\lambda} = \rho - 1 \qquad (|z| = \rho\alpha(\rho - 1)^{-1}).$$

Since $T(T - zI)^{-1}$ converges to 0 as $z \to \infty$, the maximum-modulus principle shows that the inequality above holds for $|z| \geq \rho\alpha(\rho - 1)^{-1}$. Now (**) yields the relation

$$\log\{\rho \cdot w(\rho)\} \leq \log(\rho\alpha) = \lambda \cdot \log\{\rho_1 \cdot w(\rho_1)\} + (1 - \lambda) \cdot \log\{\rho_2 \cdot w(\rho_2)\},$$

which proves the first half of the theorem. The second half follows from the first by Theorem 3.

COROLLARY 5. *The function* $\rho \cdot w(\rho)$ *is increasing on* $(1, \infty)$ *and decreasing on* $(0, 1)$.

Proof. Let $f(x) = \log\{(e^x + 1) \cdot w(e^x + 1)\}$. As we remarked in Section 1,

$$f(x) \geq \log w(1) = f(-\infty),$$

and therefore $f(x)$ must be increasing, because of its convexity. Thus $\rho \cdot w(\rho)$ is increasing on $(1, \infty)$. The remaining part now follows from Theorem 3.

On $(0, 1)$, Holbrook's majorization

$$\rho \cdot w(\rho) \leq (2\rho' - \rho) \cdot w(\rho') \qquad (0 < \rho \leq \rho' \leq 1)$$

is best possible, while on $(1, \infty)$, Corollary 5 is best possible:

$$\rho \cdot w(\rho) \leq \rho' \cdot w(\rho') \qquad (1 \leq \rho \leq \rho').$$

3. THE OPERATOR RADIUS FOR SPECIAL T

An explicit algebraic form of $w(\rho)$ is seldom known. However, Theorem 3 implies that if T is a normaloid, that is, if $w(1) = w(\infty)$, then

$$w(\rho) = \max\{\rho^{-1}(2 - \rho), 1\} w(1);$$

this was first proved by E. Durszt [2].

If $T^2 = 0$, then $T(T - zI)^{-1} = -z^{-1} T$, and from (**) it follows that

$$w(\rho) = \rho^{-1} \|T\| = \rho^{-1} w(1);$$

this was first proved by Holbrook [3].

THEOREM 6. (a) *If* T *is idempotent* (*that is, if* $T^2 = T$) *and* $T \neq 0$, *then*

$$w(\rho) = \rho^{-1} \{w(1) + |\rho - 1|\}.$$

(b) *If* T *is involutive, that is, if* $T^2 = I$, *then*

$$w(\rho) = \rho^{-1} \{w(2) + \sqrt{w(2)^2 + \rho(\rho - 2)}\}.$$

Proof. (a) Since $T^2 = T$ implies that $T(T - zI)^{-1} = (1 - z)^{-1} T$, the assertion follows immediately from (**).

(b) For each vector $h \neq 0$, the linear span $\mathcal{M}_h$ of h and Th is invariant under T, and the restriction T_h of T to $\mathcal{M}_h$ is again involutive. This observation shows that

$$w(\rho) = \sup_h w(\rho; T_h).$$

Since $w(2; T_h) \geq 1$ and the function

$$f(t) = \rho^{-1} \{t + \sqrt{t^2 + \rho(\rho - 2)}\}$$

is increasing on $(1, \infty)$, it suffices to prove the assertion for each T_h. Thus we can restrict our discussion to the case $\dim \mathcal{H} \leq 2$. Further, as we remarked at the beginning of this section, if $T = I$ or $T = -I$, then

$$w(\rho) = \max(\rho^{-1}(2 - \rho), 1) = \rho^{-1} \{w(2) + \sqrt{w(2)^2 + \rho(\rho - 2)}\}.$$

Thus it remains to treat the case where $T \neq \pm I$ and $\dim \mathcal{H} = 2$. With respect to some orthonormal basis, T has here the matrix form

$$T = \begin{bmatrix} 1 & \eta \\ 0 & -1 \end{bmatrix}.$$

Therefore $\{(\rho - 1)T - \rho z I\}^{*-1} T^* T \{(\rho - 1)T - \rho z I\}^{-1}$ has the matrix form

$$\begin{bmatrix} \dfrac{1}{|(\rho - 1) - \rho z|^2} & \dfrac{\rho \eta z}{|(\rho - 1) - \rho z|^2 \cdot \{(\rho - 1) + \rho z\}} \\ \dfrac{\rho \bar{\eta} \bar{z}}{|(\rho - 1) - \rho z|^2 \cdot \{(\rho - 1) + \rho \bar{z}\}} & \dfrac{\rho^2 |z|^2 |\eta|^2}{|(\rho - 1)^2 - \rho^2 z^2|^2} + \dfrac{1}{|(\rho - 1) + \rho z|^2} \end{bmatrix}$$

It follows from (**) that for $|z| = w(\rho)$ the maximum value of the eigenvalues of these matrices is 1. With $w = w(\rho)$, this leads to the equations

$$\{(\rho - 1)^2 - \rho^2 w^2\}^2 - 2(\rho - 1)^2 - 2\rho^2 w^2 - \rho^2 |\eta|^2 w^2 + 1 = 0$$

and

$$\rho^2 w^4 - w^2 \{2(\rho - 1)^2 + |\eta|^2 + 2\} + (\rho - 2)^2 = 0.$$

For $\rho = 2$, this takes the form

$$4\,w(2)^2 = |\eta|^2 + 4\,.$$

Substitution now yields the equation

$$\rho^2 \cdot w(\rho)^4 - 2\,w(\rho)^2 \cdot \{2\,w(2)^2 + \rho(\rho - 2)\} + (\rho - 2)^2 = 0\,.$$

By Corollary 5, $\rho \cdot w(\rho)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, and therefore the last two equations imply that

$$\rho \cdot w(\rho) = w(2) + \sqrt{w(2)^2 + \rho(\rho - 2)}\,.$$

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