THE INTEGRAL MEANS OF CLOSE-TO-CONVEX FUNCTIONS

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Let C denote the class of close-to-convex functions f analytic in the unit disc $\Delta = \{z: |z| < 1\}$ and normalized so that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Let $k(z) = z/(1 - z)^2$ denote the Koebe function. In this paper we prove the following result.

THEOREM. *If* $f \in C$, $0 \leq r < 1$, *and* p *is a real number* $(p \geq 1)$, *then*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |k(re^{i\theta})|^p \, d\theta .$$

Recently, T. H. MacGregor [2] established this result for the case where p is a positive integer.

Both MacGregor's and our proof rely on the characterization of the extreme points of the class C viewed as a compact family in the space of all analytic functions in Δ with the topology of uniform convergence on compact subsets of Δ (see [1]). The restriction $p \geq 1$ (rather than $p > 0$) is essential in our technique. In the case $p \geq 1$, the p-norm $\|f\|_p = \left[(2\pi)^{-1} \int |f(re^{i\theta})|^p d\theta \right]^{1/p}$ is subadditive, so that the maximum value is attained over the extreme points.

The proof of the theorem involves some problems in the calculus.

LEMMA 1. *Let*

$$0 \leq x \leq 1, \;\; 1/2 \leq p \leq 1, \;\; F(x, p) = [1 - (2p - p^2)x^2]^{1/2} - px - (1 - x)^p .$$

Then $F(x, p) \leq 0$.

Proof. $F(x, p) \leq 0$, provided $1 - 2px^2 \leq 2px(1 - x)^p + (1 - x)^{2p}$. Since $(1 - x)^p \geq 1 - x$ for $p \leq 1$, and since $(1 - x)^{2p} \geq 1 - 2px$ for $p \geq 1/2$, the lemma follows.

LEMMA 2. *Let* $f(z) = 1 + \sum_{n=1}^{\infty} \gamma_n z^n = (1 - z)^p$ $(1/2 \leq p \leq 1)$, *and let* $f_N(z) = 1 + \sum_{n=1}^{N} \gamma_n z^n$ *be the* Nth *partial sum of the power series. Then* $|f_N(z)| \leq 1$ *if* $|z - 1/2| \leq 1/2$.

Proof. Because $\gamma_1 = -p$ and $\gamma_n < 0$ $(n = 2, 3, \cdots)$,

$$\left| 1 + \sum_{n=1}^{N} \gamma_n z^n \right| \leq |1 - pz| + \sum_{n=2}^{N} |\gamma_n| \, |z|^n \leq |1 - pz| + 1 - p|z| - (1 - |z|)^p .$$

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We note that the circle $|z - 1/2| = 1/2$ has the parametric representation

$$z = e^{i\alpha}\cos\alpha \qquad \left(-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}\right),$$

and that with the notation $|z| = r = \cos\alpha$ we can therefore write

$$|1 - pz| + 1 - p|z| - (1 - |z|)^p = [1 - (2p - p^2)r^2]^{1/2} + 1 - pr - (1 - r)^p.$$

Thus it suffices to show that

$$[1 - (2p - p^2)r^2]^{1/2} + 1 - pr - (1 - r)^p \leq 1$$

for $0 \leq r \leq 1$ and $1/2 \leq p \leq 1$. The inequality follows from Lemma 1.

The critical step in the proof of the theorem is the following lemma.

LEMMA 3. *Let*

$$h_a(z) = \left[\frac{1 - az}{(1 - z)^2}\right]^q = \sum_{n=0}^{\infty} \sigma_n(a) z^n \qquad \left(q \geq \frac{1}{2},\ \left|a - \frac{1}{2}\right| = \frac{1}{2}\right).$$

Then $|\sigma_n(a)| \leq \sigma_n(0)$; *that is, each coefficient is maximized when* $a = 0$.

Proof. Write $q = N/2 + \varepsilon$, where N is a positive integer and $0 \leq \varepsilon \leq 1/2$. Let $S = q/N = 1/2 + \varepsilon/N$, so that $1/2 \leq S \leq 1$. Then

$$h_a(z) = \left[\frac{(1 - az)^S}{1 - z}\right]^N \frac{1}{(1 - z)^{2\varepsilon}}.$$

Note that $(1 - az)^S/(1 - z) = 1 + \sum_{n=1}^{\infty} b_n(a) z^n$, where $b_n(a) = 1 + \sum_{k=1}^{N} \gamma_k a^k$ and γ_k is as given in Lemma 2 with $p = S$. Hence $|b_n(a)| \leq 1$; that is, $|b_n(a)| \leq b_n(0)$ for $n = 1, 2, \cdots$. The same inequality clearly holds for $[(1 - az)^S/(1 - z)]^N$, and since all coefficients of $(1 - z)^{-2\varepsilon}$ are nonnegative, the inequality $|\sigma_n(a)| \leq \sigma_n(0)$ holds.

Proof of the theorem. In [1], the extreme points of C are given, up to an appropriate rotation, in the form $f_a(z) = (z - az^2)/(1 - z)^2$ with $|a - 1/2| = 1/2$. We therefore need only show that

$$\frac{1}{2\pi}\int |f_a(re^{i\theta})|^p\, d\theta \leq \frac{1}{2\pi}\int |k(re^{i\theta})|^p\, d\theta \qquad (p \geq 1).$$

Let $p = 2q$ $(q \geq 1/2)$ and $h_a(z) = \dfrac{(1 - az)^q}{(1 - z)^{2q}} = \sum_{n=0}^{\infty} \sigma_n(a) z^n$. We want to show that

$$\frac{1}{2\pi}\int |h_a(re^{i\theta})|^2\, d\theta \leq \frac{1}{2\pi}\int |h_0(re^{i\theta})|^2\, d\theta,$$

in other words, that

$$\sum_{n=0}^{\infty} |\sigma_n(a)|^2 r^{2n} \leq \sum_{n=0}^{\infty} |\sigma_n(0)|^2 r^{2n} = \sum_{n=0}^{\infty} (\sigma_n(0))^2 r^{2n}.$$

But this follows immediately from Lemma 3, and the theorem is proved.

Since the function $(1 - az)/(1 - z)^2$ is not subordinate to $1/(1 - z)^2$ for every admissible a, the inequality

$$\frac{1}{2\pi} \int |f_a(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi} \int |f_0(re^{i\theta})|^p d\theta \qquad (0 \leq r < 1,\ p \geq 1)$$

cannot be regarded as a special case of subordination theory.

REFERENCES

1. L. Brickman, T. H. MacGregor, and D. R. Wilken, *Convex hulls of some classical families of univalent functions.* Trans. Amer. Math. Soc. 156 (1971), 91-107.

2. T. H. MacGregor, *Applications of extreme-point theory to univalent functions.* Michigan Math. J. 19 (1972), 361-376.

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