A UNIFORM-BOUNDEDNESS THEOREM FOR MEASURES

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The following theorem is due to J. Dieudonné [1].

THEOREM 1. *Let* X *be a compact Hausdorff space, and let* $\mathscr{U}$ *be a collection of regular Borel measures on* X *such that for each open subset* U *of* X, $\sup\{|\mu(U)|: \mu \in \mathscr{U}\}$ *is finite. Then* $\sup\{|\mu|(X): \mu \in \mathscr{U}\}$ *is finite, where* $|\mu|$ *is the total variation of* μ.

Recently, B. B. Wells [3] has strengthened this theorem by showing that it is sufficient that $\sup\{|\mu(V)|: \mu \in \mathscr{U}\}$ is finite for regular open sets V (V is regular if $V = \mathrm{Int}(\overline{V})$). The purpose of this paper is to prove Dieudonné's theorem for the case where X is a regular (T_3) topological space and $\mathscr{U}$ is a collection of Borel measures on X with the following property: if E is a Borel subset of X, $\varepsilon > 0$, and $\mu \in \mathscr{U}$, there exists a compact subset $K \subseteq E$ with $|\mu(E \sim K)| < \varepsilon$. We shall call such measures weakly regular; clearly, regular measures (a complex measure μ is regular if $|\mu|$ is regular) are weakly regular.

LEMMA 1. *Let* U *be an open subset of* X, *and let* μ *be a nonzero weakly regular measure on* X. *Then there exists an open set* V *with* $V \subseteq U$ *and*

$$|\mu(V)| > |\mu|(U)/7 .$$

Proof. First we show the following: corresponding to each Borel set $E \subseteq U$ and each $\varepsilon > 0$, there is an open set V such that $E \subseteq V \subseteq U$ and $|\mu(V \sim E)| < \varepsilon$. Since $U \sim E$ is a Borel set, there exists a compact set $K \subseteq U \sim E$ with

$$|\mu((U \sim E) \sim K)| < \varepsilon .$$

But $V = U \sim K \supseteq E$, and

$$|\mu(V \sim E)| = |\mu((U \sim K) \sim E)| = |\mu((U \sim E) \sim K)| < \varepsilon .$$

Now choose a partition $E_1, \cdots, E_n$ of U such that

$$\sum_{k=1}^{n} |\mu(E_k)| > \frac{6}{7} |\mu|(U) .$$

By [2, p. 119], there exists a subset $\{j_1, \cdots, j_p\}$ of $\{1, \cdots, n\}$ such that $|\sum_{i=1}^{p} \mu(E_{j_i})| \geq \frac{1}{6} \sum_{k=1}^{n} |\mu(E_k)|$. Let $F = \bigcup_{i=1}^{p} E_{j_i}$; then

$$|\mu(F)| = \left|\sum_{i=1}^{p} \mu(E_{j_i})\right| \geq \frac{1}{6} \sum_{k=1}^{n} |\mu(E_k)| > \frac{1}{7} |\mu|(U) .$$

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Now let $\varepsilon = [|\mu(F)| - |\mu|(U)/7]/2$ and choose V so that $F \subseteq V \subseteq U$ and $|\mu(V \sim F)| < \varepsilon$; then $|\mu(V)| > |\mu|(U)/7$. ■

LEMMA 2. *Let $\mathcal{U}$ be a collection of Borel measures on a topological space* X. *Suppose that corresponding to each finite set $\{\mu_1, \cdots, \mu_n\}$ of measures in $\mathcal{U}$, each positive number* M, *and each collection of disjoint open sets $U_1, \cdots, U_n$ such that $\bigcup_{k=1}^n U_k \neq X$, there exist an open set U_{n+1} and a measure $\mu_{n+1} \in \mathcal{U}$ such that*

$$\bigcup_{k=1}^{n+1} U_k \neq X, \quad U_{n+1} \cap \left(\bigcup_{k=1}^{n} U_k\right) = \emptyset, \quad |\mu_{n+1}(U_{n+1})| > M,$$

and

$$|\mu_k(U_{n+1})| < 1/2^{n+1}$$

for $1 \leq k \leq n$. *Then there exists an open set* $U \subseteq X$ *with* $\sup\{|\mu(U)|: \mu \in \mathcal{U}\} = \infty$.

Proof. Assume that the number $S(V) = \sup\{|\mu(V)|: \mu \in \mathcal{U}\}$ is finite for each open set V. Choose $\mu_1 \in \mathcal{U}$, and let U_1 be an open set such that $U_1 \neq X$ and $|\mu_1(U_1)| > 2$. Having chosen $\mu_1, \cdots, \mu_n \in \mathcal{U}$ and a collection $\{U_1, \cdots, U_n\}$ of disjoint open sets with $X \neq \bigcup_{k=1}^n U_k$, choose $\mu_{n+1} \in \mathcal{U}$ and an open set U_{n+1} such that

$$U_{n+1} \cap \left(\bigcup_{k=1}^{n} U_k\right) = \emptyset, \quad \bigcup_{k=1}^{n+1} U_k \neq X, \quad |\mu_{n+1}(U_{n+1})| > \sum_{k=1}^{n} S(U_k) + n + 2,$$

and

$$|\mu_k(U_{n+1})| < 1/2^{n+1}$$

for $1 \leq k \leq n$. The set $U = \bigcup_{n=1}^\infty U_n$ is open, and since the sets U_n are disjoint, $\mu_n(U) = \sum_{k=1}^\infty \mu_n(U_k)$; consequently,

$$\begin{aligned} |\mu_n(U_n)| &\leq |\mu_n(U)| + \sum_{k=1}^{n-1} |\mu_n(U_k)| + \sum_{k=n+1}^{\infty} |\mu_n(U_k)| \\ &< |\mu_n(U)| + \sum_{k=1}^{n-1} S(U_k) + \sum_{k=n+1}^{\infty} 1/2^k \\ &< |\mu_n(U)| + \sum_{k=1}^{n-1} S(U_k) + 1. \end{aligned}$$

Therefore, $\sum_{k=1}^{n-1} S(U_k) + n + 1 < |\mu_n(U_n)| < |\mu_n(U)| + \sum_{k=1}^{n-1} S(U_k) + 1$, and hence $|\mu_n(U)| > n$. ■

LEMMA 3. *Let* X *be a regular* (T_3) *topological space, and let $\mathcal{U}$ be a collection of weakly regular Borel measures on* X *such that* $\sup\{|\mu(U)|: \mu \in \mathcal{U}\}$ *is finite for every open subset* U *of* X. *Then each* $x \in X$ *lies in an open neighborhood* U_x *such that* $\sup\{|\mu|(U_x): \mu \in \mathcal{U}\}$ *is finite.*

Proof. Assume that this is false, and let x be a point such that

$$\sup\{|\mu|(U): \mu \in \mathcal{U}\} = \infty$$

for every open set U containing x. First we show that

$$\sup\{|\mu|(U \setminus \{x\}): \mu \in \mathcal{U}\} = \infty .$$

To see this, fix an open set V containing x; then $V \setminus \{x\}$ is also open, and

$$\{x\} = V \setminus (V \setminus \{x\}) \quad \Rightarrow \quad \mu(\{x\}) = \mu(V) - \mu(V \setminus \{x\}),$$

and hence

$$\sup\{|\mu(\{x\})|: \mu \in \mathcal{U}\} \leq \sup\{|\mu(V)|: \mu \in \mathcal{U}\} + \sup\{|\mu(V \setminus \{x\})|: \mu \in \mathcal{U}\} < \infty .$$

Let $S(x) = \sup\{|\mu(\{x\})|: \mu \in \mathcal{U}\}$. Then $|\mu|(\{x\}) = |\mu(\{x\})| \leq S(x)$, and if $\mu \in \mathcal{U}$ and U is an open neighborhood of x, we have the relations

$$|\mu|(U \setminus \{x\}) = |\mu|(U) - |\mu|(\{x\}) \geq |\mu|(U) - S(x) .$$

The last member can be made arbitrarily large.

We recall the following property of a regular space: if K is compact, C is closed, and $K \cap C = \emptyset$, then we can find an open set U such that $U \supseteq K$ and $\overline{U} \cap C = \emptyset$.

Choose a neighborhood U_1 of x, and let $M > 0$. Pick $\mu_1 \in \mathcal{U}$ so that $|\mu_1|(U_1 \setminus \{x\}) > 7M$. By Lemma 1, we can find an open set $W_1 \subseteq U_1 \setminus \{x\}$ with $|\mu_1(W_1)| > M$. Since μ_1 is weakly regular, there exists a compact set $K_1 \subseteq W_1$ such that $|\mu_1(K_1)| > M$. Choose an open set O_1 with $K_1 \subseteq O_1$ and $\overline{O}_1 \cap W_1^c = \emptyset$, and apply the first part of the proof of Lemma 1 to obtain an open set V_1 such that $|\mu_1(V_1)| > M$ and $K_1 \subseteq V_1 \subseteq O_1$.

Now assume that we have found open sets $V_1, \cdots, V_n$ such that the closures $\overline{V}_n$ are disjoint and the set $\bigcup_{k=1}^n \overline{V}_k$ contains neither $\{x\}$ nor $X \setminus \{x\}$. Suppose that $\mu_1, \cdots, \mu_n \in \mathcal{U}$ and that $M > 0$. We shall show the existence of an open set V_{n+1} and a measure $\mu_{n+1} \in \mathcal{U}$ such that $\overline{V}_{n+1} \cap \left(\bigcup_{k=1}^n \overline{V}_k\right) = \emptyset$, the set $\bigcup_{k=1}^{n+1} \overline{V}_k$ contains neither $\{x\}$ nor $X \setminus \{x\}$, $|\mu_{n+1}(V_{n+1})| > M$, and $|\mu_k(V_{n+1})| < 1/2^{n+1}$ for $1 \leq k \leq n$; by Lemma 2, this will complete the proof of Lemma 3.

The set $U = X \sim \left(\bigcup_{k=1}^n \overline{V}_k\right)$ is a neighborhood of x. As above, we can find an open set Q_1 and a measure $\nu_1 \in \mathcal{U}$ such that

$$x \notin \overline{Q}_1, \quad \overline{Q}_1 \cap \left(\bigcup_{k=1}^n \overline{V}_k\right) = \emptyset, \quad \text{and} \quad |\nu_1(Q_1)| > M .$$

Repeat this procedure to find an open set Q_2 and a measure $\nu_2 \in \mathcal{U}$ such that

$$x \notin \overline{Q}_2, \quad \overline{Q}_2 \cap \left[\overline{Q}_1 \cup \left(\bigcup_{k=1}^n \overline{V}_k\right)\right] = \emptyset, \quad \text{and} \quad |\nu_2(Q_2)| > M .$$

Let N be an integer greater than $n + 2^{n+1} \sum_{k=1}^{n} |\mu_k|(X)$, and go through the procedure N times; this leads to a collection of open sets $Q_1, \cdots, Q_N$ and measures $\nu_1, \cdots, \nu_N \in \mathcal{U}$ such that

1) $x \notin \overline{Q}_k$ for $1 \leq k \leq N$,

2) $|\nu_k(Q_k)| > M$,

3) $\overline{Q}_k \cap \left[\left(\bigcup_{j=1}^{k-1} \overline{Q}_j \right) \cup \left(\bigcup_{i=1}^{n} \overline{V}_i \right) \right] = \emptyset$ for $2 \leq k \leq N$.

For $1 \leq k \leq n$, the inequality $|\mu_k(Q_j)| \geq 2^{-n-1}$ cannot hold for more than $2^{n+1} |\mu_k(X)|$ of the indices $j = 1, 2, \cdots, N$. Otherwise, the μ_k-variation for some partition of X would be greater than $|\mu_k|(X)$. Since N has been chosen sufficiently large, at least one of the $Q_1, \cdots, Q_N$, which we denote by Q_p, satisfies the condition $|\mu_k(Q_p)| < 2^{-n-1}$ for $1 \leq k \leq n$. We complete the induction by choosing $V_{n+1} = Q_p$ and $\mu_{n+1} = \nu_p$. By Lemma 2, $\sup\{|\mu(X \setminus \{x\})| : \mu \in \mathcal{U}\} = \infty$, and this contradicts our hypothesis. ■

THEOREM 2. *Suppose that* $\mathcal{U}$ *is a collection of weakly regular Borel measures on a* T_3*-topological space* X *and that* $\sup\{|\mu(U)| : \mu \in \mathcal{U}\} < \infty$ *for each open* $U \subseteq X$. *Then* $\sup\{|\mu|(X) : \mu \in \mathcal{U}\} < \infty$.

Proof. By Lemma 3, each $x \in X$ lies in a neighborhood U_x of x such that $\sup\{|\mu|(U_x) : \mu \in \mathcal{U}\} = M_x$ is finite. Suppose that V is an open set such that $\sup\{|\mu|(V) : \mu \in \mathcal{U}\} = \infty$, and let U be an open set such that $\overline{U} \subseteq \bigcup_{k=1}^{n} U_{x_k}$ for some $x_1, \cdots, x_n \in X$. To see that $\sup\{|\mu|(V \sim \overline{U}) : \mu \in \mathcal{U}\} = \infty$, suppose this supremum were M. Then, for each $\mu \in \mathcal{U}$, we would have the contradictory relations

$$|\mu|(V) \leq |\mu|(V \sim \overline{U}) + |\mu|(\overline{U}) \leq M + |\mu|\left(\bigcup_{k=1}^{n} U_{x_k}\right) \leq M + \sum_{k=1}^{n} M_{x_k}.$$

Assume that $\sup\{|\mu|(X) : \mu \in \mathcal{U}\} = \infty$; we go through a construction similar to the one in the proof of Lemma 3. For each positive M, we can find a compact subset K_1 and a measure $\mu_1 \in \mathcal{U}$ such that $|\mu_1(K_1)| > M$. Since K_1 is compact, there exists a finite set $\{x_1, \cdots, x_n\}$ such that $K_1 \subseteq \bigcup_{k=1}^{n} U_{x_k}$. Choose an open set V_1 with $K_1 \subseteq V_1 \subseteq \overline{V}_1 \subseteq \bigcup_{k=1}^{n} U_{x_k}$. By the first part of Lemma 1, we can choose an open set U_1 such that $K_1 \subseteq U_1 \subseteq V_1$ and $|\mu_1(U_1)| > M$. Now

$$\sup\{|\mu|(X \sim \overline{U}_1) : \mu \in \mathcal{U}\} = \infty,$$

and we can continue the induction.

At the nth step, suppose we have open sets $U_1, \cdots, U_n$ with disjoint closures, that $\bigcup_{k=1}^{n} \overline{U}_k$ is contained in the union of finitely many of the sets U_x $(x \in X)$, that $\mu_1, \cdots, \mu_n \in \mathcal{U}$, and that $|\mu_k(U_n)| < 2^{-n}$ for $1 \leq k \leq n-1$. Note that

$$\sup\left\{|\mu|\left(X \sim \left(\bigcup_{k=1}^{n} \overline{U}_k\right)\right) : \mu \in \mathcal{U}\right\} = \infty.$$

As in Lemma 3, choose a large number N, and then choose compact sets $F_1, \cdots, F_N$, measures $\nu_1, \cdots, \nu_N \in \mathscr{U}$, and open sets $Q_1, \cdots, Q_n$ such that each $\overline{Q}_k$ $(1 \leq k \leq N)$ is contained in a finite union of finitely many U_x, the closures $\overline{Q}_1, \cdots, \overline{Q}_N, \overline{U}_1, \cdots, \overline{U}_N$ are disjoint, and $|\nu_k(F_k)| > M$. Use the first part of Lemma 1 to find open sets $G_k \subseteq Q_k$ $(1 \leq k \leq N)$ such that $|\nu_k(G_k)| > M$; as in Lemma 3, if N is sufficiently large, then for some G_p the inequality $|\mu_k(G_p)| < 1/2^{n+1}$ holds for $1 \leq k \leq n$. We have thus established the necessary induction machinery to apply Lemma 2 and complete the proof. ■

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