PIECEWISE LINEAR EMBEDDINGS OF BOUNDED MANIFOLDS

C. H. Edwards, Jr.

1. INTRODUCTION

Let M^m and Q^q $(q \geq m + 3)$ be *bounded* piecewise linear (PL) manifolds (that is, let M and Q be compact and have nonempty boundaries ∂M and ∂Q), and write $d = 2m - q$. Let $f: M \to Q$ be a proper PL mapping (that is, suppose $f^{-1}(\partial Q) = \partial M$). J. F. P. Hudson has proved (Theorem 8.2 of [3]) that f is homotopic, as a map of pairs $(M, \partial M) \to (Q, \partial Q)$, to a proper PL embedding, provided that $(M, \partial M)$ is d-connected and $(Q, \partial Q)$ is $(d + 1)$-connected. This result is similar to the earlier PL embedding theorem of Irwin [5], who assumes instead that $f \mid \partial M: \partial M \to \partial Q$ is an embedding, M is d-connected, and Q is $(d + 1)$-connected, and proves that f is homotopic to a proper embedding *via* a homotopy that is fixed on ∂M. Thus Hudson's theorem deals with embedding *modulo* the boundary, and Irwin's with embedding *relative* to the boundary. (Section 5 contains a remark on the relation between the two types of embedding problem.)

The purpose of this paper is to prove a generalization of Hudson's theorem. We replace the hypothesis that $(Q, \partial Q)$ is $(d + 1)$-connected with the weaker assumption that $(Q, \partial Q)$ is d-connected and $f_*: \pi_{d+1}(M, \partial M) \to \pi_{d+1}(Q, \partial Q)$ is an epimorphism. This is Theorem 1 in Section 3; however the details of the proof require the hypothesis that $q \geq m + 4$ (rather than $q \geq m + 3$).

2. ENGULFING LEMMAS

The proof of Theorem 1 requires several engulfing lemmas. The first two are elementary and well-known, and we omit their proofs.

First a remark on terminology: Suppose that the polyhedron X collapses to the subpolyhedron Y in the bounded PL manifold M. Zeeman [6, Chapter 7] calls $X \searrow Y$ an *interior collapse* if $X - Y$ is contained in int M. *All* collapses in this paper will be interior collapses.

LEMMA 1 (see [3, Lemma 7.1] and [6, Lemma 37]). *Let* X_0, X, Y *be polyhedra in the bounded* PL *manifold* M *such that* $X_0 \subset X$ *and* $X \cap (Y \cup \partial M) \subset X_0$, *and such that* X *collapses to* X_0. *If* U *is a neighborhood of* X_0 *in* M, *then there exists a* PL *homeomorphism* $h: M \to M$ *such that* $X \subset h(U)$ *and* $h \mid X_0 \cup Y \cup \partial Q$ = identity.

LEMMA 2 [3, Lemma 7.3]. *If* X *and* Y *are subpolyhedra of the polyhedron* Z *and* Z *collapses to* X, *then there exists a polyhedron* $T \subset Z$ *such that*

$$X \cup Y \subset X \cup T, \qquad Z \searrow X \cup T \searrow X,$$

and $\dim T \leq \dim Y + 1$.

Received September 22, 1970.
This research was supported in part by the National Science Foundation under Grant GP-19961.

Michigan Math. J. 19 (1972).

The following formulation of the engulfing theorem of Stallings (see Corollary 7.6 of Hudson's lecture notes [3]) will be useful.

Let $Y \subset X$ *be polyhedra in the bounded* PL *manifold* M^m *with*

$$\dim X = k \leq m - 3,$$

and let U *be an open subset of* M *such that* $Y \cup (X \cap \partial M) \subset U$. *If* (M, U) *is* k-*connected, then there exists a* PL *homeomorphism* $h: M \to M$ *such that* $X \subset h(U)$ *and* $h \mid Y \cup \partial M = \text{identity}$.

The proof of the following lemma is a standard application of the engulfing theorem of Stallings.

LEMMA 3. *Let* M *be a bounded* PL m-*manifold, and let* R *be a compact* (m - 1)-*dimensional* PL *submanifold of* ∂M *such that* (M, R) *is* k-*connected* ($k \leq m - 3$). *Suppose* C *and* X *are polyhedra in* M *such that* C *collapses to* $C \cap \partial M \subset \text{int } R$, $X \cap \partial M \subset C$, *and* $\dim(X - R) \leq k$. *Then there exists a polyhedron* C^* *in* M *that collapses to* $C^* \cap \partial M \subset \text{int } R$, *with* $C \cup X \subset C^*$ *and*

$$\dim (C^* - C) \leq k + 1.$$

Proof. Let N_0 be a regular neighborhood of $C \cup R$ in M, and write

$$R^* = \text{cl}(\partial N_0 \cap \text{int } M) \quad \text{and} \quad M^* = M - \text{int}_M N_0.$$

Since $C \cup R$ collapses to R, the set N_0 is a regular neighborhood of R in M, and it follows that the pairs (M, R) and (M^*, R^*) are PL homeomorphic. Therefore (M^*, R^*) is k-connected (since (M, R) is k-connected).

If N^* is a regular neighborhood of R^* in M^*, and $U^* = \text{int}_{M^*} N^*$, then (M^*, U^*) is k-connected. Therefore Stallings' engulfing theorem gives a PL homeomorphism $h: M^* \to M^*$ such that $X \cap M^* \subset h(U^*)$ and $h \mid \partial M^* = \text{identity}$.

If we extend h to M by the identity on N_0, then $N = N_0 \cup N^*$ is a regular neighborhood of $C \cup R$, and $h: M \to M$ is a PL homeomorphism such that $h(N) \supset X$ and $h \mid C \cup R = \text{identity}$. Hence h(N) collapses to $C \cup R$, and therefore the desired polyhedron C^* is now provided by Lemma 2.

For the special case $C = \emptyset$, Lemma 3 reduces to the following proposition.

COROLLARY 1. *Let* R *be a compact codimension-zero* PL *submanifold of* ∂M *such that* (M, R) *is* k-*connected* ($k \leq m - 3$). *If* X *is a* k-*dimensional polyhedron in* M *with* $X \cap \partial M \subset \text{int } R$, *then there exists a* (k + 1)-*dimensional polyhedron* C^* *in* M *that contains* X *and collapses to* $C \cap \partial M \subset \text{int } R$.

In Zeeman's terminology, the set $C^* - C$ in Lemma 3 is a *feeler* pushed out from C to engulf the polyhedron X. Note that the dimension of this feeler is greater by 1 than the dimension of X. We shall need a special case in which the dimension of the feeler can be lowered by 1 (compare with Zeeman's Corollary to Theorem 20 of [6]).

LEMMA 4. *Let* M, R, *and* C *satisfy the conditions in Lemma* 3. *Let* X *be a polyhedron in* M *that collapses to a polyhedron* X_0 *with*

$$\dim(X_0 - R) \leq k < \dim(X - R),$$

and suppose that

$$X \cap \partial M = X_0 \cap \partial M \subset C \cap \operatorname{int} R \quad \textit{and} \quad X \cap C = X_0 \cap C .$$

Then some polyhedron C^* *in* M *collapses to* $C^* \cap \partial M \subset \operatorname{int} R$, *with* $C \cup X \subset C^*$ *and* $\dim(C^* - C) \leq \dim X$.

Proof. Since $\dim(X_0 - R) \leq k$, Lemma 3 provides a polyhedron C' that collapses to $C' \cap \partial M \subset \operatorname{int} R$, with $C \cup X_0 \subset C'$ and $\dim(C' - C) \leq k + 1 \leq \dim X$. The only problem is that C' and X may intersect in such a way that $C' \cup X$ does not collapse to C'.

But $C \cup X$ collapses to $C \cup X_0 \subset C'$, because $C \cap X = C \cap X_0$. We can therefore use the (interior) elementary expansions from $C \cup X_0$ to $C \cup X$ to push C' to a polyhedron C'' such that $(C \cup X) \cup C''$ collapses to C'', the push being an ambient isotopy of M that is fixed on $C \cup X_0 \cup \partial M$ (this is the proof of Lemma 42 of [6], and it is a slight generalization of the elementary construction for the proof of Lemma 1 above).

If $C^* = (C \cup X) \cup C''$, then C^* collapses to $C^* \cap \partial M \subset \operatorname{int} R$, because C^* collapses to C'' and the pairs $(C'', C'' \cap \partial M)$ and $(C', C' \cap \partial M)$ are PL homeomorphic. Since $C^* - C \subset (X - X_0) \cup C''$, it is clear that $\dim(C^* - C) \leq \dim X$.

LEMMA 5. *Let* $f\colon M^m \to Q^q$ *be a nondegenerate* PL *mapping of bounded* PL *manifolds. Let* R *and* S *be compact, codimension-zero* PL *submanifolds of* ∂M *and* ∂Q, *respectively, such that* (M, R) *is* k-*connected and* (Q, S) *is* $(k+1)$-*connected, where* $k \leq m - 3$ *and* $m \leq q - 3$. *Let* C *and* D *be polyhedra in* M *and* Q *that collapse to* $C \cap \partial M \subset \operatorname{int} R$ *and* $D \cap \partial Q \subset \operatorname{int} S$, *respectively. Finally, let* X *and* Y *be polyhedra in* M, *with* $\dim X = k$ *and* $f^{-1}(D) = C \cup X$.

Then there exist polyhedra C^* *and* D^* *in* M *and* Q *that collapse to* $C^* \cap \partial M \subset \operatorname{int} R$ *and* $D^* \cap \partial Q \subset \operatorname{int} S$, *respectively, and such that*

$$C \cup X \subset C^* = f^{-1}(D^*), \quad \dim(C^* - C) \leq k + 1, \quad \dim(D^* - D) \leq k + 2,$$

and $C^* - (C \cup X)$ *is in general position with respect to* Y.

Proof. This lemma is essentially the same as Lemma 5 of [2], except that collapsible polyhedra are here replaced by polyhedra that collapse to their intersections with the boundary. The proof is basically the inductive process used originally in the proof of Irwin's embedding theorem [5].

Starting with $C_0 = C$, $D_0 = D$, and $X_0 = X$, we define inductively polyhedra C_i, D_i, and X_i such that

$$C \cup X \subset C_i \searrow C_i \cap \partial M \subset \operatorname{int} R, \quad D \cup f(C_i) \subset D_i \searrow D_i \cap \partial Q \subset \operatorname{int} S,$$

$$\dim X_i \leq k - i, \quad \dim(C_i - C) \leq k + 1, \quad \dim(D_i - D) \leq k + 2,$$

$(C_i \cup X_i) - (C \cup X)$ is in general position with respect to Y, and $f^{-1}(D_i) = C_i \cup X_i$ for $i > 0$. The induction stops when $i = k + 1$, for then $X_{k+1} = \emptyset$, so that the polyhedra $C^* = C_{k+1}$ and $D^* = D_{k+1}$ satisfy the desired conditions. We omit the details of this construction, since they are identical with those in Hudson's proof, with the conditions $C_i \searrow 0$ and $D_i \searrow 0$ replaced throughout by

$$C_i \searrow C_i \cap \partial M \subset \operatorname{int} R \quad \text{and} \quad D_i \searrow D_i \cap \partial Q \subset \operatorname{int} S,$$

and with our Lemma 3 playing the role of Hudson's Lemma 4.

Finally, we quote for reference the following elementary result on the collapsing of the image of a product $P \times I$ to the image of a vertical cylinder through the singular set.

LEMMA 6. *Let* P *be a polyhedron,* M *a bounded* PL *manifold, and* $h: P \times I \to M$ *a* PL *map such that* $h^{-1}(\partial M) = P \times \{1\}$. *Let* Q *be a subpolyhedron of* $P \times I$ *that contains the singular set* $S(h)$, *and denote by* J *the union of all vertical line segments in* $P \times I$ *through points of* Q. *Then* $h(P \times I)$ *collapses to* $h(P \times \{1\}) \cup h(J)$.

See, for example, Lemma 38 and Corollary 2 to Lemma 45 of [6].

3. EMBEDDING MODULO THE BOUNDARY

Recall that a mapping $f: X \to Y$ is said to be k-*connected* if the pair (C_f, X) is k-connected, that is, if $\pi_r(C_f, X) = 0$ for $r \leq k$, where C_f denotes the mapping cylinder of the mapping f. We formulate a similar definition of connectivity for maps of pairs.

Let (X; A, B) be a triad; that is, let A and B be subsets of X with $A \cap B \neq \emptyset$. Recall that the Blakers-Massey triad homotopy group $\pi_r(X; A, B)$ is the set of all homotopy classes of maps of triples $(D^r, D_+^{r-1}, D_-^{r-1}) \to (X, A, B)$, where D^r is the standard unit r-ball in E^r, and where D_+^{r-1} and D_-^{r-1} are the upper and lower hemispheres of $S^{r-1} = \partial D^r$.

Now let $f: (X, A) \to (Y, B)$ be a map of pairs, and write $\partial f = f \mid A: A \to B$. We then say that f is k-*connected* if and only if $\pi_r(C_f; C_{\partial f}, X) = 0$ for $r \leq k$.

LEMMA 7. *The map of pairs* $f: (X, A) \to (Y, B)$ *is* k-*connected if and only if the induced homomorphism* $f_*: \pi_r(X, A) \to \pi_r(Y, B)$ *is an isomorphism for* $r < k$ *and an epimorphism for* $r = k$.

Proof. Noting that $C_{\partial f} \cap X = A$ and that the pair $(C_f, C_{\partial f})$ deformation retracts to the pair (Y, B), we deduce the lemma immediately from the exact homotopy sequence [1, p. 176]

$$\to \pi_{r+1}(C_f; C_{\partial f}, X) \to \pi_r(X, C_{\partial f} \cap X) \to \pi_r(C_f, C_{\partial f}) \to \pi_r(C_f; C_{\partial f}, X) \to$$

of the triad $(C_f; C_{\partial f}, X)$.

We are now ready to prove that if, in Hudson's theorem on embedding modulo the boundary, the $(d+1)$-connectivity of $(Q, \partial Q)$ is relaxed to d-connectivity $(d = 2m - q)$, then we can still obtain the conclusion on embedding if in addition we suppose that f is $(d+1)$-connected, that is (see Lemma 7), that

$$f_*: \pi_{d+1}(M, \partial M) \to \pi_{d+1}(Q, \partial Q)$$

is an epimorphism. For convenience in applications, we state this result in a relative form.

THEOREM 1. *Let* M^m *and* Q^q *be bounded* PL *manifolds with* $q \geq m + 4$, *and let* $f: M \to Q$ *be a proper mapping. Let* R *and* S *be compact codimension-zero* PL *submanifolds of* ∂M *and* ∂Q, *respectively, such that* $f(R) \subset S$ *and* $f \mid (\partial M - \text{int } R)$ *is a* PL *embedding of* $\partial M - \text{int } R$ *into* $\partial Q - \text{int } S$. *Let* $d = 2m - q$. *If*

(a) *the pair* (Q, S) *is* d-*connected, and*

(b) *the mapping* $f: (M, R) \to (Q, S)$ *is* $(d+1)$-*connected,*

then f *is homotopic, as a map of triples* $(M, \partial M, R) \to (Q, \partial Q, S)$, *to a proper* PL *embedding of* M *into* Q, *with the homotopy being fixed on* $\partial M - \text{int } R$.

Remark. Under the assumption (b), Lemma 7 implies that d-connectivity of (Q, S) is equivalent to d-connectivity of (M, R).

Proof. We may suppose that f is a nondegenerate PL map in *general position.* For our purposes, it will suffice for this to mean that the dimension of the *singular set* $S(f) = \text{cl}\{x \in M: f^{-1}f(x) \neq x\}$ is at most $d = 2m - q$, and that the set of all those points of S(f) that are not nice double points (a point $x \in S(f)$ is a *nice double point* if $f^{-1}f(x) = x \cup x'$, with x and x' having neighborhoods that are embedded by f, with their images being transverse in Q) is a subpolyhedron of S(f) of dimension at most $d - 1$ (see Lemma 31 of [6] and Lemma 23 of [2]).

Let K and L be triangulations of M and Q, respectively, with respect to which f is simplicial, with $S(f) = |K_0|$, where K_0 is a full subcomplex of K. Then f embeds and identifies pairwise the open d-simplexes of K_0.

Step 1. In order to pinpoint the place in Step 3 where codimension 4 is actually needed for the proof, we shall now merely assume that $q \geq m + 3$.

By the remark above, (M, R) is d-connected. If (Q, S) were $(d+1)$-connected, we could proceed to engulf the singular set S(f), that is, to find polyhedra $C \subset M$ and $D \subset Q$ collapsing to $C \cap \partial M \subset \text{int } R$ and $D \cap \partial Q \subset \text{int } S$, respectively, with $S(f) \subset C = f^{-1}(D)$ (this is the method of proof of Theorem 8.2 of [3]). However, since we only assume that (Q, S) is d-connected, we can at first engulf only the $(d-1)$-skeleton X_0 of K_0. We would like to do this in such a way that $C \cap S(f)$ collapses to X_0.

Corollary 1 gives a d-dimensional polyhedron C_0 in M that collapses to $C_0 \cap \partial M \subset \text{int } R$ and $X_0 \subset C_0$. Another application of Corollary 1 gives a $(d+1)$-dimensional polyhedron D_0 in Q that collapses to $D_0 \cap \partial Q \subset \text{int } S$ and $f(C_0) \subset D_0$. We may assume that $C_0 - X_0$ is in general position with respect to $Y = |K_0| = S(f)$, and that $D_0 - f(C_0)$ is in general position with respect to both f(M) and f(Y). Then

$$\dim (C_0 \cap (Y - X_0)) \leq 2d - m \leq d - 3,$$

$$\dim ((f^{-1}(D_0) - C_0) \cap (Y - X_0)) \leq (d+1) + d - q \leq d - 5,$$

and

$$\dim (f^{-1}(D_0) - C_0) \leq (d+1) + m - q \leq d - 2.$$

Therefore Lemma 5 (with $k = d - 2$) gives polyhedra $C_0^* \subset M$ and $D_0^* \subset Q$ that collapse to $C_0^* \cap \partial M \subset \text{int } R$ and $D_0^* \cap \partial Q \subset \text{int } S$, respectively, and such that

$$C_0 \subset f^{-1}(D_0) \subset C_0^* = f^{-1}(D_0^*), \quad \dim (C_0^* - C_0) \leq d - 1, \quad \dim (D_0^* - D_0) \leq d,$$

and $C_0^* - f^{-1}(D_0)$ is in general position with respect to Y.

The first step would be finished if $C_0^* \cap Y$ collapsed to X_0. However, all we know at this stage is that $C_0^* \cap Y = X_0 \cup X_1$, where X_1 is a polyhedron such that

$$\dim X_1 \leq 2d - m \leq d - 3.$$

From this and Lemma 1 it follows easily that some polyhedron $Z_1 \subset Y$ collapses to X_0 and contains $C_0^* \cap Y = X_0 \cup X_1$, where

$$\dim (Z_1 - X_0) \leq (\dim X_1) + 1 \leq d - 2 .$$

By two applications of Lemma 3, there exist polyhedra $C_1 \subset M$ and $D_1 \subset Q$ that collapse to $C_1 \cap \partial M \subset \operatorname{int} R$ and $D_1 \cap \partial Q \subset \operatorname{int} S$, respectively, with

$$C_0^* \cup Z_1 \subset C_1, \quad \dim (C_1 - C_0^*) \leq d - 1, \quad C_1 \subset f^{-1}(D_1), \quad \dim (D_1 - D_0^*) \leq d .$$

We may assume that $C_1 - (C_0^* \cup Z_1)$ is in general position with respect to Y, and that $D_1 - f(C_1)$ is in general position with respect to both $f(M)$ and $f(Y)$, so that

$$\dim (C_1 - (C_0^* \cup Z_1)) \cap Y \leq (d - 1) + d - m \leq d - 4$$

and

$$\dim (f^{-1}(D_1) - C_1) \leq d + m - q \leq d - 3 .$$

Therefore Lemma 5 (with $k = d - 3$) gives polyhedra $C_1^* \subset M$ and $D_1^* \subset Q$ that collapse to $C_1^* \cap \partial M \subset \operatorname{int} R$ and $D_1^* \cap \partial Q \subset \operatorname{int} S$, respectively, and such that

$$C_1 \subset f^{-1}(D_1) \subset C_1^* = f^{-1}(D_1^*), \quad \dim (C_1^* - C_1) \leq d - 2, \quad \dim (D_1^* - D_1) \leq d - 1,$$

and $C_1^* - f^{-1}(D_1)$ is in general position with respect to Y. Therefore

$$\dim (C_1^* - C_1) \cap Y \leq (d - 2) + d - m \leq d - 5 .$$

Therefore $C_1^* \cap Y = X_0 \cup Z_1 \cup X_2$, where $\dim X_2 \leq d - 4$.

Thus, in constructing C_1^* from C_0^*, we have reduced by 1 the dimension of the polyhedron X_{i+1} that prevents $C_i^* \cap Y$ from collapsing to X_0. After a finite number of repetitions of this tedious construction, we finally obtain the desired polyhedra $C^* \subset M$ and $D^* \subset Q$ that collapse to $C^* \cap \partial M \subset \operatorname{int} R$ and $D^* \cap \partial Q \subset \operatorname{int} S$, respectively, with $X_0 \subset C^* = f^{-1}(D^*)$, and such that $C^* \cap S(f)$ collapses to the $(d - 1)$-skeleton X_0 of $S(f)$.

Step 2. We can now greatly simplify the singular set of f by shrinking C^* and D^* into R and S, respectively. Let K^* and L^* be subdivisions of K and L, with respect to which f is simplicial and X_0, $S(f)$, C^*, D^*, R, S are all subcomplexes. Let V and W be the second barycentric derived neighborhoods of $C^* \cup R$ (mod ∂R) and $D^* \cup Q$ (mod ∂Q) in K^* and L^*, respectively. Then, since

$$C^* \searrow C^* \cap \partial M \subset \operatorname{int} R \quad \text{and} \quad D^* \searrow D^* \cap \partial Q \subset \operatorname{int} S ,$$

V and W are relative regular neighborhoods of R (mod ∂R) and S (mod ∂S), respectively, and $f^{-1}(W) = V$.

Next we construct a PL homotopy $G_t \colon M \to M$ $(t \in [0, 1])$ such that

G_0 is the identity and $G_t \mid (\partial M - R)$ is the identity for all $t \in [0, 1]$,

G_1 is a homeomorphism of M onto $\operatorname{cl}(M - V)$, and

$G_t(R) \subset V$ for all $t \in [0, 1]$,

and a PL homotopy $H_t \colon Q \to Q$ such that

H_0 = identity and $H_t \mid \partial Q$ = identity for all $t \in [0, 1]$,

H_1 is a homeomorphism of $cl(Q - W)$ onto Q,

$H_1(W) = S$ and $H_t(W) \subset W$ for all $t \in [0, 1]$.

These homotopies are constructed by stretching and shrinking across the relative product neighborhoods V and W in the obvious manner.

Then the composition $g = H_1 \circ f \circ G_1$ is homotopic to $H_0 \circ f \circ G_0 = f$ by a homotopy of triples $(M, \partial M, R) \to (Q, \partial Q, S)$ that is fixed on $\partial M - R$. The result of this deformation is that the singular set $S(f)$ is very attractive. Because $C^* \cap S(f)$ collapses to the $(d - 1)$-skeleton of $S(f)$, which contains all the points of $S(f)$ that are not nice double points, $S(f) - \mathrm{int}_M V$ is a finite collection of mutually disjoint closed d-balls that are pairwise identified and properly embedded into $Q - \mathrm{int}_Q W$ by f. It follows that $S(g)$ is the union of a collection of mutually disjoint d-balls $A_1^+, A_1^-, \cdots, A_p^+, A_p^-$ properly embedded in M with their boundaries in int R, and that g embeds each of them properly in Q, so that $g(S(g))$ is a collection of d-balls $A_1, \cdots, A_p$ with their boundaries in int S and with $A_i = g(A_i^+) = g(A_i^-)$ $(i = 1, \cdots, p)$.

If we were working in the metastable range of dimensions $q > 3(m + 1)/2$, the remainder of the proof would be quite simple. We could find mutually disjoint $(d + 1)$-balls B_1^+ and B_1^- in M, intersecting neither A_i^+ nor A_i^- for $i > 1$, such that

$$A_1^{\pm} \subset \partial B_1^{\pm} \quad \text{and} \quad \partial B_1^{\pm} - A_1^{\pm} \subset \mathrm{int}\, R\,.$$

Moreover we could use hypothesis (b) to choose B_1^+ and B_1^- so that the proper $(d + 1)$-ball $B_1 = g(B_1^+) \cup g(B_1^-) \subset Q$ represents the trivial element of $\pi_{d+1}(Q, S)$. We could then embed a $(d + 2)$-ball D in Q such that $B_1 \subset \partial D$, $\partial D - B_1 \subset \mathrm{int}\, S$, and $g^{-1}(D) = C = B_1^+ \cup B_1^-$. Since C and D would then obviously collapse to their intersections with R and S, respectively, we could eliminate the two components A_1^+ and A_1^- by shrinking C and D into R and S, as above. A finite number of such steps would complete the proof.

Step 3. We attempt to carry through the program indicated above, keeping track of the dimensions of the singularities resulting from the fact that we are only assuming $q \geq m + 4$.

Let I^{d+1} be the cube $[-1, 1]^{d+1} \subset E^{d+1}$, and I_0^d the spanning d-cube consisting of those points of I^{d+1} whose last coordinate x_{d+1} is 0. Denote by I_+^{d+1}, I_-^{d+1}, I_*^{d+1}, I_*^d the sub-balls of I^{d+1} determined by the conditions

$$x_{d+1} \geq 0, \qquad x_{d+1} \leq 0, \qquad x_{d+1} \leq -\frac{1}{2}, \qquad x_{d+1} = -\frac{1}{2},$$

respectively. Finally, write $I_{\pm}^d = I_{\pm}^{d+1} \cap \partial I^{d+1}$, so that $\partial I^{d+1} = I_+^d \cup I_-^d$.

Since $\pi_d(M, R) = 0$, each of the d-balls $A_i^{\pm}$ can be deformed (relative to its boundary) into R. It follows that for each $i = 1, \cdots, p$, there exist mappings $\phi_i^{\pm}\colon I_{\pm}^{d+1} \to M$ such that

$$\phi_i^+(I_0^d) = A_i^+ \quad \text{and} \quad \phi_i^-(I_0^d) = A_i^-,$$

$$\phi_i^+(I_+^d) \cup \phi_i^-(I_-^d) \subset \mathrm{int}\, R\,,$$

and

$$\phi_i^-(I_*^{d+1}) = p_i \in R.$$

Since g maps each of the balls A_i^+ and A_i^- homeomorphically onto A_i, we may assume that $g\phi_i^+ \mid I_0^d = g\phi_i^- \mid I_0^d$; therefore

$$\psi_i = g \circ (\phi_i^+ \cup \phi_i^-): I^{d+1} \to Q$$

is a (well-defined) mapping with $\psi_i(\partial I^{d+1}) \subset S$. Let $\alpha_i = [\psi_i] \in \pi_{d+1}(Q, S)$.

Since g is homotopic (as a map of pairs) to f, and f: $(M, R) \to (Q, S)$ is $(d+1)$-connected, it follows from Lemma 7 that $g_*: \pi_{d+1}(M, R) \to \pi_{d+1}(Q, S)$ is epimorphic. Therefore, for $i = 1, \cdots, p$, there exists a mapping $\phi_i^*: (I_*^{d+1}, \partial I_*^{d+1}) \to (M, R)$ such that $\phi_i^*(I_*^d)$ is the basepoint p_i and the composition

$$g \circ \phi_i^*: (I_*^{d+1}, \partial I_*^{d+1}) \to (Q, S)$$

represents the element $-\alpha_i$ in $\pi_{d+1}(Q, S)$. If we redefine ϕ_i^- (only) on I_*^{d+1}, so that it equals ϕ_i^* there, then the altered mapping $\psi_i = g \circ (\phi_i^+ \cup \phi_i^-)$ represents the *trivial* element of $\pi_{d+1}(Q, S)$.

If the mappings $\phi_i^{\pm}$ were PL embeddings with mutually disjoint images, then the polyhedron

$$X = \bigcup_{i=1}^{p} \phi_i^{\pm}(I_{\pm}^{d+1})$$

would contain S(g) and collapse to its intersection with R. The set of singularities that prevents such a collapse is

$$S_\phi = \bigcup_{i=1}^{p} S(\phi_i^{\pm}) \cup \bigcup_{i=1}^{p} (\phi_i^{\pm})^{-1}\left(\bigcup_{j\neq i} \phi_j^{\pm}(I_{\pm}^{d+1}) \cup \bigcup_{j=1}^{p} \phi^{\mp}(I_{\mp}^{d+1})\right).$$

Assuming that the $\phi_i^{\pm}$ are PL general-position maps with $(\phi_i^{\pm})^{-1}(\partial M) = I_{\pm}^d$, we conclude that the set S_ϕ is a polyhedron with

$$\dim S_\phi \leq 2(d+1) - m = d + 2 + (d - m) \leq d - 2,$$

because $d - m = m - q \leq -4$ (this is our only essential use of the codimension-4 hypothesis).

Denote by J_ϕ the union of all "vertical" line segments in I^{d+1} through points of S_ϕ; then $\dim J_\phi \leq d - 1$. If

$$X_0 = \bigcup_{i=1}^{p} \phi_i^{\pm}(I_{\pm}^d \cup J_\phi),$$

then X collapses to X_0, by Lemma 6, and

$$X \cap \partial M = X_0 \cap \partial M = \bigcup_{i=1}^{p} \phi_i^{\pm}(I_{\pm}^d) \subset \operatorname{int} R.$$

An application of Lemma 4 (with $k = d - 1$ and $C = \emptyset$) then yields a polyhedron $C \subset M$ that contains X and collapses to $C \cap \partial M \subset \text{int } R$, with $\dim C = d + 1$. In fact, the proof of Lemma 4 shows that $C = X \cup C'$, where $\dim C' \leq d$ and $\dim (C' \cap X) \leq d - 1$.

Next we want to construct a polyhedron $D \subset Q$ that contains $f(C)$ and collapses to $D \cap \partial Q \subset \text{int } S$. Since the mapping $\psi_i: (I^{d+1}, \partial I^{d+1}) \to (Q, S)$ is nullhomotopic as a map of pairs, there exists for each $i = 1, \cdots, p$ a mapping

$$\Psi_i: (I^{d+2} = I^{d+1} \times [0, 1]) \to Q$$

such that

$$\Psi_i(x, 0) = \psi_i(x) \quad \text{and} \quad \Psi_i(x, 1) \in \text{int } S \qquad \text{for all } x \in I^{d+1},$$

$$\Psi_i(x, t) \in \text{int } S \qquad \text{for all } x \in \partial I^{d+1} \quad \text{and} \quad t \in [0, 1],$$

and $\Psi_i^{-1}(\partial Q) = I^{d+1} \times \{1\} \cup \partial I^{d+1} \times [0, 1]$. If the mappings Ψ_i were PL embeddings with mutually disjoint images, then the polyhedron $\bigcup_{i=1}^{p} \Psi_i(I^{d+2})$ would contain $g(X)$ (but not $g(C')$), and it would collapse to its intersection with S. Let

$$S_\Psi = \bigcup_{i=1}^{p} [\Psi_i^{-1} g(S_\phi) \cup S(\Psi_i) \cup \Psi_i^{-1}(g(C') \cap \Psi_i(I^{d+2})) \cup \bigcup_{j \neq i} \Psi_i^{-1}(\Psi_i(I^{d+2}) \cap \Psi_j(I^{d+2}))].$$

We can easily compute that if the singular balls $\Psi_i(I^{d+2})$ are moved into PL general position with respect to each other and $g(C')$, but without moving the faces $\psi_i(I^{d+1})$, then

$$\dim S_\Psi \leq d - 1$$

(actually, $\dim S(\Psi_i) \leq 2(d + 2) - q \leq d - 4$; the larger dimension $d - 1$ comes from $g(X \cap C')$). Therefore, if J_Ψ is the union of all vertical line segments in I^{d+2} through points of S_Ψ, then $\dim J_\Psi \leq d$. Now, if

$$Y = g(C') \cup \bigcup_{i=1}^{p} (I^{d+2}) \qquad \text{and} \qquad Y_0 = g(C') \cup \bigcup_{i=1}^{p} (\Psi_i(J_\Psi) \cup (\partial Q \cap \Psi_i(I^{d+2}))),$$

then Y collapses to Y_0 (by Lemma 6), $Y_0 \cap \partial Q = Y \cap \partial Q \subset \text{int } S$, and

$$\dim (Y_0 - \partial Q) \leq d < \dim Y = d + 2.$$

Consequently, an application of Lemma 4 (with $k = d$ and $C = \emptyset$) gives a polyhedron $D \subset Q$ that collapses to $D \cap \partial Q \subset \text{int } S$, with $g(C) \subset Y \subset D$ and $\dim D = \dim Y = d + 2$. This uses the d-connectedness of (Q, S) and hence requires that we know that $\dim (Y_0 - \partial Q) = \dim C' \leq d$ (note that we have finally applied codimension 4).

Assuming that D is in general position with respect to $g(M)$ modulo $g(C)$, we find that

$$\dim (g^{-1}(D) - C) \leq (d + 2) + m - q \leq d - 2.$$

Hence an application of Lemma 5 (with $k = d - 2$) provides polyhedra C^* and D^* in M and Q that collapse to $C^* \cap \partial M \subset \text{int } R$ and $D^* \cap \partial Q \subset \text{int } S$, respectively, and such that $S(g) \subset X \subset C \subset C^* = f^{-1}(D^*)$.

Finally, by shrinking C^* and D^* into R and S by the construction of Step 2, we obtain the desired proper PL embedding of M into Q.

Setting $R = \partial M$ and $S = \partial Q$ in Theorem 1, we obtain the following special case.

COROLLARY 2. *Let* M^m *and* Q^q *be bounded* PL *manifolds* $(q \geq m + 4)$, *and let* $f: (M, \partial M) \to (Q, \partial Q)$ *be a* PL *map. If* $(M, \partial M)$ *is* d-*connected* $(d = 2m - q)$ *and* f *is* $(d + 1)$-*connected, then* f *is homotopic (as a map of pairs) to a proper embedding.*

4. THE CORRESPONDING UNKNOTTING THEOREM

Theorem 1 is stated and proved in a relative form that enjoys the familiar virtue of combining with Hudson's "concordance-implies-isotopy" theorem [4] to produce immediately an unknotting theorem.

THEOREM 2. *Let* M^n *and* Q^q *be bounded* PL *manifolds with* $q \geq m + 4$, *and let* R *and* S *be compact, codimension*-0 PL *submanifolds of* ∂M *and* ∂Q, *respectively. Let* f *and* g *be two proper* PL *embeddings of* M *into* Q *that are homotopic as maps of pairs* $(M, R) \to (Q, S)$, *via a homotopy that keeps the image of* $\partial M - \operatorname{int} R$ *fixed in* $\partial Q - \operatorname{int} S$. *If* (Q, S) *is* $(d + 1)$-*connected and* $f: (M, R) \to (Q, S)$ *is* $(d + 2)$-*connected, then* f *and* g *are ambient isotopic, as maps of pairs* $(M, \partial M) \to (Q, \partial Q)$, *keeping* $f(\partial M - \operatorname{int} R)$ *fixed.*

Proof. By Corollary 1.4 of [4], it suffices to construct an allowable concordance between f and g that is fixed on $\partial M - \operatorname{int} R$; in other words, a PL embedding $F: M \times I \to Q \times I$ such that

$$F^{-1}(Q \times 0) = M \times 0, \quad F^{-1}(Q \times 1) = M \times 1, \quad F^{-1}(\partial Q \times I) = \partial M \times I, \quad F_0 = f, \quad F_1 = g,$$

and $F(x, t) = (f(x)), t)$ for all $x \in \partial M - \operatorname{int} R$ and $t \in I$.

The assumed homotopy between f and g provides a proper PL mapping $H: M \times I \to Q \times I$ such that

$$H_0 = f, \quad H_1 = g, \quad H(R \times I) \subset S \times I, \quad H((\partial M - \operatorname{int} R) \times I) = (f(\partial M - \operatorname{int} R)) \times I.$$

Let $\overline{M} = M \times I$, $\overline{Q} = Q \times I$, $\overline{R} = R \times I$, $\overline{S} = S \times I$, and $\overline{d} = 2(m + 1) - (q + 1) = d + 1$.

Then $H: \overline{M} \to \overline{Q}$ is a proper PL mapping such that $H(\overline{R}) \subset \overline{S}$ and $H \mid (\partial\overline{M} - \operatorname{int} \overline{R})$ is a PL embedding into $\partial\overline{Q} - \operatorname{int} \overline{S}$. From the commutative diagram

$$\begin{array}{ccc} \pi_i(\overline{M}, \overline{R}) & \xrightarrow{H_*} & \pi_i(\overline{Q}, \overline{S}) \\ \approx \uparrow & & \uparrow \approx \\ \pi_i(M, R) & \xrightarrow[f_*]{} & \pi_i(Q, S) \end{array}$$

in which the vertical isomorphisms are induced by inclusion, we see that $(\overline{Q}, \overline{S})$ is $\overline{d}$-connected because (Q, S) is $(d + 1)$-connected, and (by Lemma 7) that $H: (\overline{M}, \overline{R}) \to (\overline{Q}, \overline{S})$ is $(\overline{d} + 1)$-connected because $f: (M, R) \to (Q, S)$ is $(d + 2)$-connected. Theorem 1 therefore provides the desired allowable concordance between f and g.

With $R = \partial M$ and $S = \partial Q$ we obtain the following result.

COROLLARY 3. *Let* f *and* g *be two homotopic proper embeddings* $(M^m, \partial M) \to (Q^q, \partial Q)$ $(q \geq m + 4)$. *If* $(M, \partial M)$ *is* $(d+1)$*-connected and* f *is* $(d+2)$*-connected, then* f *and* g *are ambient isotopic as maps of pairs.*

5. REMARKS

There is the obvious question whether the assumption $q \geq m + 4$ (rather than $q \geq m + 3$) and the d-connectivity of (M, R) and (Q, S) in Theorem 1 are necessary hypotheses.

Hudson has proved the following generalization of Irwin's theorem on embedding relative to the boundary. Let M^m and Q^q be compact PL manifolds $(q \geq m + 3)$, and let $f: M \to Q$ be a proper PL map such that $f \mid \partial M$ is an embedding. If M is $(t+2)$-connected $(t = 3m - 2q)$ and f is $(d+1)$-connected $(d = 2m - q)$, then f is homotopic (rel ∂M) to a proper PL embedding [2].

The analogous generalization of Theorem 1 would be obtained by replacing the codimension-4 hypothesis by $q \geq m + 3$ and the d-connectivity of (M, R) by $(t+2)$-connectivity (of course, $t + 2 < d$ if $q \geq m + 3$).

Notice that this generalization of Theorem 1 would imply the theorem of Hudson. To see this, suppose M, Q, f are defined as above. Let $A^m \subset \text{int } M$ and $B^q \subset \text{int } Q$ be PL balls such that $f^{-1}(B) = A$, and write

$$\overline{M} = M - \text{int } A, \quad \overline{R} = \partial A, \quad \overline{Q} = Q - \text{int } B, \quad \overline{S} = \partial B .$$

Since $f: M \to Q$ is $(d+1)$-connected, the commutative diagram

$$\begin{array}{ccc} \pi_k(M) & \xrightarrow{f_*} & \pi_k(Q) \\ \approx \uparrow & & \uparrow \approx \\ \pi_k(\overline{M}) & \xrightarrow{\overline{f}_*} & \pi_k(\overline{Q}) \end{array}$$

implies that $f = f \mid \overline{M}: \overline{M} \to \overline{Q}$ is $(d+1)$-connected. Therefore Lemma 7 and the commutative diagram

$$\begin{array}{ccccccc} 0 = \pi_k(\overline{R}) & \longrightarrow & \pi_k(\overline{M}) & \xrightarrow{\approx} & \pi_k(\overline{M}, \overline{R}) & \longrightarrow & \pi_{k-1}(\overline{R}) = 0 \\ & & \overline{f}_* \downarrow & & \downarrow \overline{f}_* & & \\ 0 = \pi_k(\overline{S}) & \longrightarrow & \pi_k(\overline{Q}) & \xrightarrow{\approx} & \pi_k(\overline{Q}, \overline{S}) & \longrightarrow & \pi_{k-1}(\overline{S}) = 0 \end{array}$$

imply that $(\overline{M}, \overline{R})$ is $(t+2)$-connected and that the map of pairs $\overline{f}: (\overline{M}, \overline{R}) \to (\overline{Q}, \overline{S})$ is $(d+1)$-connected.

REFERENCES

1. A. L. Blakers and W. S. Massey, *The homotopy groups of a triad. I.* Ann. of Math. (2) 53 (1951), 161-205.

2. J. F. P. Hudson, *Piecewise linear embeddings.* Ann. of Math. (2) 85 (1967), 1-31.

3. ———, *Piecewise linear topology.* W. A. Benjamin, Inc., New York, 1969.

4. ———, *Concordance, isotopy, and diffeotopy.* Ann. of Math. (2) 91 (1970), 425-448.

5. M. C. Irwin, *Embeddings of polyhedral manifolds.* Ann. of Math. (2) 82 (1965), 1-14.

6. E. C. Zeeman, *Seminar on combinatorial topology.* Mimeographed notes, Institut des Hautes Études Scientifiques, Paris, 1965.

The University of Georgia
Athens, Georgia 30601