A MINKOWSKI AREA HAVING NO CONVEX EXTENSION

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Let $A = E^4 \wedge E^4$, and let M be the set of all simple (decomposable) elements in A. Let $\{\lambda_i\}$ denote a set of nonnegative numbers whose sum is 1. Let h be a real-valued function on M. Then [3, p. 20] h is *weakly convex* on M if

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 h(x_1) + \lambda_2 h(x_2)$$

whenever $x_1, x_2 \in M$ and $\lambda_1 x_1 + \lambda_2 x_2 \in M$. Also, h is *convex* on M if

(a) $$h\left(\sum_{i=1}^{k} \lambda_i x_i\right) \leq \sum_{i=1}^{k} \lambda_i h(x_i)$$

whenever $\{x_1, \cdots, x_k\}$ is a finite set of points in M and $\sum_{i=1}^k \lambda_i x_i \in M$, and

(b) there exists a linear function L on A with $L \leq h$ on M.

Suppose that h is convex on M, and let $q(x) = \inf \sum_{i=1}^k \lambda_i h(x_i)$, where the infimum is taken over all k-tuples $\{x_i\}_1^k$ of points in M and over all $\{\lambda_i\}_1^k$ such that $x = \sum_{i=1}^k \lambda_i x_i$. Then (see [3, p. 21]) q extends h to A and is convex.

Let K be a central convex body in E^4 with its center at the origin. If $R \in M$ and $\mathscr{R}$ is the plane determined by R, let $f(R) = |R|/e(K \cap \mathscr{R})$, where $e(K \cap \mathscr{R})$ is the Euclidean area of $K \cap \mathscr{R}$. It is known that the Minkowski area f is weakly convex [4, p. 62]. We shall show that, for suitable K, f is not convex; this answers a question of Busemann and Petty [2, Problem 10]. This problem was discussed in greater detail in [4] and listed again in [3, p. 33].

Let $r^i = (r^i_1, r^i_2)$ $(i \in I = \{1, 2, 3, 4\})$ be linear functions on E^2 that are linearly independent. Let p_1 and p_3 in E^2 be determined by the equations $r^1(p_1) = r^2(p_1) = 1$ and $r^2(p_3) = r^3(p_3) = 1$. Then it is not difficult to verify that, except possibly for sign, the area of the parallelogram spanned by p_1 and p_3 is

$$\frac{[1\,2] + [2\,3] + [3\,1]}{[1\,2]\,[2\,3]}, \qquad \text{where } [i\,j] = \det \begin{vmatrix} r^i_1 & r^i_2 \\ r^j_1 & r^j_2 \end{vmatrix}.$$

Thus, if P is a symmetric octagon whose consecutive sides, in appropriate order, are

$$-r^4 = 1, \quad r^1 = 1, \quad r^2 = 1, \quad r^3 = 1, \quad r^4 = 1, \quad -r^1 = 1, \quad -r^2 = 1, \quad -r^3 = 1,$$

then area $P = A = A_1 + A_2 + A_3 + A_4$, where

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$$A_1 = \frac{[1\,4]+[1\,2]+[4\,2]}{[1\,4]\,[1\,2]}, \qquad A_2 = \frac{[1\,2]+[2\,3]+[3\,1]}{[1\,2]\,[2\,3]},$$

$$A_3 = \frac{[2\,3]+[3\,4]+[4\,2]}{[2\,3]\,[3\,4]}, \qquad A_4 = \frac{[3\,4]+[1\,4]+[3\,1]}{[3\,4]\,[1\,4]}.$$

Now let $K = \{x \in E^4 |\ |x^i| \le 1,\ i \in I\}$, and with the notation $\alpha = 2^{-1/2}$, let

$$a_1 = (\alpha, 1, \alpha, 0), \quad a_2 = (-\alpha, 0, \alpha, 1), \quad b_1 = (\alpha, 0, -\alpha, 1), \quad b_2 = (\alpha, -1, \alpha, 0).$$

Then

$$a_1 + sb_1 = (\alpha(1+s), 1, \alpha(1-s), s)$$

and

$$a_2 + sb_2 = (-\alpha(1-s), -s, \alpha(1+s), 1)$$

so that

$$\begin{aligned} R(s) &= (a_1 + sb_1) \wedge (a_2 + sb_2) \\ &= \{\alpha(1 - 2s - s^2), 1 + s^2, \alpha(1 + 2s - s^2), \alpha(1 + 2s - s^2), 1 + s^2, \alpha(1 - 2s - s^2)\} \end{aligned}$$

for each real number s. Let $\mathscr{R}(s)$ be the plane determined by $R(s)$, and suppose that $\delta \in (0, 2\alpha - 1)$. If $|s| < \delta$, then $K \cap \mathscr{R}(s)$ is a symmetric octagon, and if $T: \mathscr{R}(s) \to E^2$ is defined by the equation

$$T(u^1(a_1 + sb_1) + u^2(a_2 + sb_2)) = (u^1, u^2),$$

then the linear transformation T takes $K \cap \mathscr{R}(s)$ into an octagon P whose consecutive sides are in the order given earlier. Hence

$$\begin{aligned} \text{area } P = A &= \frac{4\{2\alpha(1 - s^2) - 1 - s^2\}}{\alpha^2(1 + 2s - s^2)(1 - 2s - s^2)} \\ &= \frac{8\{(2\alpha - 1) - (2\alpha + 1)s^2\}}{(1 - s^2)^2 - 4s^2} = \frac{8}{(2\alpha + 1) - (2\alpha - 1)s^2}, \end{aligned}$$

so that $f(R(s)) = e(T\mathscr{R}(s))\,e^{-1}(P) = A^{-1} = [(2\alpha + 1) - (2\alpha - 1)s^2]/8$. Thus the function $\phi = f \circ R$ on $(-\delta, \delta)$ is not convex.

THEOREM. *The Minkowski area* f *defined for the central convex body* K *is not convex.*

Proof. Suppose that f is convex on M. Then there exists a convex function g on all of A that extends f. Thus there exist real-valued C^∞-functions g_n that are convex and converge uniformly to g on compact subsets of A. Let $\psi_n = g_n \circ R$ on $(-\delta, \delta)$. Evidently, the sequence $\{\psi_n\}$ converges uniformly to ϕ. However,

$$\psi_n' = (g_n' \circ R)R' \quad \text{and} \quad \psi_n'' = (g_n'' \circ R)R'(2) \ge 0,$$

since $R'(s) = a_1 \wedge b_2 + b_1 \wedge a_2$ is independent of s, and since g_n is convex. Thus ψ_n, and hence ϕ, is also convex, so that f cannot be convex.

The essential part of the proof that f is weakly convex if K is symmetric appears in [1]. If K is not symmetric, then f need not be weakly convex, as can be seen from the following example. Let

$$M(x, y, z) = \max\{x^+, y^+, z^+\} \quad \text{and} \quad K = \{(x, y, z) \in E^3 \mid M(x, y, z) \leq 1\},$$

where $r^+ = \max\{r, 0\}$ when r is a real number. Let

$$a = (1, 0, -1), \quad c = (1, 0, -3), \quad e = (1, 0, -2),$$

$$b = (0, 1, -1), \quad d = (0, 1, -5), \quad m = (0, 1, -3),$$

so that $e \wedge m = (a \wedge b + e \wedge d)/2$. It is easy to see that $f(a \wedge b) = 2/9$, $f(c \wedge d) = 10/27$, and

$$f(e \wedge m) = \frac{1}{3} > \frac{8}{27} = \frac{1}{2}\left[\frac{2}{9} + \frac{10}{27}\right] = \frac{1}{2}[f(a \wedge b) + f(c \wedge d)].$$

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