LOCALLY COMPACT, TOTALLY DISCONNECTED, SOLVABLE GROUPS

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In this note, we continue our study of the structure of locally compact, totally disconnected groups [1]. Let G be a topological group, and let the inner automorphisms act naturally on G. An element g in G has a relatively compact orbit if the closure of the orbit of g under the actions of inner automorphisms is compact. As we know, the set P(G) of compact elements in a locally compact, totally disconnected, solvable group G need not be a subgroup of G. We shall prove, however, that the set $P(G) \cap B(G)$ is a characteristic subgroup of G, where

$$B(G) = \{g \mid g \text{ has relatively compact orbit}\}.$$

We shall apply this result to obtain information about the structure of a solvable group G that has a finitely-generated, free abelian subgroup Z^n such that G/Z^n is compact.

From now on, G denotes a locally compact, totally disconnected topological group with identity e. Let B(G) denote the subset of G consisting of all points whose orbits are relatively compact under the action of the group of inner automorphisms of G; in other words, $B(G) = \{x \in G \mid \overline{I_G(x)} \text{ is compact}\}$, where $I_G(x) = \{gxg^{-1} \mid g \in G\}$. An element x in G is a compact element if x is contained in a compact subgroup of G. Let

$$P(G) = \{x \mid x \in G, \ x \text{ is a compact element}\}.$$

Let H be a topological group, and let H' denote the closure of the commutator (derived) subgroup of H. A topological group H is solvable if there exists an integer n such that

$$H^0 = H, \qquad H^{i+1} = (H^i)' \quad (0 \leq i \leq n-1), \qquad \text{and } H^n = \{e\}.$$

Remark 1. A locally compact, totally disconnected group contains compact-open subgroups; but, in general, even a nilpotent group need not contain a compact-open, normal subgroup. It is known that G contains compact-open, normal subgroups if and only if B(G) is an open subgroup of G [1].

Remark 2. In general, P(G) does not form a subgroup. The group G generated by two elements a, b satisfying the relation $a^2 = b^2 = e$, together with the discrete topology, provides an example. (The group G can also be obtained as the semidirect product of the integers Z with the automorphism $\theta\colon Z \to Z$, defined by $\theta(n) = -n$.)

PROPOSITION 1. *Suppose* G *is a solvable group. Then* $R(G) = P(G) \cap B(G)$ *is a subgroup of* G.

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Proof. We shall prove the proposition by induction on the length of the solvability chain of G. An abelian group G is also compact. Now assume that if G is generated by a and b, where a, b are elements in $P(G) \cap B(G)$, and if the length of the solvability chain does not exceed n, then G is compact. Let G be a solvable group of length $n+1$ that is generated by $a, b \in R(G) = P(G) \cap B(G)$. If $H = G_{n+1}$, then H is a characteristic subgroup of G, the group H is abelian, and G/H is compact. Define the functions $\theta_a\colon H \to H$, $\theta_b\colon H \to H$ by

$$\theta_a(h) = hah^{-1}a^{-1}, \qquad \theta_b(h) = hbh^{-1}b^{-1} \qquad (h \in H).$$

Since $a, b \in B(G)$, the sets $\overline{\theta_a(H)}$ and $\overline{\theta_b(H)}$ are compact. Because H is an abelian and normal subgroup of G, we have the relations

$$\begin{aligned}\theta_a(h_1 h_2) &= h_1 h_2 a h_2^{-1} h_1^{-1} a^{-1} = h_1(h_2 a h_2^{-1} a^{-1}) a h_1^{-1} a^{-1} \\ &= (h_1 a h_1^{-1} a^{-1})(h_2 a h_2^{-1} a^{-1}) = \theta_a(h_1)\theta_a(h_2).\end{aligned}$$

Hence θ_a is a homomorphism, and similarly θ_b is a homomorphism.

We shall prove that $B(G) = G$. Since G/H is compact, we have that $G = KH$ for some compact subset K of G. Let $g \in G$; then $g = \lim g_n$, where g_n is of the form

$$a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_\ell} b^{\beta_\ell} \qquad (\alpha_i, \beta_i \text{ are integers}).$$

Now

$$\begin{aligned} I_G(g_n) &= \{khg_n h^{-1}k^{-1} \mid k \in K,\ h \in H\} \\ &= \{k(hg_n h^{-1} g_n^{-1}) g_n k^{-1} \mid k \in K,\ h \in H\} \subset K\{hg_n h^{-1} g_n^{-1} \mid h \in H\} g_n K.\end{aligned}$$

It is easy to see that $\{hg_n h^{-1} g_n^{-1} \mid h \in H\} \subset \overline{\theta_a(H)}\,\overline{\theta_b(H)}$. Thus

$$I_G(g_n) \subset K\overline{\theta_a(H)}\,\overline{\theta_b(H)}\, g_n K,$$

and $I_G(g) \subset K\,\overline{\theta_a(H)}\,\overline{\theta_b(H)}\, gK$ $(g \in B(G))$. By a theorem in [1] (see Remark 1 above), G contains a compact-open, normal subgroup F. It follows that G/F is a discrete group generated by torsion elements aF and bF. Moreover, $B(G) = G$ and $B(G/F) = G/F$, and therefore G/F is an FC-group [2]. Hence G/F is finite, and G is compact.

COROLLARY 1. *Suppose* G *is a solvable group with the property that* $\overline{B(G)} = G$. *If there exists a torsion-free, discrete, abelian subgroup* A *of* G *such that* G/A *is compact, then* $\overline{R(G)} = \overline{P(G) \cap B(G)}$ *is a compact-open, normal subgroup of* G.

Proof. Let K be a compact-open subgroup of G. Since $\overline{B(G)} = G$, the set $R(G) \cap K$ is dense in K, the inclusion $\overline{R(G)} \supseteq K$ holds, and $\overline{R(G)}$ is an open subgroup of G. Set $R(G) = P(G) \cap B(G)$; then R(G) is an open subgroup of B(G), and R(G) is also closed (with respect to the relative topology). Hence

$$\overline{R(G)} \cap B(G) = R(G) \cap B(G).$$

Because $A \subset B(G)$, the group A is torsion-free and discrete, which implies that $\overline{R(G)} \cap A = \{e\}$. The relations above, together with the fact that G/A is compact, imply that $\overline{R(G)}$ is compact.

PROPOSITION 2. *Suppose* G *is solvable and contains a finitely-generated* (*say* n *generators*), *torsion-free, abelian subgroup* Z^n *such that* G/Z^n *is compact. Then* $Z^n \subset B(G) \subset \overline{B(G)}$, *the group* $G/\overline{B(G)}$ *is compact, and* $\overline{B(G)}$ *contains a compact-open, normal subgroup* F (*with respect to* $\overline{B(G)}$) *such that* $\overline{B(G)}/F$ *is a* (*discrete*) FC-*group*.

Proof. It is clear that the inclusions $Z^n \subset B(G) \subset \overline{B(G)}$ hold and that $G/\overline{B(G)}$ is compact. Corollary 1 guarantees the existence of G. Hence $\overline{B(G)}/F$ is discrete. Since $\overline{B(G)}/Z^n$ is compact, the group $\overline{B(G)}/Z^n F$ is compact, the group $\overline{B(G)}/F = L$ contains a subgroup isomorphic with Z^n such that L/Z^n is finite, and $\overline{B(G)}/F$ is an FC-group.

Remark 3. Clearly, one can state the theorems above for locally projective, solvable groups.

Remark 4. It follows from the proof of Proposition 1 that we actually have the following result: If $a, b \in P(G) \cap B(G)$ and a, b are elements of solvable normal subgroups of G, then $ab \in P(G) \cap B(G)$.

Definition 1. A locally compact, totally disconnected, topological group G is a *generalized-solvable group of class one* if and only if the subgroup

$$S(G) = \{g \in G \mid g \text{ belongs to a solvable normal subgroup of } G\}$$

is dense in G.

Definition 2. An element $g \in G$ is *solvable-compact* if and only if there exists a closed, solvable, normal subgroup H such that gH is a compact element in G/H. We use Sp(G) to denote the set $\{g \mid g \in G,\ g \text{ is solvable compact}\}$.

We shall use Z^n to denote the free abelian group whose basis consists of n elements.

PROPOSITION 3. *Suppose* G *is a generalized-solvable group of class one, and suppose there exists a closed subgroup* A *isomorphic with* Z^n *such that* G/A *is compact. Then there exist closed, normal subgroups* H, N, M *of* G *with the following properties:*

(1) $H \subset N \subset M$, *and* H *is a closed, solvable subgroup;*

(2) G/M *is finite;*

(3) N/H *is compact, and* M/N *is a discrete* FC-*group; in other words, the group* M/N *has finite conjugate classes.*

Proof. It is clear that $F = Sp(G) \cap A$ is a subgroup of A. Since A is a free, abelian, finitely-generated group, F is also free. Moreover, F is a direct summand in A. Let D be a direct complement of F in A. Let $\{f_1, \cdots, f_r\}$ be a basis of F, and suppose $H_1, \cdots, H_r$ are closed, solvable, normal subgroups of G such that $f_i H_i$ is a compact element in G/H_i $(1 \leq i \leq r)$. Let

$$H = C\ell_G(H_1 H_2 \cdots H_r);$$

then H is a closed, solvable, normal subgroup of G. It follows that DH is closed in G, for otherwise D would contain nontrivial, solvable, compact elements of G. Set $G' = G/H$ and $D' = DH/H$. Since $\overline{FH}/H$ is compact, DH/H is closed. Hence $D\overline{FH}/H$ is closed in G/H, the set $D\overline{FH}$ is closed in G, and $\overline{AH} = D\overline{FH}$. It follows that G'/D' is compact, and G' is a generalized-solvable group of class one. Furthermore, for every closed, normal subgroup H' of G', the group $H'D'$ is closed in G'.

Let K be a compact-open subgroup of G' with the property that $K \cap D' = \{e'\}$, where e' denotes the identity of G'. Since G' is a generalized-solvable group of class one, $S(G') \cap K$ is dense in K. For each $g' \in S(G')$, the solvable normal subgroup H' of G' is closed. Hence $H'D'$ is closed, $H' \cap D' = \{e'\}$, and H' is compact. This implies that $H' \subset B(G')$ and $K \subset \overline{B(G')}$, and $\overline{B(G')}$ is open and closed in G'. Since $D' \subset B(G')$, we have that G'/D' is compact. This implies that $G'/\overline{B(G')}$ is finite. Furthermore,

$$S(G') \cap P(G') \cap B(G') = N'$$

is a characteristic subgroup, by Remark 4. The set $N' \cap K$ is dense in K', and $\overline{N'}$ is open in G'. Moreover, $\overline{N'} \cap D' = \{e'\}$, and $\overline{B(G)}/N$ is an FC-group. Hence $\overline{N'}$ is a compact-open, normal subgroup in G'. Let $\pi\colon G \to G/H$ be the canonical map, and set $M = \pi^{-1}(\overline{B(G')})$, $N = \pi^{-1}(\overline{N'})$. It is now clear that M/N is an FC-group and G/M is finite.

The following example sheds some light on Proposition 3.

Example. Let H be the semidirect product of Z with the automorphism $\theta\colon Z \to Z$ $(\theta(n) = -n)$. Let $\{X_i \mid i = 1, 2, \cdots\}$ denote a family of finite solvable groups such that the length of the chain of solvability of X_i is strictly increasing as i increases, provided i does not exceed some fixed integer. Let

$$Y = \prod_{i=1}^{\infty} X_i,$$

and give Y the product topology. Let

$$Z = \prod_{-\infty}^{\infty} Y_j,$$

where each Y_j is a copy of Y, and let ϕ denote the shifting isomorphism of Y. Define the semidirect product L of Z with the abelian group generated by ϕ. Let R denote some finite subgroup of Y, and let the inner automorphisms of R act on L. We use this action to define the semidirect product of L and R. Now define

$$G = H \times (L \circledS R).$$

The subgroup $Z \times \{\phi\}$ is abelian and isomorphic with Z^2, and G/Z^2 is compact.

REFERENCES

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