ON A NONLINEAR VOLTERRA EQUATION

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1. INTRODUCTION

We study the asymptotic behavior of solutions of the integrodifferential equation

$$x'(t) = -\int_0^t a(t-\tau)g(x(\tau))d\tau - b(t) + f(t) \qquad (0 \leq t < \infty) \tag{1.1}$$

(primes denote differentiation with respect to t), where a(t) satisfies the conditions

$$\begin{aligned} &a(t) \in C(0, \infty) \cap L_1(0, 1);\ a(t) \text{ is nonnegative, nondecreasing,} \\ &\text{and convex on } (0, \infty);\text{ and } 0 < a(0+) \leq \infty . \end{aligned} \tag{1.2}$$

The functions g and f will be subject to the conditions

$$g(x) \in C(-\infty, \infty), \qquad xg(x) \geq 0, \qquad G(x) = \int_0^x g(\xi)d\xi \to \infty \quad (|x| \to \infty) \tag{1.3}$$

and

$$f(t) \in C[0, \infty), \qquad K_0 = \int_0^\infty |f(t)|\, dt < \infty . \tag{1.4}$$

We first find conditions ensuring that all solutions x(t) of (1.1) satisfy the condition

$$\lim_{t \to \infty} x(t) = 0 . \tag{1.5}$$

Our result extends a theorem of J. J. Levin and J. A. Nohel [6, Theorem 1(ii)], which deals with the case where $a(t) \in C[0, \infty)$ and $(-1)^k a^{(k)}(t) \geq 0$ $(0 < t < \infty;\ k = 0, 1, 2, 3)$.

For the linear case $(g(x) = x)$ with $f(t) \equiv 0$ and $b(t) \equiv$ constant, we showed in [3] that there exist kernels a(t), satisfying (1.2), for which a solution x(t) does not satisfy (1.5); indeed there exists a nonconstant periodic function $\omega(t)$ such that $[x(t) - \omega(t)] \to 0$ as $t \to \infty$. These kernels satisfy the equation

$$a(t) = \delta_0 + \sum_{k=1}^{\infty} \delta_k \left(1 - \frac{\min\{t, kt_0\}}{kt_0}\right), \tag{1.6}$$

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where $\delta_k \geq 0$ and $t_0 > 0$. In studying the nonlinear equation (1.1) with kernels a(t) that satisfy (1.6), we find asymptotic behavior similar to that found by Levin and Nohel in [5] for the delay equation

$$x'(t) = -\frac{1}{R}\int_{t-R}^{t} [R - (t - \tau)]g(x(\tau))d\tau,$$

where R is a positive constant. Our proof of this result is adapted from [5].

Note that when a(t) satisfies conditions (1.2), we may write ([7, p. 230])

$$a(t) = a(\infty) + \int_{\infty}^{t} \alpha(\tau)d\tau, \tag{1.7}$$

where $\alpha(t)$ is a nonpositive, nondecreasing function for which $\alpha(t) = \alpha(t+)$.

When $a(0) < \infty$, we obtain the equation

$$x''(t) + a(0)g(x(t)) = -\int_0^t \alpha(t - \tau)g(x(\tau))d\tau - b'(t) - f'(t) \tag{1.8}$$

from (1.1) by differentiation. We also introduce the differential equation

$$x''(t) + a(0)g(x(t)) = 0 \tag{1.9}$$

and the equivalent system

$$x' = y, \qquad y' = -a(0)g(x). \tag{1.10}$$

In connection with equations (1.9) and (1.10), we use the notation of [5]. We let $\phi(t) = \phi(t, t_0, \alpha, \beta)$ denote the solution of (1.9) for which $\phi(t_0) = \alpha$ and $\phi'(t_0) = \beta$. Then $x = \phi(t)$, $y = \phi'(t)$ is the solution of (1.10) that passes through the point (α, β) at $t = t_0$. For $(\alpha, \beta) \neq (0, 0)$, let $\rho = \rho(\alpha, \beta) > 0$ denote the (common) least period of all solutions of (1.10) passing through (α, β). For each pair (α, β), let

$$\Gamma(\alpha, \beta) = \{(x, y) \mid x = \phi(t, t_0, \alpha, \beta),\ y = \phi'(t, t_0, \alpha, \beta) \quad (-\infty < t, t_0 < \infty)\}.$$

Then $\Gamma(\alpha, \beta)$ is the orbit of (1.10) passing through (α, β). Finally, for any two pairs (α_1, β_1) and (α_2, β_2), we define $D(\alpha_1, \beta_1; \alpha_2, \beta_2)$ as the closed, connected set whose boundary is composed of the two curves $\Gamma(\alpha_1, \beta_1)$ and $\Gamma(\alpha_2, \beta_2)$.

In the following, K denotes a finite, *a priori* bound; its value may change from line to line. By $LAC[T, \infty)$ we denote the class of functions that are absolutely continuous on every bounded subinterval of $[T, \infty)$.

THEOREM 1. (i) *Let* a(t) *satisfy the conditions* (1.2), *and let* $\alpha(t)$ *be as in* (1.7). *Assume that the hypotheses* (1.3) *and* (1.4) *hold. In addition, suppose that*

$$|g(x)| \leq K_1(1 + G(x)) \quad (|x| < \infty) \quad \textit{for some } K_1 < \infty, \tag{1.11}$$

$$b(t) \in LAC[0, \infty), \tag{1.12}$$

(1.13) *there exists* $c(t) \in LAC[0, \infty)$ *such that* $b^2(t) \le a(t)c(t)$ $(0 < t < \infty)$ *and* $[b'(t)]^2 \le \alpha(t)c'(t)$ a.e. *on* $0 < t < \infty$.

If $x(t)$ *is a solution of* (1.1) *on* $0 \le t < \infty$, *then*

(1.14) $$|x(t)| \le M_0 \quad (0 \le t < \infty) \quad \text{for some } M_0 < \infty .$$

(ii) *In addition, suppose that*

(1.15) $$xg(x) > 0 \qquad (x \ne 0)$$

and that either

(1.16) $$\begin{cases} \text{(a) } a(0) = a(0+) < \infty, \\ \text{(b) } f(t) \in LAC[R, \infty) \text{ for some } R < \infty, \text{ and} \\ \qquad |f'(t)| + |b'(t)| \le K \text{ a.e. on } R \le t < \infty, \end{cases}$$

or

(1.17) $$\begin{cases} \text{(a) } a(t) \in L_1(0, \infty), \\ \text{(b) } |f(t)| \le K \qquad (0 \le t < \infty). \end{cases}$$

Finally, suppose (1.5) *does not hold.*

Then there exist a $\delta > 0$ *and sequences* $\{\xi_k\}_{k=1}^{\infty}$ *and* $\{\delta_k\}_{k=1}^{\infty}$ *such that*

(1.18) $$\xi_1 > \delta, \quad \xi_{k+1} > \xi_k + \delta, \quad \delta_k \ge 0 \qquad (k = 1, 2, \cdots)$$

and

(1.19) $$a(t) = a(\infty) + \sum_{k=1}^{\infty} \delta_k \left(1 - \frac{\min\{t, \xi_k\}}{\xi_k}\right).$$

Moreover, if $\Delta = \{k \mid \delta_k > 0\}$, *then*

(1.20) $$\lim_{t \to \infty} \int_{t-\xi_k}^{t} g(x(\tau))\,d\tau = 0 \qquad (k \in \Delta).$$

(iii) *Suppose the hypotheses of parts* (i) *and* (ii) *are satisfied; assume further that* $f(t) \in LAC[R, \infty)$,

(1.21) $$\lim_{t \to \infty} \left(\operatorname*{ess\,sup}_{t \le \tau < \infty} |f'(\tau) - b'(\tau)|\right) = 0,$$

and

(1.22) $$g(x) \text{ is locally Lipschitzian}$$

(*in other words, for each positive number* A *there exists a number* $N = N(A)$ *such that* $|g(x_1) - g(x_2)| \le N|x_1 - x_2|$ *whenever* $|x_1| + |x_2| < A$). *Let* Ω *be the limit set (for* $t \to \infty$) *of the curve* $x = x(t)$, $y = x'(t)$. *Then*

(1.23) $\Omega = D(\alpha_1, \beta_1; \alpha_2, \beta_2)$ *for some pairs* (α_1, β_1) *and* (α_2, β_2).

Moreover, there exists a positive number L *such that*

(1.24) $$\xi_k = jL \text{ for some integer } j = j(k) \qquad (k \in \Delta)$$

and

(1.25) $$L = \rho(\alpha, \beta) \text{ with } (\alpha, \beta) \in \Omega, \; (\alpha, \beta) \neq (0, 0).$$

If in addition

(1.26) $$|tb(t)| + |t^2 a(t)| \leq K \qquad (0 \leq t < \infty)$$

and

(1.27) $$B_0 = -\int_0^\infty t^2 \alpha(t)\,dt < \infty,$$

then there exists a pair (α_0, β_0) *such that*

(1.28) $$\Omega = \Gamma(\alpha_0, \beta_0);$$

and whenever $(\alpha, \beta) \in \Omega$ *and* $0 < K_3 < \infty$, *then the relations*

(1.29) $$\lim_{n\to\infty} \Big[\max_{0 \leq t \leq K_3} |x^{(j)}(t + nL) - \phi^{(j)}(t, t_n, \alpha, \beta)| \Big] = 0 \qquad (j = 0, 1)$$

hold for some sequence $\{t_n\}$ $(0 \leq t_n = t_n(\alpha, \beta) \leq \rho(\alpha, \beta))$.

As in [6], we may use the estimate (1.14) to prove the existence of a solution x(t) of (1.1) on $0 \leq t < \infty$.

As in [6], we may omit condition (1.11) in Theorem 1 if $f(t) \equiv b(t) \equiv 0$. A. Halanay [1] treated this case of (1.1) under the hypothesis that $a(t) - \varepsilon_0 e^{-\alpha t}$ defines a positive kernel for some $\varepsilon_0 > 0$ and $\alpha > 0$.

Suppose now that a(t) satisfies conditions (1.2) and that $a(R) = 0$ for some positive number R. Consider the functional differential equation

(1.30) $$x'(t) = -\frac{1}{R} \int_{t-R}^{t} a(t - \tau) g(x(\tau))\,d\tau - b(t) + f(t)$$

with initial data $x(t) = \psi(t)$ $(-R \leq t \leq 0)$, where $\psi(t)$ is a prescribed function in $C[-R, 0]$. Setting

$$f^*(t) = f(t) - \frac{1}{R} \int_{\min\{0, t-R\}}^{0} a(t - \tau) g(\psi(\tau))\,d\tau,$$

one sees that (1.30) is of the form (1.1), so that Theorem 1 applies. When $f(t) \equiv b(t) \equiv 0$, we can obtain a stronger result (see [5, Theorems 1 and 2]) by applying our method to the energy function introduced by Levin and Nohel in [5] (the method of J. K. Hale [2] will also work).

To prove parts (i) and (ii) of Theorem 1, we combine Lemma 2 with the method used by Levin in [4] and by Levin and Nohel in [6]. Similarly, one can generalize the other theorems of [6] to the case of convex kernels a(t).

2. LEMMAS

LEMMA 1. *Let* a(t) *satisfy conditions* (1.2), *and let* $\alpha(t)$ *be as in* (1.7). *Then*

(i) $$ta(t) \to 0 \quad as \quad t \to 0+,$$

(ii) $$B_1 = -\int_0^1 t\alpha(t)\,dt < \infty,$$

(iii) $$B_2(v) = \int_v^\infty t\,d\alpha(t) < \infty \qquad (v > 0),$$

(iv) $$-t^2\alpha(t) \le 2\int_0^t a(\tau)\,d\tau \le 2\int_0^1 a(\tau)\,d\tau \qquad (0 < t < 1), \quad and$$

(v) $$B_3 = \int_0^1 t^2\,d\alpha(t) < \infty.$$

Proof. The assertions (i), (ii), and (iii) are easily obtained with the aid of conditions (1.2) and (1.7) and integration by parts. The inequality

$$a(\tau) = a(t) - \int_\tau^t \alpha(s)\,ds \ge a(t) - \alpha(t)[t - \tau] \qquad (0 < \tau \le t < 1)$$

implies the relation $\int_0^t a(\tau)\,d\tau \ge ta(t) - t^2\alpha(t)/2$, so (iv) holds. By (iv), $t^2\alpha(t) \to 0$ as $t \to 0+$; this, together with (ii) and integration by parts, yields (v). This completes the proof of Lemma 1.

LEMMA 2. *Suppose* $x(t) \in C'[0, \infty)$ *and* $|x(t)| + |x'(t)| \le K$ $(0 \le t < \infty)$. *Let* a(t) *and* g(x) *satisfy conditions* (1.2), (1.3), *and* (1.15), *and let* $\alpha(t)$ *be as in* (1.7). *Let*

$$H(t) = \int_0^t \left[\int_{t-\tau}^t g(x(s))\,ds\right]^2 d\alpha(\tau),$$

and suppose that $\int_0^\infty H(t)\,dt < \infty$. *Then either* (1.5) *holds, or the conclusions of Theorem* 1(ii) *hold.*

Proof. For $0 \le y_1 < y_2 < \infty$ and $t > y_2$, define

$$S(t; y_1, y_2) = \int_{y_1}^{y_2} \left[\int_{t-\tau}^{t} g(x(s))\,ds \right]^2 d\alpha(\tau).$$

Then

$$(2.1) \qquad 0 \leq S(t; y_1, y_2) \leq H(t) \qquad (y_2 < t < \infty).$$

Since $|x(t)|$ is bounded and $g(x)$ is continuous, there exists a finite number M such that $|g(x(t))| \leq M$ $(0 \leq t < \infty)$. Using Lemma 1(iii), we find that

$$(2.2) \qquad \begin{aligned} \left| \frac{d}{dt} S(t; y_1, y_2) \right| &= 2 \left| \int_{y_1}^{y_2} \left[\int_{t-\tau}^{t} g(x(s))\,ds \right] [g(x(t)) - g(x(t-\tau))]\,d\alpha(\tau) \right| \\ &\leq 4M^2 B_2(y_1) \qquad (0 < y_1 < y_2 < \infty). \end{aligned}$$

From (2.1), (2.2), the mean-value theorem, and the inequality $\int_0^\infty H(t)\,dt < \infty$, we conclude that

$$(2.3) \qquad S(t; y_1, y_2) \to 0 \text{ as } t \to \infty \qquad (0 < y_1 < y_2 < \infty).$$

Now suppose that (1.5) does not hold. Then there exist a positive constant λ and a sequence $\{t_n\}$ such that $t_n \uparrow \infty$ and $|x(t_n)| \geq \lambda$. The inequality $|x'(t)| \leq K$, together with conditions (1.3) and (1.15), implies the existence of positive constants δ and η such that $|g(x(t))| \geq \eta$ provided $|t_n - t| < \delta$ for some n.

Now write $\alpha(t) = \beta(t) + \gamma(t)$, where $\beta(t)$ is continuous and $\gamma(t)$ is the saltus function of $\alpha(t)$. Then $\beta(t)$ and $\gamma(t)$ are nonpositive, nondecreasing functions.

Suppose either $\beta(t)$ is not identically 0 or $\gamma(t)$ has discontinuities at t_1 and t_2 $(0 < t_1 < t_2 < \infty$ and $t_2 - t_1 \leq \delta)$. Then there exist positive numbers v_1 and v_2 such that $0 < v_2 - v_1 < 2\delta$ and

$$(2.4) \qquad \rho = \min\{\alpha(v_2) - \alpha(v_2 - \varepsilon), \alpha(v_1 + \varepsilon) - \alpha(v_1)\} > 0,$$

where $\varepsilon = (v_2 - v_1)/4$. On the other hand, we shall show that $\rho = 0$, contrary to (2.4).

Set $v_0 = (v_1 + v_2)/2$ and $T_n = t_n + v_0$. Then $|T_n - \tau - t_n| = |v_0 - \tau| < \delta$ when $v_1 \leq \tau \leq v_2$. Thus, for each sufficiently large n, either the inequality $g(x(T_n - \tau)) \geq \eta$ $(v_1 \leq \tau \leq v_2)$ or the inequality $g(x(T_n - \tau)) \leq -\eta$ $(v_1 \leq \tau \leq v_2)$ holds. Since

$$\frac{d}{d\tau} \int_{T_n - \tau}^{T_n} g(x(s))\,ds = g(x(T_n - \tau)),$$

we have for each sufficiently large n either the inequality

$$\left| \int_{T_n - \tau}^{T_n} g(x(s))\,ds \right| \geq \varepsilon\eta \qquad (v_1 \leq \tau \leq v_1 + \varepsilon)$$

or the inequality

$$\left| \int_{T_n - \tau}^{T_n} g(x(s))\,ds \right| \geq \varepsilon\eta \qquad (v_2 - \varepsilon \leq \tau \leq v_2).$$

We conclude that $S(T_n ; v_1 , v_2) \geq \rho(\varepsilon\eta)^2$ for sufficiently large n. Relation (2.3) implies that $\rho = 0$. Thus $\beta \equiv 0$, and

$$\alpha(t) = \gamma(t) = \sum_{t < \xi_k} \gamma_k \qquad (0 < t < \infty), \tag{2.5}$$

where $\gamma_k < 0$, $\sum_k \gamma_k > -\infty$, $0 < \xi_1$, and $\xi_{k+1} > \xi_k + \delta$ $(k = 1, 2, \cdots)$. Note also that

$$S(t; \xi_1/2, \xi_k) = S(t; 0, \xi_k) = -\sum_{j=1}^{k} \gamma_j \left(\int_{t-\xi_j}^{t} g(x(s))\,ds \right)^2 \qquad (k = 1, 2, \cdots). \tag{2.6}$$

Setting $k = 1$ in (2.6) and arguing as above, we obtain the inequality $\xi_1 > \delta$. Using (1.7) and (2.5), we obtain relation (1.19) with $\delta_k = -\xi_k \gamma_k$ (set $\delta_k = 0$ for large k, if there are only finitely many γ_k). An inductive argument, involving relations (2.3) and (2.6) and the fact that $\gamma_j < 0$, yields (1.20). This completes the proof of Lemma 2.

3. PROOF OF THEOREM 1

(i) For $0 \leq t < \infty$, define

$$E(t) = G(x(t)) + \frac{1}{2} a(t) \left(\int_0^t g(x(s))\,ds \right)^2 + b(t) \int_0^t g(x(s))\,ds$$

$$+ \frac{1}{2} c(t) - \frac{1}{2} \int_0^t \left[\int_{t-\tau}^{t} g(x(s))\,ds \right]^2 \alpha(\tau)\,d\tau,$$

$$F(t) = \int_0^t |f(\tau)|\,d\tau, \qquad V(t) = [1 + E(t)]e^{-K_1 F(t)}, \quad \text{and}$$

$$H(t) = \int_0^t \left[\int_{t-\tau}^{t} g(x(s))\,ds \right]^2 d\alpha(\tau).$$

From conditions (1.2), (1.7), and (1.13), and from Lemma 1, we see that $0 \leq E(t), V(t), H(t) < \infty$. Now define $E_1(t)$ almost everywhere on $0 < t < \infty$ by the expression

$$E_1(t) = g(x(t))f(t) + \frac{1}{2} \alpha(t) \left(\int_0^t g(x(s))\,ds \right)^2$$

$$+ b'(t) \int_0^t g(x(s))\,ds + \frac{1}{2} c'(t) - \frac{1}{2} H(t).$$

Then (1.1), together with integration by parts (justified where necessary by Lemma 1), yields the relation

$$E_1(t) = \frac{d}{dt}\left\{G(x(t)) + b(t)\int_0^t g(x(\tau))\,d\tau + \frac{1}{2}c(t)\right\}$$

$$-\frac{1}{2}\int_0^t \frac{d}{dt}\left(\int_{t-\tau}^t g(x(s))\,ds\right)^2 \alpha(\tau)\,d\tau$$

$$+\frac{1}{2}a(t)\frac{d}{dt}\left(\int_0^t g(x(\tau))\,d\tau\right)^2 .$$

Using Fubini's theorem, integration by parts, and Lemma 1, we now find that

$$V(t) = V(0) + \int_0^t V_1(\tau)\,d\tau , \tag{3.1}$$

where

$$V_1(t) = -K_1\,|f(t)|\,V(t) + E_1(t)\,e^{-K_1F(t)} .$$

Inequality (1.11) shows that

$$-K_1\,|f(t)|\,[1 + G(x(t))] + g(x(t))f(t) \le 0 ,$$

hence conditions (1.2), (1.13), and (1.4) imply the formulas

$$V_1(t) \le -\frac{1}{2}H(t)\,e^{-K_1K_0} \le 0 \text{ a.e.} \tag{3.2}$$

and

$$G(x(t)) \le \left[\,1 + G(x(0)) + \frac{1}{2}c(0)\,\right]e^{K_1K_0} ;$$

therefore, by (1.3), (1.14) holds. Relations (3.1) and (3.2), together with the inequality $V(t) \ge 0$, yield

$$\int_0^\infty H(t)\,dt < \infty . \tag{3.3}$$

(ii) In view of (1.14), (3.3), the hypotheses, and Lemma 2, we need only show that $|x'(t)|$ is bounded on $0 \le t < \infty$.

If (1.16) holds, then $\alpha(t) \in L_1(0, \infty)$. Integration of (1.8), together with (1.16b) and (1.1), shows that $x'(t) - x'(0) = \int_0^t x''(\tau)\,d\tau$, where $|x''(t)| \le K$ a.e. on $0 < t < \infty$. Thus $|x'(t_1) - x'(t_2)| \le K\,|t_1 - t_2|$; in view of (1.14) and the mean-value theorem, boundedness of $|x'(t)|$ follows.

When (1.17) holds, we note first that (1.12) and (1.13) imply that $|b(t)| \leq K$ $(0 \leq t < \infty)$. Then (1.1) and (1.17) show that $x'(t)$ is bounded, as claimed. This proves (ii).

(iii) We set $M = \sup_{0 \leq t < \infty} |g(x(t))|$. Then M is finite. Using (1.19), we can write

$$h(t) = -\int_0^t \alpha(t-\tau)g(x(\tau))\,d\tau = \sum_{k=1}^{\infty} \frac{\delta_k}{\xi_k} \int_{\max\{0,\, t-\xi_k\}}^t g(x(\tau))\,d\tau\,.$$

It follows that

$$|h(t)| \leq \sum_{k=1}^{n} \frac{\delta_k}{\xi_k} \left| \int_{t-\xi_k}^t g(x(\tau))\,d\tau \right| + M \sum_{k=n+1}^{\infty} \delta_k \qquad (t > \xi_n)\,.$$

In view of (1.20), we see that $h(t) \to 0$ as $t \to \infty$. Using (1.21), we may thus write equation (1.8) as a system

$$x'(t) = y(t), \qquad y'(t) = -a(0)g(x(t)) + z(t),$$

where $\operatorname{ess\,sup}_{t \leq \tau < \infty} |z(\tau)| \to 0$ as $t \to \infty$, and where y is absolutely continuous. Assertions (1.23), (1.24), and (1.25) now follow from an argument adapted from [5, pp. 38-41]; we give an outline indicating the modifications.

Using Gronwall's inequality, we can find, for each $(\alpha, \beta) \in \Omega$, sequences $\{t_n\}$ and $\{T_n\}$ $(t_n \to \infty,\ T_n \to \infty$ as $n \to \infty)$ such that

$$\lim_{n\to\infty} \left(\max_{t_n \leq t \leq t_n+T_n} (|x(t) - \phi(t, t_n, \alpha, \beta)| + |x'(t) - \phi'(t, t_n, \alpha, \beta)|) \right) = 0\,. \tag{3.4}$$

An immediate consequence of (3.4) is that $\Gamma(\alpha, \beta) \subset \Omega$ if $(\alpha, \beta) \in \Omega$, and an argument involving the definition of Ω yields relation (1.23).

Using (3.4), (1.20), (1.3), periodicity of $\phi(t, t_0, \alpha, \beta)$, and the identity $\phi(t + t_0, t_0) = \phi(t, 0)$, we find, for $(\alpha, \beta) \in \Omega$ and $(\alpha, \beta) \neq (0, 0)$, that

$$\int_{t-\xi_k}^t g(\phi(\tau, t_0, \alpha, \beta))\,d\tau = 0 \qquad (k \in \Delta,\ -\infty < t,\ t_0 < \infty)\,. \tag{3.5}$$

Differentiation of (3.5), together with (1.9), yields the relation

$$\phi(t + \xi_k, t_0, \alpha, \beta) = \phi(t, t_0, \alpha, \beta) \qquad (k \in \Delta,\ -\infty < t,\ t_0 < \infty);$$

hence $\xi_k = j(k; \alpha, \beta)\rho(\alpha, \beta)$ $((0, 0) \neq (\alpha, \beta) \in \Omega,\ k \in \Delta,\ j$ an integer). From (1.23) and the continuity of $\rho(\alpha, \beta)$ for $(\alpha, \beta) \neq (0, 0)$, we conclude that $j(k; \alpha, \beta) = j(k)$; thus (1.24) and (1.25) hold.

For the final assertions, we claim first that $t^{-1} \int_0^t g(x(s))\,ds$ tends to 0 as $t \to \infty$. Choose $\varepsilon > 0$ and let $k \in \Delta$. Using (1.20), choose t' so large that the inequality $t \geq t'$ implies that

$$\left| \int_{t-\xi_k}^{t} g(x(s))\,ds \right| < \varepsilon \xi_k / 2 .$$

Choose $t'' \geq t'$ so that $t'' \geq 2M(t' + \xi_k)/\varepsilon$. Finally, for $t > t''$, let $n = n(t)$ be the integer satisfying the condition $t' \leq t - n\xi_k < t' + \xi_k$. For such t we then have the inequalities

$$\left| \int_0^t g(x(s))\,ds \right| \leq \left| \int_{t-n\xi_k}^{t} g(x(s))\,ds \right| + \left| \int_0^{t-n\xi_k} g(x(s))\,ds \right|$$

$$\leq n\varepsilon\xi_k/2 + M(t' + \xi_k) \leq \varepsilon t .$$

This proves our claim.

From (1.26) it now follows that

$$b(t) \left| \int_0^t g(x(s))\,ds \right| + a(t) \left(\int_0^t g(x(s))\,ds \right)^2 \to 0 \quad \text{as } t \to \infty . \tag{3.6}$$

From (3.1), (3.2), and the fact that $V(t) \geq 0$, we see that the function $V(t)$ decreases to V_0 ($V_0 \geq 0$) as t increases to ∞. Then, by (1.4),

$$E(t) \to V_0\, e^{K_1 K_0} - 1 = E_0 \qquad (t \to \infty) .$$

From (1.13), we see that $c(t)$ decreases to $c(\infty)$ ($c(\infty) \geq 0$) as t increases to ∞. Set $D_0 = E_0 - c(\infty)/2$, and define $D(t)$ by the expression

$$D(t) = G(x(t)) - \frac{1}{2} \int_0^t \left[\int_{t-\tau}^{t} g(x(s))\,ds \right]^2 \alpha(\tau)\,d\tau \geq 0 .$$

Using (3.6) and the definition of $E(t)$, we find that $D(t) \to D_0 \geq 0$ as $t \to \infty$.

Let $\phi_A(t, t_0) = \phi(t, t_0, -A, 0)$ $(0 < A < \infty)$. From the study of (1.10), it is well known that

$$G(-A) = G(\phi_A(t, t_0)) + \frac{1}{2a(0)} [\phi'_A(t, t_0)]^2 \qquad (0 < A < \infty,\ -\infty < t,\ t_0 < \infty) .$$

This, together with (1.25) and the substitution

$$\phi'(t) - \phi'(t - \tau) = -a(0) \int_{t-\tau}^{t} g(\phi(s))\,ds ,$$

shows that

$$W(t, t_0, A) = G(\phi_A(t, t_0)) + \frac{a(0)}{2L} \int_0^L \left[\int_{t-\tau}^{t} g(\phi_A(s, t_0))\,ds \right]^2 d\tau$$

$$= G(-A) + \frac{1}{2a(0)L} \int_{t-L}^{t} [\phi_A'(\tau, t_0)]^2 d\tau ,$$

if $A \in Q = \{A > 0 | (-A, 0) \in \Omega\}$ and $-\infty < t, t_0 < \infty$. As in [5], using (1.25), one can show that on Q the last expression is a strictly increasing function of A alone. Thus, by (1.23), to prove (1.28) we need only show that

$$\inf_{-\infty < t, t_0 < \infty} |W(t, t_0, A) - D_0| = 0 \quad (A \in Q). \tag{3.7}$$

Note that inequality (1.26) implies that $a(\infty) = 0$. Using relations (1.19), (1.24), and (1.25), we compute the identities

$$\frac{a(0)}{2L} \int_0^L \left[\int_{t-\tau}^{t} g(\phi_A(s, t_0))\, ds \right]^2 d\tau$$

$$= -\frac{1}{2} \sum_{k=1}^{\infty} \alpha(kL-) \int_{(k-1)L}^{kL} \left[\int_{t-\tau}^{t} g(\phi_A(s, t_0))\, ds \right]^2 d\tau$$

$$= -\frac{1}{2} \int_0^{\infty} \alpha(\tau) \left[\int_{t-\tau}^{t} g(\phi_A(s, t_0))\, ds \right]^2 d\tau \qquad (A \in Q,\ -\infty < t,\ t_0 < \infty).$$

Fix $A \in Q$. For $\{t_n\}$ and $\{T_n\}$ as in (3.4), and for $(\alpha, \beta) = (-A, 0)$, set $\sigma_n = t_n + T_n$. Then

$$|W(\sigma_n, t_n, A) - D_0| \leq |W(\sigma_n, t_n, A) - D(\sigma_n)| + |D(\sigma_n) - D_0|$$

$$\leq |G(\phi_A(\sigma_n, t_n)) - G(x(\sigma_n))|$$

$$+ \frac{1}{2} \left| \int_0^{T_n} \alpha(\tau) \left\{ \left[\int_{\sigma_n - \tau}^{\sigma_n} g(\phi_A(s, t_n))\, ds \right]^2 - \left[\int_{\sigma_n - \tau}^{\sigma_n} g(x(s))\, ds \right]^2 \right\} d\tau \right|$$

$$+ \frac{1}{2} \left| \int_{T_n}^{\infty} \alpha(\tau) \left[\int_{\sigma_n - \tau}^{\sigma_n} g(\phi_A(s, t_n))\, ds \right]^2 d\tau \right|$$

$$+ \frac{1}{2} \left| \int_{T_n}^{\sigma_n} \alpha(\tau) \left[\int_{\sigma_n - \tau}^{\sigma_n} g(x(s))\, ds \right]^2 d\tau \right| + |D(\sigma_n) - D_0|$$

$$\leq \max_{t_n \leq t \leq \sigma_n} |G(\phi_A(t, t_n)) - G(x(t))| + \varepsilon_n N M B_0$$

$$+ M^2 \int_{T_n}^{\infty} \tau^2 |\alpha(\tau)|\, d\tau + \max_{t \geq t_n} |D(t) - D_0| ,$$

where $\varepsilon_n = \max_{t_n \leq t \leq \sigma_n} |\phi_A(t, t_n) - x(t)|$, B_0 is from (1.27), and N is a local Lipschitz constant for g(x). By (1.27) and (3.4), we have (3.7), and therefore (1.28) holds.

As in [5], relation (1.29) follows from (1.28) by means of an argument similar to the proof of (3.4).

This completes the proof of Theorem 1.

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