THE COMPACTNESS OF THE SET OF ARC CLUSTER SETS

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Let f be a continuous, complex-valued function defined in the unit disk D, let C be the unit circle, and let W be the Riemann sphere. For each point $p \in C$, let $\mathfrak{T}(p)$ be the set of all Jordan arcs contained in $D \cup \{p\}$ and having one endpoint at p. For each $t \in \mathfrak{T}(p)$, define the *cluster set of* f *at* p *relative to the arc* t by

$$C_t(f, p) = \bigcap_{r>0} \overline{f(t \cap \{z: |z - p| < r\})}.$$

By a continuum we shall mean a closed, connected, nonempty subset of W. We remark that under our definition, a set with exactly one element is a continuum, and that for each continuous function f and each $t \in \mathfrak{T}(p)$, the cluster set $C_t(f, p)$ is a continuum.

If A and B are two nonempty closed subsets of W, define

$$M(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)),$$

where $d(w_1, w_2)$ is the chordal distance between w_1 and w_2. The distance M(A, B) is a metric on the set of all nonempty closed subsets of W. If we define

$$\mathfrak{C}_f(p) = \{C_t(f, p): t \in \mathfrak{T}(p)\},$$

that is, if $\mathfrak{C}_f(p)$ is the set whose elements are the sets $C_t(f, p)$, then the metric M topologizes the set $\mathfrak{C}_f(p)$ with what we shall call the M-topology. The purpose of this paper is to investigate conditions under which $\mathfrak{C}_f(p)$ is compact in the M-topology.

By an *ambiguous point* p *for the function* f we mean a point $p \in C$ for which there exist two arcs t_1 and t_2 in $\mathfrak{T}(p)$ such that $C_{t_1}(f, p) \cap C_{t_2}(f, p) = \emptyset$. Our main result is the following theorem.

THEOREM 1. *Let* f *be a continuous function in* D, *and let* p *be a point of* C. *If* p *is not an ambiguous point for* f, *then* $\mathfrak{C}_f(p)$ *is a compact set in the* M-*topology.*

Proof. Suppose $\mathfrak{C}_f(p)$ is not a compact set in the M-topology. Then there exist a sequence of continua $\{K_n\}$ and a continuum K such that $K_n \in \mathfrak{C}_f(p)$ for each positive integer n, and such that $K \notin \mathfrak{C}_f(p)$ and $M(K_n, K) \to 0$. For each positive integer n, let

$$H_n = \{z \in D: d(f(z), K_n) < 1/n \text{ and } |z - p| < 1/n\}.$$

Since $K_n \in \mathfrak{C}_f(p)$, there exist a component G_n of H_n and an arc $t_n \in \mathfrak{T}(p)$ such that $C_{t_n}(f, p) = K_n$ and $t_n \subset G_n \cup \{p\}$.

Received December 7, 1968.
Peter Lappan acknowledges support from the National Science Foundation under grant No. GP8183.

Suppose that $G_n \cap G_{n+1} \neq \emptyset$ for each n. Then there exists a Jordan curve $t_0 \in \mathfrak{T}(p)$ such that t_0 passes through the consecutive G_n and such that

$$M(\overline{f(t_0 \cap G_n)}, K_n) \to 0 .$$

But this means that $C_{t_0}(f, p) = K$, in violation of our supposition that $K \notin \mathfrak{C}_f(p)$. Thus there exists an integer n such that $G_n \cap G_{n+1} = \emptyset$. For this integer n, the boundary of the component G_n contains a set L such that $L \cup \{p\}$ is a closed connected set. Because f is uniformly continuous on each compact subset of D, we can choose a sequence $\{s_j\}$ of points on L such that $s_j \to p$ and such that for each point z on any rectilinear segment $[s_j, s_{j+1}]$ the condition $d(f(z), K_n) > 1/2n$ is satisfied. Some subset s of the union of the segments $[s_j, s_{j+1}]$ constitutes an element of $\mathfrak{T}(p)$, and since

$$C_s(f, p) \cap C_{t_n}(f, p) = C_s(f, p) \cap K_n = \emptyset,$$

p is an ambiguous point for f.

In view of Bagemihl's Ambiguous-Point Theorem [1, Theorem 2, p. 380], we obtain the following result.

COROLLARY 1. *Let* f *be a continuous function in* D, *and let* E *be the set of points* p *for which* $\mathfrak{C}_f(p)$ *is not compact in the* M-*topology. Then* E *is a countable set.*

Let $\pi(f, p) = \bigcap C_t(f, p)$, where the intersection is taken over all $t \in \mathfrak{T}(p)$. We then obtain the following result.

COROLLARY 2. *Let* f *be a continuous function in* D, *and let* p *be a point in* C *such that* $\pi(f, p) \neq \emptyset$. *Then* $\mathfrak{C}_f(p)$ *is a compact set in the* M-*topology.*

The following corollary is related to a result of McMillan [2, Theorem 1, p. 495].

COROLLARY 3. *Let* f *be a continuous function in* D. *Then for each point* $p \in C$, *either* p *is an ambiguous point,* f *has an asymptotic value at* p, *or there exists a positive number* h *such that for each* $t \in \mathfrak{T}(p)$, *the diameter of* $C_t(f, p)$ *is greater than* h.

Remark 1. McMillan [3, Theorem 3, p. 496] has given an example of a meromorphic function f in D such that $\mathfrak{C}_f(1)$ is not compact.

Remark 2. There exists a holomorphic function f in D for which some set $\mathfrak{C}_f(p)$ is not compact in the M-topology. M. Heins [2] has proved the existence of an entire function for which the set of asymptotic values is not closed. Let g be such an entire function, let P be the complex plane slit along an asymptotic path of g, and let h be a conformal mapping of D onto P. The set of asymptotic values of $f(z) = g(h(z))$ at some point $z = p$ is precisely the set of asymptotic values of the entire function g. Since this set is not a closed set, $\mathfrak{C}_f(p)$ is not compact in the M-topology.

Remark 3. If p is an ambiguous point for the continuous function f, it is possible that $\mathfrak{C}_f(p)$ is compact. For example, it is easily verified that

$$f(z) = \exp\{1/(z - 1)\}$$

has an ambiguous point at $p = 1$, while $\mathfrak{C}_f(1)$ is compact in the M-topology.

Theorem 1 is not true if the condition that f is a continuous function is removed, as the following theorem shows.

THEOREM 2. *There exists a function* f *in* D *such that* $\pi(f, 1) \neq \emptyset$ *and* $\mathfrak{C}_f(1)$ *is not compact in the* M-*topology*.

Proof. Let

$$R_0 = \{z = x + iy: 0 < x \leq 1,\ 0 < y \leq 1\},$$

$$A_0 = \{z = x + iy: x = 1,\ 0 < y \leq 1/3 \text{ or } 2/3 \leq y \leq 1\},$$

$$B_0 = E_0 \cup F_0 \cup G_0,$$

where

$$E_0 = \{z = x + iy: 0 < x < 1,\ y = 1\},$$

$$F_0 = \bigcup_{n=1}^{\infty} \{z = x + iy: x = 1/2n,\ 0 < y < 2/3\},$$

$$G_0 = \bigcup_{n=1}^{\infty} \{z = x + iy: x = 1/(2n+1),\ 1/3 < y < 1\}.$$

For each positive integer k, let R_k, A_k, and B_k be the image of R_0, A_0, and B_0, respectively, under the translation $L(z) = z + k$.

Define a function F(z) on the first quadrant Q such that F is periodic with period i and such that for each nonnegative integer k

$$F(z) = \begin{cases} 1/(k+1) & \text{for } z \in A_k, \\ 1 & \text{for } z \in B_k, \\ 0 & \text{for } z \in R_k - (A_k \cup B_k). \end{cases}$$

Figure 1 shows a typical square in the k^{th} column of squares in Q, together with some adjacent territory. The images of the set B_0 under horizontal and vertical translations are indicated by light line segments, and the translates of A_0 appear as heavily drawn segments.

Let B(z) be a conformal mapping from D onto Q such that $B(1) = \infty$, and let $f(z) = F(B(z))$. It is easy to see that $0 \in \pi(f, 1)$. Also, if $t \in \mathfrak{T}(1)$ and $1 \notin C_t(f, 1)$, then t can be shortened so that B(t) is contained in a strip $n - 1 < x \leq n$, for some positive integer n. In this case, $C_t(f, 1) = \{0, 1/(n+1)\}$. But, with the notation $K_n = \{0, 1/(n+1)\}$ and $K = \{0\}$, we see that $K_n \in \mathfrak{C}_f(1)$ for each positive integer n, that $M(K_n, K) \to 0$, and that $K \notin \mathfrak{C}_f(1)$. Hence it follows that $\mathfrak{C}_f(1)$ is not compact in the M-topology, and thus f is the desired function.

We wish to thank the referee for suggesting several improvements in style, and particularly for suggesting to us the example used in Remark 2.

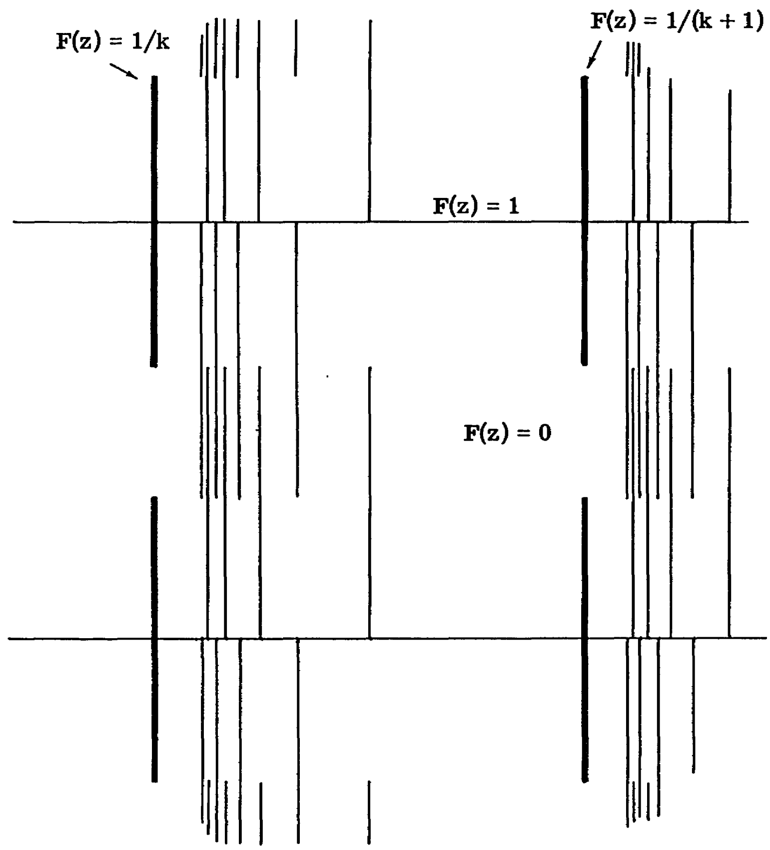

Figure 1.

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