ON THE COEFFICIENTS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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1. INTRODUCTION

In this note, we discuss the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the unit disc U, satisfy the condition $f'(z) \neq 0$ in U, and map U onto a domain with bounded boundary rotation (for a definition of this concept, see [3]). In particular, we denote by V_k the family of functions that satisfy the above conditions and map U onto a domain with boundary rotation at most $k\pi$. V. Paatero [3] showed that $f \in V_k$ if and only if

$$f(z) = \int_0^z \exp\left\{ \int_0^{2\pi} \log(1 - ze^{-it})^{-1} d\mu(t) \right\} dz, \tag{1.1}$$

where $\mu(t)$ is real-valued and of bounded variation on $[0, 2\pi]$ and satisfies the conditions

$$\text{i)} \int_0^{2\pi} d\mu(t) = 2, \qquad \text{ii)} \int_0^{2\pi} |d\mu(t)| \leq k.$$

V_2 is precisely the class of normalized univalent functions that map U onto a convex domain, and it is known [3] that for $2 \leq k \leq 4$, V_k consists only of univalent functions.

In spite of considerable effort, the problem of determining

$$A_n(k) = \max_{f \in V_k} |a_n| \tag{1.2}$$

remains unsolved, except for $k = 2$ and $k = 4$.

K. Loewner [2] proved that $A_n(2) = 1$, and A. Rényi [5] proved that $A_n(4) = n$. Rényi's result shows that

$$A_n(k) \leq n \qquad (k \leq 4); \tag{1.3}$$

in addition Rényi proved that

$$A_n(k) \leq n^{k-2}, \tag{1.4}$$

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which in view of (1.3) is of interest only when $k \leq 3$. O. Lehto [1] proved that $A_2(k) = k/2$ and $A_3(k) = (k^2 + 2)/6$ (Lehto states that $A_2(k) = k/2$ had been proved earlier by G. Pick). Lehto also proved that

$$A_n(k) \sim \frac{k^{n-1}}{n!} \qquad (k \to \infty).$$

Finally, necessary conditions on the extremal function for this extremal problem have recently been obtained in [4] and [6].

The purpose of this note is to prove the estimate

$$A_n(k) < e\left(2 - \frac{1}{n}\right)^{k/2-2} n^{k/2-1}, \tag{1.5}$$

and to show that the order $n^{k/2-1}$ cannot be improved for any k. We note that for sufficiently large n, (1.5) is an improvement on (1.3) and (1.4).

2. COEFFICIENT ESTIMATES

Our main result is a consequence of the following theorem, which is of some interest in itself.

THEOREM 1. *Let* $f(z) \in V_k$. *If* $\lambda \geq 1$, *then*

$$I_\lambda(r, f') = \int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta \leq 2\pi \left(\frac{1}{1 - r^2}\right)^\lambda \left(\frac{1 + r}{1 - r}\right)^{\frac{1}{2}k\lambda - 1} \qquad \textit{for } 0 \leq r < 1.$$

Proof. It follows from (1.1) that

$$f'(z) = \exp\left\{\int_0^{2\pi} \log(1 - ze^{-it})^{-1} d\mu(t)\right\}, \tag{2.1}$$

where

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k. \tag{2.2}$$

If $\gamma(t)$ denotes the variation of $\mu(t)$ over $[0, t]$, then

$$\mu(t) = \frac{1}{2}\{\gamma(t) + \mu(t)\} - \frac{1}{2}\{\gamma(t) - \mu(t)\} = \nu(t) - \sigma(t)$$

is a decomposition of $\mu(t)$ into a difference of increasing functions. Moreover, by (2.2),

$$2 \leq \nu(2\pi) - \nu(0) = a \leq \frac{1}{2}k + 1, \qquad 0 \leq \sigma(2\pi) - \sigma(0) \leq \frac{1}{2}k - 1. \tag{2.3}$$

It is clear from (2.1) and (2.3) that if $z = re^{i\theta}$, then

$$I_\lambda(r, f') = \int_0^{2\pi} \exp\left\{\lambda \int_0^{2\pi} \log|1 - ze^{-it}|^{-1}\, d(\nu(t) - \sigma(t))\right\} d\theta$$

$$\leq (1+r)^{\lambda(k/2-1)} \int_0^{2\pi} \exp\left\{\lambda \int_0^{2\pi} \log|1 - ze^{-it}|^{-1}\, d\nu(t)\right\} d\theta .$$

Applying Jensen's inequality in the previous estimate we may write

$$I_\lambda(r, f') \leq (1+r)^{\lambda(k/2-1)} \int_0^{2\pi} \exp\left\{a\lambda \int_0^{2\pi} \log|1 - ze^{-it}|^{-1}\, d\left(\frac{\nu(t)}{a}\right)\right\} d\theta$$

$$\text{(2.4)} \qquad \leq (1+r)^{\lambda(k/2-1)} \int_0^{2\pi}\int_0^{2\pi} |1 - ze^{-it}|^{-a\lambda}\, d\left(\frac{\nu(t)}{a}\right) d\theta$$

$$= (1+r)^{\lambda(k/2-1)} \int_0^{2\pi}\int_0^{2\pi} |1 - ze^{-it}|^{-a\lambda}\, d\theta\, d\left(\frac{\nu(t)}{a}\right) .$$

Since $2 \leq a\lambda \leq \lambda(k/2+1)$,

$$\int_0^{2\pi} |1 - ze^{-it}|^{-a\lambda}\, d\theta \leq \frac{1}{(1-r)^{a\lambda-2}} \int_0^{2\pi} |1 - ze^{-it}|^{-2}\, d\theta$$

$$= \frac{2\pi}{(1-r^2)(1-r)^{a\lambda-2}} \leq \frac{2\pi}{(1+r)(1-r)^{\lambda(k/2+1)-1}} .$$

Using this inequality in (2.4), we obtain the estimate

$$I_\lambda(r, f') \leq 2\pi \left(\frac{1}{1-r^2}\right)^{\lambda} \left(\frac{1+r}{1-r}\right)^{\frac{1}{2}k\lambda-1} ,$$

and the proof of Theorem 1 is complete.

THEOREM 2. *Let* $A_n(k)$ *be defined by* (1.2); *then*

$$\text{(2.5)} \qquad A_n(k) < e\left(2 - \frac{1}{n}\right)^{k/2-2} n^{k/2-1} .$$

Proof. If $f(z) \in V_k$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then (with the notation $z = re^{i\theta}$)

$$n|a_n| = \frac{1}{2\pi r^n}\left|\int_0^{2\pi} zf'(z)e^{-in\theta}\, d\theta\right| \leq \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} |f'(re^{i\theta})|\, d\theta .$$

Using the estimate of Theorem 1 and taking $r = \frac{n-1}{n}$, we get the inequality

(2.6) $$n|a_n| \leq \left(1+\frac{1}{n-1}\right)^{n-1}\left(2-\frac{1}{n}\right)^{k/2-2} n^{k/2},$$

which implies that

$$|a_n| < e\left(2-\frac{1}{n}\right)^{k/2-2} n^{k/2-1}.$$

The theorem follows from this inequality.

Since we are only interested in estimating $A_n(k)$ for $n \geq 4$, we can simplify the estimate in Theorem 2 as follows.

COROLLARY. *For* $n \geq 4$,

$$A_n(k) < \begin{cases} e\cdot\left(\frac{4}{7}\right)^{2-k/2} n^{k/2-1} & (k \leq 4), \\ e\cdot 2^{k/2-2} n^{k/2-1} & (k \geq 4). \end{cases}$$

Proof. For $n \geq 4$,

$$\left(2-\frac{1}{n}\right)^{k/2-2} \leq \begin{cases} \left(\frac{4}{7}\right)^{2-k/2} & (k \leq 4), \\ 2^{k/2-2} & (k \geq 4), \end{cases}$$

and the result follows from (2.5).

It is interesting to compare (2.5) with the actual value for $A_n(k)$ in the two cases where the value is known. For $k = 2$, (2.5) yields

$$1 = A_n(2) < e\cdot\frac{n}{2n-1},$$

and for $k = 4$,

$$n = A_n(4) < e\cdot n.$$

Four is the smallest value of n for which $A_n(k)$ is not known. We use (2.6) to give the following estimate.

COROLLARY. $A_4(k) \leq \frac{4^4}{3^3}\cdot 7^{k/2-2}$.

It remains to show that the order $n^{k/2-1}$ in (2.5) cannot be improved. Let $\mu(t)$ be a step function on $[0, 2\pi]$, with two discontinuities. One discontinuity occurs at $t = 0$, with positive jump $1 + k/2$, and the other at $t = \pi$, with negative jump $1 - k/2$. The function f(z) defined by (1.1) belongs to V_k, and it is given by

$$f(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right] = z + \sum_{n=2}^{\infty} A_n z^n.$$

Since

$$(1 - z)^{-k/2} = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}k\left(\frac{1}{2}k+1\right) \cdots \left(\frac{1}{2}k+n-1\right)}{n!} z^n$$

and

$$(1+z)^{k/2} = \sum_{n=0}^{\infty} \binom{k/2}{n} z^n,$$

it follows that

$$\text{(2.7)} \qquad A_n = \frac{1}{k}\left[\sum_{j=0}^{n-1} \binom{k/2}{j} \frac{\frac{1}{2}k\left(\frac{1}{2}k+1\right) \cdots \left(\frac{1}{2}k+n-j-1\right)}{(n-j)!}\right] + \frac{1}{k}\binom{k/2}{n}.$$

Now $\binom{k/2}{j}$ is nonnegative if $j \leq [k/2]+1$ (greatest-integer notation). If $j > [k/2]+1$, then $\binom{k/2}{j}$ is nonnegative or nonpositive according as $j - [k/2] - 1$ is an even or an odd integer. Thus, if $[k/2]$ is not an integer, the sign of $\binom{k/2}{j}$ is alternating for $j \geq [k/2]+1$. For $j \leq n$,

$$\frac{\frac{1}{2}k\left(\frac{1}{2}k+1\right) \cdots \left(\frac{1}{2}k+n-j-1\right)}{(n-j)!} \geq \frac{\frac{1}{2}k\left(\frac{1}{2}k+1\right) \cdots \left(\frac{1}{2}k+n-j-2\right)}{(n-j-1)!},$$

and for $j \geq [k/2]+1$,

$$\left|\binom{k/2}{j}\right| \geq \left|\binom{k/2}{j+1}\right|.$$

It follows that the terms of the series in (2.7) are decreasing in absolute value if $j \geq [k/2]+1$. The term corresponding to $j = [k/2]+1$ is nonnegative. Therefore, the sum of the series in (2.7) is larger than its first term. Consequently, we have the inequality

$$A_n(k) > \frac{1}{k}\,\frac{\frac{1}{2}k\left(\frac{1}{2}k+1\right) \cdots \left(\frac{1}{2}k+n-1\right)}{n!} + \frac{1}{k}\binom{k/2}{n}$$

$$= \frac{1}{k}\,\frac{\frac{1}{2}k\left(\frac{1}{2}k+1\right) \cdots \left(\frac{1}{2}k+n-1\right)}{(n-1)^{k/2}\,(n-1)!} \cdot \frac{(n-1)^{k/2}}{n} + \frac{1}{k}\binom{k/2}{n} \sim \frac{n^{k/2-1}}{k\,\Gamma(k/2)} \quad (n \to \infty).$$

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