A NOTE ON INVARIANT SUBSPACES

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In [4] Bernstein and Robinson proved that every polynomially compact operator on a Hilbert space has a nontrivial invariant subspace. A Banach-space version proved in [3] generalizes a theorem of Aronszajn and Smith. Halmos [6] simplified the proof [4], eliminating the strong dependence on metamathematical techniques, and Feldman extracted from his work a proof of the following result, announced in [5]: if T is a quasi-nilpotent operator on Hilbert space, and if some nonzero compact operator can be norm-approximated by polynomials in T, then T has a nontrivial invariant subspace. In this note, the latter theorem is proved, by a somewhat simpler and more natural variant due to Arveson.

We remark that it is easy to deduce the results of [4] from this theorem.

In the following, $B(\mathscr{H})$ stands for the ring of all bounded operators on a separable infinite-dimensional Hilbert space $\mathscr{H}$. Given $T \in B(\mathscr{H})$, let e be a vector in $\mathscr{H}$ such that $e, Te, T^2 e, \cdots$ are linearly independent, let R_n be the projection on $[e, Te, \cdots, T^n e]$ $(n \geq 1)$, and let $d_n = \|T^n e - R_{n-1} T^n e\|$ be the distance from $T^n e$ to $[e, Te, \cdots, T^{n-1} e]$. Of course, d_n is positive. The following lemma simply states the result of a computation.

LEMMA. $\|TR_n - R_n TR_n\| = d_{n+1}/d_n$.

Proof. Take $f \in \mathscr{H}$. Then $R_n f$ can be written in the form

$$R_n f = a_0 e + a_1 Te + \cdots + a_n T^n e.$$

Thus $TR_n f = a_0 Te + \cdots + a_n T^{n+1} e$, and

$$R_n TR_n f = a_0 Te + \cdots + a_{n-1} T^n e + a_n R_n T^{n+1} e.$$

Hence, $TR_n f - R_n TR_n f = a_n(T^{n+1} e - R_n T^{n+1} e)$, and we have the relations

$$\|TR_n f - R_n TR_n f\| = |a_n| d_{n+1} = |a_n| d_n \cdot d_{n+1}/d_n.$$

Now $|a_n| d_n = |a_n| \cdot \|T^n e - R_{n-1} T^n e\| = \|a_n T^n e - R_{n-1} a_n T^n e\|$. Since $R_{n-1} T^j e = T^j e$ for $j \leq n - 1$, we see that

$$a_n T^n e - R_{n-1} a_n T^n e = R_n f - R_{n-1} R_n f = R_n f - R_{n-1} f.$$

Hence,

$$\|TR_n f - R_n TR_n f\| = \|R_n f - R_{n-1} f\| \cdot d_{n+1}/d_n.$$

The conclusion follows, because $R_n - R_{n-1}$ is a nonzero projection, and therefore has norm 1.

COROLLARY. *Let* T *be a bounded operator having a cyclic vector* e *such that* $\lim_n \inf \|T^n e\|^{1/n} = 0$. *Then there exists an increasing sequence* $\{P_n\}$ *of finite-dimensional projections such that* $P_n \to I$ *strongly, and* $\|TP_n - P_n TP_n\| \to 0$.

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Moreover, if for each n, Q_n *is a subprojection of* P_n *that is invariant under the finite-dimensional operator* $T_n = P_n T P_n$, *then also* $\|TQ_n - Q_n TQ_n\| \to 0$.

Proof. If R_n and d_n are defined as in the paragraph before the lemma, then $R_n \to I$ strongly, and $d_n = \|T^n e - R_{n-1} T^n e\| \leq \|T^n e\|$. Using an elementary inequality from calculus [1, p. 383], we see that

$$\liminf d_{n+1}/d_n \leq \liminf (d_n)^{1/n} \leq \liminf \|T^n e\|^{1/n} = 0 .$$

So, by the lemma, $\liminf \|TR_n - R_n TR_n\| = 0$, and the required sequence $\{P_n\}$ can be taken as an appropriate subsequence of $R_1, R_2, \cdots$.

For the second statement, we have the equations

$$TQ_n - Q_n TQ_n = TP_n Q_n - Q_n T_n Q_n = TP_n Q_n - T_n Q_n = (TP_n - P_n TP_n) Q_n .$$

Thus, $\|TQ_n - Q_n TQ_n\| \leq \|TP_n - P_n TP_n\| \to 0$ as $n \to \infty$, and the proof is complete.

This corollary implies that certain operators have an "approximately invariant" sequence $\{P_n\}$ of finite-dimensional projections that converges strongly to the identity. In the following arguments, we shall not need the full strength of the convergence $P_n \to I$, and the fact that $P_n \leq P_{n+1}$ will not enter at all. According to the second statement of the corollary, once we have an approximately invariant sequence, an entire family of such sequences is evident. We turn now to the main result.

THEOREM. *Let* T *be a bounded operator such that* $\liminf \|T^n e\|^{1/n} = 0$ *for some* $e \neq 0$ *in* $\mathscr{H}$, *and such that the norm-closed algebra generated by* T *and the identity contains a nonzero compact operator* C. *Then* T *has a nontrivial invariant subspace.*

Proof. If the nullspace of C is nonzero, then already this provides us with a nontrivial T-variant subspace. Therefore we may assume that $Cf = 0$ implies $f = 0$, for every $f \in \mathscr{H}$. For an equally obvious reason, we can assume that the given vector e is cyclic: $[e, Te, T^2 e, \cdots] = \mathscr{H}$.

Let ρ be a normal state on the algebra $B(\mathscr{H})$ of all bounded operators such that $\rho(E) \leq 1/2$ for every one-dimensional projection E; for example, one may take $\rho(A) = \frac{1}{2}(Au, u) + \frac{1}{2}(Av, v)$, where u and v are orthogonal unit vectors. Let $[a, b]$ be a closed subinterval of the open unit interval $(0, 1)$, with $b - a \geq 1/2$.

Let $\{P_n\}$ be the sequence of projections described in the preceding corollary. Since $P_n \to I$ strongly, we see that $\rho(P_n) \to \rho(I) = 1 > b$; so we may, by discarding a finite number of terms, assume that $\rho(P_n) > b$ for every n. We claim that for each n there exists a $P_n TP_n$-invariant subprojection Q_n of P_n such that $\rho(Q_n) \in [a, b]$. Indeed, let

$$0 = Q^0 \leq Q^1 \leq \cdots \leq Q^N = P_n$$

be a sequence of $P_n TP_n$-invariant subprojections, increasing one dimension at a time (of course, the sequence $\{Q^j\}$ depends on n). The numbers

$$\rho(Q^0), \rho(Q^1), \cdots, \rho(Q^N)$$

increase from 0 to $\rho(P_n) > b$, and by the choice of ρ, the distance between any two *consecutive* elements is no more than 1/2. Therefore, at least one of these numbers must lie in the interval $[a, b]$, and the corresponding Q^j we take as Q_n.

By passing to a subsequence, if necessary, we may assume that $\{Q_n\}$ is convergent in the weak-operator topology (this by weak sequential compactness of the unit ball of $B(\mathscr{H})$). Of course, the limit Q need not be a projection, but by weak continuity of ρ on bounded sets, we do have the relation $\rho(Q) \in [a, b]$. In particular, $Q \neq 0$ and $Q \neq I$, since $[a, b]$ contains neither 0 nor 1.

Let $\mathscr{M} = \{f \in \mathscr{H}: Qf = f\}$. $\mathscr{M}$ is clearly a closed subspace, and $\mathscr{M} \neq \mathscr{H}$ because $Q \neq I$. To see that $\mathscr{M}$ is invariant under T, take $f \in \mathscr{M}$. Since $Q_n^2 = Q_n = Q_n^*$, we see that

$$\|Q_n f - f\|^2 = (Q_n f, Q_n f) - 2(Q_n f, f) + (f, f) = -(Q_n f, f) + (f, f) \to 0,$$

so that $Q_n f \to f$ in norm. Thus $TQ_n f \to Tf$ in norm, so that $Q_n TQ_n f \to QTf$ weakly. According to the previous corollary, $\|Q_n TQ_n f - TQ_n f\| \to 0$, and therefore $TQ_n f \to QTf$ weakly. But we have just seen that the left side, $TQ_n f$, tends in norm to Tf. Hence, $Tf = QTf$, and therefore $Tf \in \mathscr{M}$.

It remains to show that $\mathscr{M} \neq \{0\}$, and for this we must use the compact operator C. First, note that $\|CQ_n - Q_n CQ_n\| \to 0$. To see this, write

$$\mathscr{A} = \{S \in B(\mathscr{H}): \|SQ_n - Q_n SQ_n\| \to 0\};$$

then $\mathscr{A}$ contains T (by the corollary) and the identity, so that $\mathscr{A}$ contains C provided it is a norm-closed algebra. Clearly, $\mathscr{A}$ is closed under addition and scalar multiplication. If $S \in \mathscr{A}$ and S' is arbitrary, then

$$\begin{aligned}\|S'Q_n - Q_n S'Q_n\| &\leq \|SQ_n - Q_n SQ_n\| + \|Q_n(S' - S)Q_n\| + \|(S' - S)Q_n\| \\ &\leq \|SQ_n - Q_n SQ_n\| + 2\|S' - S\|.\end{aligned}$$

Thus $\lim_n \sup \|S'Q_n - Q_n S'Q_n\| \leq 2\|S' - S\|$, and it follows that $\mathscr{A}$ is norm-closed. From the identity

$$S_1 S_2 Q_n - Q_n S_1 S_2 Q_n = (S_1 Q_n - Q_n S_1 Q_n) S_2 Q_n + (S_1 - Q_n S_1)(S_2 Q_n - Q_n S_2 Q_n)$$

it follows that $\mathscr{A}$ is closed under products, and so C has the stated property.

Finally, we show that $\mathscr{M}$ contains $CQ\mathscr{H}$, or, what is the same, that $QCQ = CQ$. This will complete the proof; for $Q \neq 0$ implies $CQ\mathscr{H} \neq \{0\}$, since C has trivial nullspace. Choose $f \in \mathscr{H}$. Then $Q_n f \to Qf$ weakly, so that $CQ_n f \to CQf$ in norm (here we use compactness of C). Hence, $Q_n CQ_n f \to QCQf$ weakly, since $Q_n \to Q$ in the weak operator topology. But $\|Q_n CQ_n f - CQ_n f\| \to 0$, by the preceding lines, so that in fact $CQ_n f \to QCQf$ weakly. We have just seen that the left side converges in norm to CQf. Hence, $CQf = QCQf$, and the theorem is proved.

Naturally, it would be of interest to remove the hypothesis $\lim\inf \|T^n e\|^{1/n} = 0$. It is not very difficult to see that a valid (and weaker) replacement for the hypothesis is that there be a sequence $e_1, e_2, \cdots$ of nonzero vectors such that $\lim_n \inf \|T^n e_j\|^{1/n}$ tends to 0 as $j \to \infty$. However, we have not yet been able to find a substitute that is both significantly weaker and natural.

As an obvious corollary, we have the result of [5]:

COROLLARY 1. *If* T *is a quasi-nilpotent operator on* $\mathscr{H}$ *such that the norm-closed algebra generated by* T *and* I *contains a nonzero compact operator, then* T *has a nontrivial invariant subspace.*

By use of standard devices, the theorem of Bernstein and Robinson is easily deduced from Corollary 1. Although the gist of this argument resembles an argument in [3], we include a proof for completeness.

COROLLARY 2 (Bernstein and Robinson). *Let* $T \in B(\mathscr{H})$ *be such that some nonzero polynomial in* T *is compact. Then* T *has a nontrivial invariant subspace.*

Proof. Let $p \neq 0$ be a polynomial such that $C = p(T)$ is compact. If $C = 0$, then T has obvious finite-dimensional invariant subspaces. So assume that $C \neq 0$. If C has positive spectral radius, then the spectrum of C has an isolated point different from 0, and this gives rise to a nontrivial idempotent E that commutes with C and T (in fact, E commutes with every operator that commutes with C). Thus $E\mathscr{H}$ is a proper invariant subspace for T.

If the spectral radius of C is 0, then by the spectral mapping theorem, $p(\sigma(T)) = \sigma(p(T)) = \sigma(C) = \{0\}$, so that $\sigma(T)$ is contained in the finite set of roots of the equation $p = 0$. If $\sigma(T)$ has more than one isolated point, an argument as in the last paragraph produces an invariant subspace. Therefore we can assume that $\sigma(T)$ consists of a single point λ. But then $T - \lambda I$ is quasi-nilpotent, and now Corollary 1 applies to this operator.

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