MAXIMAL LINEAR MAPPINGS AND SMOOTH SELECTION OF MEASURES ON CHOQUET BOUNDARIES

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1. This paper answers a question raised by L. Bungart and H. Bauer [2, p. 155] concerning the algebra A(D) of functions that are holomorphic in a bounded open set $D \subseteq C^n$ and continuous in $\overline{D}$. Roughly, the question is *whether we can select positive, mutually absolutely continuous representing measures for the points of* D *in a smooth manner, and in a way that keeps the measures concentrated on the Choquet (minimal) boundary of* $\overline{D}$. Bungart showed that such a selection is possible if we replace "Choquet boundary" by "Šilov boundary" and interpret "smooth" to mean (norm-) harmonic. In his dissertation [6] (see also [5]), Hinrichsen showed that we can even replace "smooth" by "holomorphic," provided we make the obviously necessary change from "positive" to "complex," and alternatively, that we can replace "smooth" by "pluriharmonic" and merely change "positive" to "real."

We shall show here that the original question has an affirmative answer if we interpret "smooth" to mean "harmonic," and that with this interpretation of smoothness, we can replace "representing measures" with "Jensen measures" provided we thicken the Choquet boundary sufficiently to make it capable of supporting Jensen measures at all [4]. Most of the theorems below belong to the study of vector lattices; the argument that selects the measures as harmonic functions of the points they represent pivots on two facts: the bounded harmonic functions on D form an order-complete vector lattice, and the maximality techniques that produce scalar measures concentrated on distinguished boundaries can be adapted to linear mappings taking their values in order-complete vector lattices. Proofs that resemble closely the arguments used in working with scalar measures will be sketched rather than given in detail.

We follow [10] in terminology and notation for vector spaces (especially ordered vector spaces), and we shall often use [10] in place of primary references. Similarly, we shall follow [1] for integration theory, and [7] for the approach to the Choquet boundary that has become standard. However, our Choquet-boundary arguments will follow those of [4], since these are more general and can be used in the construction of Jensen measures. We shall need three further notational conventions: if E and F are ordered (real) vector spaces and $T: E \to F$ is a linear mapping, T will be called *order-bounded* if it is a difference of positive linear mappings. If L is an order-complete vector lattice, then the subspace of L^* consisting of all $F \in L^*$ for which

$$\langle \sup \mathfrak{F}, F \rangle = \lim_{f \in \mathfrak{F}} \langle f, F \rangle$$

whenever $\mathfrak{F}$ is an upward-directed majorized subset of L will be called the *order subdual* of L and denoted by L_*; the cone of nonnegative elements of L_* will be called L_*^+. (We have chosen these definitions in order to eliminate unnecessary discussions of existence and uniqueness of positive decompositions.) Finally, if a

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topological space is present in a context, the cone of bounded lower-semicontinuous functions on that space will be denoted by $\mathscr{I}$ (the bounded upper-semicontinuous functions will therefore constitute $-\mathscr{I}$).

2. We begin with three lemmas. Each is based on the observation that the proof of its special case in which $L = R$ uses only the fact that R is an order-complete vector lattice. The first two are theorems of Riedl [8, pp. 112-118], recalled here for the reader's convenience; for the third we sketch a proof.

(2.1) LEMMA [8, Theorem 9.2]. *Let* E *be a real vector space,* L *an order-complete vector lattice, and* $p\colon E \to L$ *a function satisfying the two conditions*

$$p(x+y) \leq p(x) + p(y),$$

$$p(\alpha \cdot x) = \alpha \cdot p(x) \quad \textit{for all } x, y \in E \textit{ and } 0 \leq \alpha \in R.$$

Let U *be a linear mapping defined on a subspace* $F \subseteq E$, *with* $Ux \leq p(x)$ *for every* $x \in F$. *Then there exists a linear mapping* $T\colon E \to L$ *that extends* U *and also satisfies the condition* $Tx \leq p(x)$ *for all* $x \in E$.

(2.2) LEMMA. *Let* X *be a compact Hausdorff space,* B *a subspace of* $\mathscr{C}_R(X)$ *containing the constants, and* $U\colon B \to L$ *a positive linear mapping of* B *into an order-complete vector lattice* L. *Then there exists an extension of* U *to* $\mathscr{C}_R(X)$, *say* $T\colon \mathscr{C}_R(X) \to L$, *that is also a positive linear mapping.*

This lemma can be deduced from its predecessor: set $e = U(1) \in L$, and let $p(f) = \|f\| \cdot e$ for $f \in \mathscr{C}_R(X)$. Things look more familiar at this point if we observe that (with X, B, U, T, and e as in the last lemma) the inequalities

$$-\|f\| \cdot 1 \leq f \leq \|f\| \cdot 1$$

imply that $-\|f\| \cdot e \leq Uf \leq \|f\| \cdot e$ for each $f \in B$, and thus that U takes its values in the union of order-intervals $\bigcup_{0 \leq \alpha \in R} [-\alpha \cdot e, \alpha \cdot e]$. This set (called L_e [10, p. 232]) is itself an order-complete vector lattice (with the relativized lattice operations) in the relativized order; in fact, it is an order-complete Banach lattice when normed by the gauge of $[-e, e]$ (see [1, p. 31, no. 13]). L_e is an (AM)-space with unit e in the sense of Kakutani [10, p. 246], and so it is isomorphic as an (AM)-space with unit to a suitable $\mathscr{C}_R(Z)$ with unit 1, Z being a compact Hausdorff space. T as constructed above must also take its values in L_e, and it can be regarded as a linear mapping from $\mathscr{C}_R(X)$ to $\mathscr{C}_R(Z)$ that sends 1 to 1 and is positive; thus it is continuous, with operator norm 1. Norm-continuity makes it simpler to prove the next lemma; it will also be needed (along with norm-completeness of L_e) in the proof of (2.6) below. Z is extremally disconnected, because L_e is order-complete; we observe that any measure on Z that is normal in the sense of Dixmier [3, p. 156] is in $(L_e)_*$.

(2.3) LEMMA. *Let* X *be a compact Hausdorff space,* L *an order-complete vector lattice, and* $T\colon \mathscr{C}_R(X) \to L$ *a positive linear transformation. For* $g \in \mathscr{I}$, *define a mapping* $\overline{T}\colon \mathscr{I} \to L$ *by*

$$\overline{T}g = \sup\{Tf \mid g \geq f \in \mathscr{C}_R(X)\}.$$

Then $\overline{T}$ *extends* T, *is positively homogeneous, additive, and monotone on* $\mathscr{I}$, *and thus has a unique linear extension to the subspace* $\mathscr{I} - \mathscr{I}$ *of* R^X. $\overline{T}$ *has the further* *property that if* $\mathfrak{G}$ *is an upward-directed family of elements of* $\mathscr{I}$ *with supremum* $h \in \mathscr{I}$*, then the relation*

$$(*) \qquad \overline{T}h = \sup\{\overline{T}g \mid g \in \mathfrak{G}\}$$

holds (along with its obvious dual for $-\mathscr{I}$*). Finally, if* $e = T(1)$, $F \in (L_e)_*$, *and* $\mu = T'F \in \mathscr{M}(X)$, *then*

$$\int g\,d\mu = F(\overline{T}g)$$

for every $g \in \mathscr{I} - \mathscr{I}$.

Proof (after [1, Chapter IV, Section 1, Theorems 1 and 2]). We may assume that $L = L_e$, so that T is Banach-lattice-valued and norm-continuous. Then (*) certainly holds if $\mathfrak{G} \subseteq \mathscr{C}_R(X)$ and $h \in \mathscr{C}_R(X)$, for under these circumstances $\mathfrak{G}$ converges to h in the uniform norm, by the Dini theorem, and the supremum of a *convergent* upward-directed family in L_e is its norm limit. The evident monotonicity of $\overline{T}$ implies that $\overline{T}h \geq \sup\{\overline{T}g \mid g \in \mathfrak{G}\}$ in general; therefore it suffices to prove the reverse inequality. To this end, set

$$\mathfrak{F}_g = \{f \mid g \geq f \in \mathscr{C}_R(X)\} \quad \text{for each } g \in \mathfrak{G}, \qquad \mathfrak{F} = \bigcup_{g \in \mathfrak{G}} \mathfrak{F}_g .$$

Then $\mathfrak{F}$ is directed upward with limit h, so that again $\overline{T}h \geq \sup\{\overline{T}f \mid f \in \mathfrak{F}\}$. But here the reverse inequality also holds, because for any continuous $f_0 \leq h$ the set $\{f_0 \wedge f\}_{f \in \mathfrak{F}}$ converges upward to f_0 and all the functions involved are continuous; therefore

$$T(f_0) = \sup\{T(f_0 \wedge f) \mid f \in \mathfrak{F}\} \leq \sup\{Tf \mid f \in \mathfrak{F}\}.$$

Taking the supremum on f_0, we see that $\overline{T}h \leq \sup\{\overline{T}f \mid f \in \mathfrak{F}\}$. Finally, for each $f \in \mathfrak{F}$ there is some $g \in \mathfrak{G}$ with $f \in \mathfrak{F}_g$, in other words, with $f \leq g$, so that $\overline{T}f \leq \overline{T}g$ and thus $\overline{T}f \leq \sup\{\overline{T}g \mid g \in \mathfrak{G}\}$, whence taking the supremum on f gives $\overline{T}h = \sup\{\overline{T}f \mid f \in \mathfrak{F}\} \leq \sup\{\overline{T}g \mid g \in \mathfrak{G}\}$.

That $\overline{T}$ is positively homogeneous is obvious; that it is additive follows from (*) and the observation that if $g_1, g_2 \in \mathscr{I}$, then the family

$$\{f_1 + f_2 \mid g_i \geq f_i \in \mathscr{C}_R(X),\ i = 1, 2\}$$

converges upward to $g_1 + g_2$ pointwise. Finally, if $F \in (L_e)_*$ and $\mu = T'F$, then for each $g \in \mathscr{I}$

$$F(\overline{T}g) = F[\sup\{Tf \mid g \geq f \in \mathscr{C}_R(X)\}] = \lim_f \{F(Tf) \mid g \geq f \in \mathscr{C}_R(X)\}$$
$$= \lim_f \left\{ \int f\,d\mu \mid g \geq f \in \mathscr{C}_R(X) \right\} = \int g\,d\mu,$$

by the defining property of the order subdual and the definition of the scalar integral of a function in $\mathscr{I}$.

We now place ourselves in the setting of Edwards' paper [4], and more specifically in the setting of [4, Section 3]; accordingly, we assume that there is given a min-stable separating wedge $\mathscr{G}$ of continuous real-valued functions on the compact Hausdorff space X, that is, a cone (possibly improper) of functions distinguishing the points of X and closed under the operation $\wedge$; we assume that $\mathscr{G}$ contains the constant functions. As in [4, p. 303], the Stone-Weierstrass theorem implies that the sublattice $\mathscr{G} - \mathscr{G}$ of $\mathscr{C}_R(X)$ is dense. Diverging slightly from Edwards' approach (and coming closer to that of [7]), we adopt the following definition.

(2.4) *Definition.* For any $f \in \mathscr{C}_R(X)$, the *$\mathscr{G}$-upper envelope of* f, denoted by $\bar{f}$, is the (pointwise) infimum of the family of all functions in $\mathscr{G}$ that dominate f: if $\mathscr{G}_f = \{g \mid f \leq g \in \mathscr{G}\}$, then $\bar{f}(x) = \inf_{g \in \mathscr{G}_f} g(x)$.

It is clear that $\bar{f} \in -\mathscr{I}$ for each f. Since $\mathscr{G}$ is a wedge, the map $f \to \bar{f}$ is positively homogeneous, monotone and sublinear, and thus

$$\alpha \cdot \bar{f} \leq \overline{\alpha \cdot f} \qquad \text{for each } f \text{ and each } 0 \geq \alpha \in R.$$

Moreover, the min-stability condition implies that the family $\mathscr{G}_f$ of functions is directed downward with limit $\bar{f}$, and consequently (by the dual of (*) in (2.3) above) $\overline{T}(\bar{f}) = \inf_{g \in \mathscr{G}_f} Tg$ whenever $T: \mathscr{C}_R(X) \to L$ is a positive linear mapping of $\mathscr{C}_R(X)$ into an order-complete vector lattice L (see [4, p. 304]).

We now define an ordering for positive linear mappings $T: \mathscr{C}_R(X) \to L$ (where L is an ordered vector space) by analogy with the ordering used for measures in the usual Choquet boundary treatments:

(2.5) *Definition.* If T_1 and T_2 are positive linear mappings of $\mathscr{C}_R(X)$ into an ordered vector space L, then $T_1 \succ T_2$ means that $T_1(g) \leq T_2(g)$ for every $g \in \mathscr{G}$.

Just as in the scalar case, we see that if $f \in \mathscr{G} \cap -\mathscr{G}$, then $T_1 \succ T_2$ implies $T_1(f) = T_2(f)$. In particular, $T_1(1) = T_2(1)$ whenever T_1 and T_2 are comparable under $\succ$, which means that they both take their values in the subspace L_e of L, where $e = T_1(1) = T_2(1)$. We have already observed that T_1 and T_2 are norm-continuous from $\mathscr{C}_R(X)$ into L_e if the latter is normed by the gauge of $[-e, e]$ (provided that that gauge is a norm, as it will be if L is an order-complete lattice); thus (with the same proviso), just as in the scalar case, the density of $\mathscr{G} - \mathscr{G}$ in $\mathscr{C}_R(X)$ implies that whenever $T_1 \succ T_2$ and $T_2 \succ T_1$, then T_1 and T_2 must be equal. Transitivity of $\succ$ is just as easy to prove as in the scalar case. And now that the order $\succ$ is defined, it is meaningful to talk about positive linear mappings that are maximal with respect to it; these maximal objects can be obtained, as usual, by Zornification.

(2.6) PROPOSITION. *If* $U: \mathscr{C}_R(X) \to L$ *is a positive linear mapping whose range is a complete vector lattice, then there exists a* $\succ$*-maximal positive linear mapping* $T \succ U$.

Proof. It will suffice to show that if $\mathfrak{B}$ is a $\succ$-upward-directed set of positive linear mappings $\succ$-larger than U, then $\mathfrak{B}$ has a $\succ$-supremum. In order to do this, define $T: \mathscr{G} \to L$ by $Tg = \inf\{Vg \mid V \in \mathfrak{B}\}$ for each $g \in \mathscr{G}$. Then T is clearly positively homogeneous, and a straightforward argument using the upward-directedness of $\mathfrak{B}$ (that is, the downward-directedness of each $\{Vg \mid V \in \mathfrak{B}\}$) will show that T is additive. T therefore has a unique linear extension to the subspace $\mathscr{G} - \mathscr{G}$ of $\mathscr{C}_R(X)$, and since this extension (which we shall also call T) is positive and $T(1) = e = U(1)$, T can be extended, by denseness of $\mathscr{G} - \mathscr{G}$, its own continuity (with L_e normed by the gauge of $[-e, e]$), and norm-completeness of L_e, so that it is defined on all of $\mathscr{C}_R(X)$. The definition of T on $\mathscr{G}$ insures that T is the $\succ$-supremum of $\mathfrak{B}$.

One would now like the maximal mappings to take the same values on the upper envelopes of continuous functions as on the functions themselves. The mappings are not defined on upper envelopes in general, since upper envelopes are not continuous in general; but since the extensions $\overline{T}$ of (2.3) are defined on bounded upper-semi-continuous functions, the following result is the appropriate one.

(2.7) PROPOSITION. *If* T *is a* $\succ$*-maximal linear mapping from* $\mathscr{C}_R(X)$ *into an order-complete vector lattice* L, *then* $Tf = \overline{T}(\bar{f})$ *for every* $f \in \mathscr{C}_R(X)$.

Proof (after [7, Proposition 4.2]). Fix $f \in \mathscr{C}_R(X)$, and define the linear mapping $U: Rf \to L$ by $U(\alpha \cdot f) = \alpha \cdot \overline{T}(\bar{f})$; define a function $p: \mathscr{C}_R(X) \to L$ by $p(h) = \overline{T}(\bar{h})$. It is clear that p is sublinear, and if $0 \leq \alpha \in R$, then

$$U(\alpha \cdot f) = \alpha \cdot \overline{T}(\bar{f}) = \overline{T}(\alpha \cdot \bar{f}) = \overline{T}(\overline{\alpha \cdot f}) = p(\alpha \cdot f),$$

while if $0 \geq \alpha \in R$, then

$$U(\alpha \cdot f) = \alpha \cdot \overline{T}(\bar{f}) = \overline{T}(\alpha \cdot \bar{f}) \leq \overline{T}(\overline{\alpha \cdot f}) = p(\alpha \cdot f).$$

Thus, by (2.1) above, there is an extension of U to all of $\mathscr{C}_R(X)$ (this extension we shall also denote by U), with $Uh \leq p(h)$ for all $h \in \mathscr{C}_R(X)$. Since $g \in \mathscr{G}$ implies $g = \bar{g}$, we see that

$$Ug \leq p(g) = \overline{T}(\bar{g}) = Tg \qquad \text{for every } g \in \mathscr{G},$$

or $U \succ T$; by maximality, $U = T$ and $Tf = Uf = \overline{T}(\bar{f})$.

(2.8) COROLLARY. *Let* T *be a* $\succ$*-maximal positive linear mapping of* $\mathscr{C}_R(X)$ *into an order-complete vector lattice* L. *If* $F \in (L_e)_*$ (*where* $e = T(1)$) *and* $\mu = T'F \in \mathscr{M}(X)$, *then*

$$\int f\,d\mu = \int \bar{f}\,d\mu \qquad \textit{for all } f \in \mathscr{C}_R(X).$$

Indeed, this is just an application of the last part of (2.3).

In particular, if $F \in (L_e)_*^+$, then the measure $\mu = T'F \in \mathscr{M}(X)^+$ is a maximal scalar measure (by [4, p. 305]) and thus has the properties which [4] establishes for those measures; among other things, μ in some sense lives on a boundary $\partial_{\mathscr{G}} X$ of Choquet type.

Remark. In order to consider the problem raised in [2] from the present viewpoint, it is necessary to know that when X is a compact convex subset of a locally convex space E, then a min-stable separating wedge $\mathscr{G}$ of continuous functions on X can be chosen for which the maximal measures in the sense of [4] are the same as those defined in [7, p. 25]. The reader will find it easy to see that $\mathscr{G}$ can simply be taken to be the cone of continuous concave functions on X: with this choice of $\mathscr{G}$, the order $\succ$ defined in [4] and the order $\succ$ defined in [7, p. 24] are identical.

(2.9) COROLLARY. *Let* X *be a compact Hausdorff space,* B *a separating linear subspace of* $\mathscr{C}_R(X)$ *with* $1 \in B$, *and* $U: B \to L$ *an order-bounded linear mapping of* B *into an order-complete vector lattice* L. *Suppose* $U = U^+ - U^-$ (*with* U^+ *and* U^- *positive*), *and let* $e = U^+(1) + U^-(1)$. *Then there exists an order-bounded (and therefore norm-continuous) linear mapping* T' *from* $(L_e)_*^+ - (L_e)_*^+$ *into* $\mathscr{M}(X)$ *for which the two statements*

(A) $T'F|_B = U'F$,

(B) *the variation* $|T'F|$ *of* $T'F$ *is maximal in the scalar sense and therefore annihilates every Baire subset of* X *disjoint from the Choquet boundary of* X *with respect to* B

hold for every $F \in (L_e)^+_* - (L_e)^+_*$. *If* $U = U^+$, *then* T' *can be taken positive.*

Proof. Identify X (under evaluation) with a subset of B', and let Y be its $\sigma(B', B)$-closed convex hull; identify B with the separating subspace (containing 1) of $\mathscr{C}_R(Y)$ to which it naturally corresponds. Suppose that $U = U^+$ for the present, and using (2.2), extend U to all of $\mathscr{C}_R(Y)$. If $T \succ U$ is maximal, then by (2.8) the measure $T'F$ is maximal in the scalar sense, for every $F \in (L_e)^+_*$, and thus [7, p. 30 and Chapter VI passim] is supported by X and annihilates every Baire subset of X disjoint from the Choquet boundary. If $F = F_1 - F_2$ with F_1 and F_2 in $(L_e)^+_*$, then the relation $|T'F| \leq T'F_1 + T'F_2$ implies that $|T'F|$ annihilates $\bar{f} - f$ for every $f \in \mathscr{C}_R(Y)$; therefore $|T'F|$ is also maximal. T' thus sends elements of $(L_e)^+_* - (L_e)^+_*$ back to measures supported by X, and it may thus be regarded as having range contained in $\mathscr{M}(X)$; clearly, this T' satisfies requirements (A) and (B). The straightforward extension to order-bounded U can be left to the reader.

Remark. Suppose that $(L_e)^+_*$ distinguishes points of L_e; then the transformation $T: \mathscr{C}_R(Y) \to L_e$ constructed above may (by a slight abuse of language) be thought of as defined on $\mathscr{C}_R(X)$, and T' will then simply be its transpose. To prove this, it suffices to show that if $g \in \mathscr{C}_R(Y)$ and $g \mid X = 0$, then $Tg = 0$; but if $g \mid X = 0$, then for every $F \in (L_e)^+_*$ we have (with $\mu = T'F$) the relation

$$\langle Tg, F \rangle = \int g \, d\mu = 0,$$

since μ is supported by X. All concrete cases encountered below will have enough elements in $(L_e)^+_*$ to distinguish points of L_e, and we shall without further notice consider the T constructed above to be defined on $\mathscr{C}_R(X)$.

3. It remains only to apply (2.9) to the situation envisioned in [2]. The question raised there can be viewed as a particular case of a problem in selecting representing measures for a vector space of bounded "harmonic" functions in Brelot's axiomatic theory of harmonic functions, and we shall treat that more general problem. Let W be a connected and locally connected locally compact Hausdorff space, $\mathscr{H}$ a sheaf of real-valued "harmonic" functions satisfying the axioms I, II, and III of the Brelot theory [see for example 11, p. 687] with $1 \in \mathscr{H}$. Let D be an open subset of W, and let $\mathscr{B}\mathscr{H}_D$ denote the vector space of bounded $\mathscr{H}$-harmonic functions on D. $\mathscr{B}\mathscr{H}_D$ is an order-complete vector lattice; the order-completeness is a consequence of Harnack's convergence principle, which implies that the pointwise supremum of an upward-directed majorized subset of $\mathscr{B}\mathscr{H}_D$ belongs to $\mathscr{B}\mathscr{H}_D$. Indeed, $\mathscr{B}\mathscr{H}_D$ is an (AM)-space with unit 1, and for each point $z \in D$, the evaluation functional $\varepsilon_z: h \to h(z)$ is an element of $(\mathscr{B}\mathscr{H}_D)^+_*$, so that $(\mathscr{B}\mathscr{H}_D)^+_*$ certainly distinguishes points of $\mathscr{B}\mathscr{H}_D$. If D is connected, then for each $z_0 \in D$ and each relatively compact neighborhood N of z_0, there exists a constant c_N (Harnack's constant) such that

$$\frac{1}{c_N} \cdot \varepsilon_{z_0} \leq \varepsilon_z \leq c_N \cdot \varepsilon_{z_0} \qquad \text{for every } z \in N.$$

We can make c_N as close to 1 as we wish by taking N sufficiently small.

(3.1) PROPOSITION. *Let* X *be a compact Hausdorff space, and let* $T: \mathscr{C}_R(X) \to \mathscr{B}\mathscr{H}_D$ *be a positive linear mapping that sends* 1 *to* 1. *For each* $z \in D$, *let* $\mu_z = T'\varepsilon_z \in \mathscr{M}(X)^+$. *Then*

$$Tf(z) = \int f\,d\mu_z \qquad \textit{for each } f \in \mathscr{C}_R(X) \textit{ and each } z \in D,$$

and the following four assertions hold.

(A) *The function* $z \to \mu_z$ *can be approximated uniformly on compacta in* D *by functions of the form* $z \to \sum_{j=1}^n h_j(z)\cdot\nu_j$, *where* $h_j \in \mathscr{H}_D$ *and* $\nu_j \in \mathscr{M}(X)$ $(j = 1, \cdots, n)$; *in particular, the function is (norm-) continuous.*

(B) *If* D *is connected, then for each* $z_0 \in D$, *each relatively compact neighborhood* N *of* z_0, *and each* $z \in N$,

$$\frac{1}{c_N}\cdot\mu_{z_0} \leq \mu_z \leq c_N\cdot\mu_{z_0},$$

where c_N *is Harnack's constant for* N.

(C) *If* D *is connected and countable at* ∞ *(*σ*-compact), then for each* $z_0 \in D$ *there exists a function*

$$G(z, x) = \sum_{j=1}^{\infty} \lambda_j\cdot h_j(z)\cdot g_j(x),$$

where $\sum_{j=1}^\infty |\lambda_j| < \infty$, $h_j \in \mathscr{H}_D$ $(j = 1, 2, \cdots)$, *and* $h_j \to 0$ *uniformly on compacta as* $j \to \infty$, *and where* $g_j \in \mathscr{L}^\infty(\mu_{z_0})$ $(j = 1, 2, \cdots)$ *and* $g_j \to 0$ *uniformly as* $j \to \infty$. *The function* G *can be chosen so that*

(1) *for each* $z \in D$, $\mu_z = G(z, \cdot)\cdot\mu_{z_0}$ $\left(\textit{hence } \int G(z,\cdot)\,d\mu_{z_0} \equiv 1\right)$;

(2) *for each neighborhood* N *of* z_0, $\frac{1}{c_N} \leq G(z,\cdot) \leq c_N$ *for all* $z \in N$ *(thus* G *is everywhere positive);*

(3) $x \to G(\cdot, x)$ *is a measurable* $\mathscr{H}_D$*-valued function of* $x \in X$ *with respect to the measure* μ_{z_0} (see [1, Chapter IV, Section 5]); *consequently, it is harmonic and measurable in its respective variables.*

(D) *If* X *is metrizable, then* G *can be taken identically* 1 *outside an appropriate* G_δ*-set supporting* μ_{z_0}.

Proof. $\mathscr{H}_D$ is nuclear [11, Theorem 2] and complete; therefore its bounded closed sets are compact [10, p. 101, Corollary 2]. Thus $T'': \mathscr{C}_R(X)'' \to \mathscr{H}_D$, and so the function

$$z \to T''\phi(z) = \langle T''\phi, \varepsilon_z\rangle = \langle \phi, T'\varepsilon_z\rangle = \langle \phi, \mu_z\rangle$$

is $\mathscr{H}$-harmonic for each $\phi \in \mathscr{C}_R(X)''$; the approximation assertion (A) now follows from [11, Proposition 1]. (B) is an immediate consequence of the positivity of T and the Harnack inequality relating ε_z and ε_{z_0}. (B) implies that $T: \mathscr{C}_R(X) \to \mathscr{H}_D$

is continuous when $\mathscr{C}_R(X)$ is given the relativized $L^1(\mu_{z_0})$-norm topology and $\mathscr{H}_D$ is given the topology of uniform convergence on compacta, and so T can be extended to map $L^1(\mu_{z_0})$ continuously into $\mathscr{H}_D$ by density of $\mathscr{C}_R(X)$ and completeness of $\mathscr{H}_D$. (C) and (D) then follow directly from [11, Theorem 2, second corollary], with G being modified on an appropriate set of μ_{z_0}-measure zero if necessary. See also the exhaustive discussion of nuclearity and kernels in [6, Section 5].

(3.2) THEOREM. *Let* D *be a relatively compact open subset of* W, *and let* B *be a linear subspace of* $\mathscr{C}_R(\overline{D})$ *with the three properties*

(1) $1 \in B$,

(2) $f \in B$ *implies* $f \mid D \in \mathscr{H}$,

(3) *there exists a closed subset* $X \subseteq \overline{D}$ *for which the mapping* $f \to f \mid X$ *is an isometry of* B *with a separating linear subspace of* $\mathscr{C}_R(X)$.

Let $\mathscr{G}$ *be any min-stable wedge (necessarily separating) of continuous functions on* X, *with* $B \subseteq \mathscr{G} \cap (-\mathscr{G})$. *Then there exists a* $\mathscr{G}$*-maximal* $T: \mathscr{C}_R(X) \to \mathscr{B}\mathscr{H}_D$ *whose restriction to* $B \mid X$ *is the "identity," that is, which sends* $f \mid X$ *to* f, *and thus has the property that* $\mu_z = T'\varepsilon_z$ *is a* $\mathscr{G}$*-maximal representing measure for each* $z \in D$ *on* B. *Similarly, there exists a positive* $T_1: \mathscr{C}_R(X) \to \mathscr{B}\mathscr{H}_D$ *whose restriction to* $B \mid X$ *is the "identity" for which each representing measure* $\mu_z = T_1'\varepsilon_z$ *is concentrated on the Choquet boundary* bX *of* X *with respect to* B *("concentrated" having its usual meaning: each* μ_z *annihilates each Baire set disjoint from* bX). *In either case, the function* $z \to \mu_z$ *has the properties listed in* (3.1) *above.*

Proof. One may think of the "identity" mapping $U: B \mid X \to \mathscr{B}\mathscr{H}_D$ as a positive linear transformation (sending 1 to 1) of a subspace of $\mathscr{C}_R(X)$ into the order-complete vector lattice $\mathscr{B}\mathscr{H}_D$. (2.2) above will extend U to all of $\mathscr{C}_R(X)$, (2.6) will find a maximal T $\succ$-larger than the extension, and by (2.8) each scalar measure $\mu_z = T'\varepsilon_z$ is $\mathscr{G}$-maximal. Similarly, applying (2.9) and the remark following, we can easily construct T_1. We may now simply apply (3.1) to T or T_1 directly.

If $W = C^n$, $\mathscr{H}$ is the sheaf of harmonic functions in the usual sense, B is the linear space of real parts of functions in the algebra A(D) (see Section 1), and X is the Šilov boundary of $\overline{D}$ with respect to B (or, equivalently, with respect to A(D)), then we have the setting of [2]. The Choquet boundary bX is the set of peak points of elements of A(D). Applying the second part of (3.2), we get a function $z \to \mu_z$ from D to the probability measures supported by bX which represents the points of D for functions in B and therefore for functions in A(D); this function has the properties listed in (3.1), and therefore it is norm-harmonic and thus norm-real-analytic. Similarly, if D is connected, then the kernel function $z \to G(z, \cdot)$ is a norm-harmonic and norm-real-analytic function into the Banach space formed by $\mathscr{L}^\infty(\mu_{z_0})$ equipped with the supremum norm. On the other hand, placing ourselves in the setting of [4, Section 5], taking Edwards' X to be our $\overline{D}$, and taking $\mathscr{G}$ to be the min-stable wedge $\mathscr{F} \cap \mathscr{C}_R(X)$ constructed there, then applying (3.2) (with $X = \overline{D}$), we get a norm-harmonic selection $z \to \mu_z$ of Jensen measures for the points of D with respect to the algebra A(D), with each μ_z supported by Edwards' boundary $\partial_{\mathcal{E}} X$; if D is connected, then again this function $z \to \mu_z$ can be given by a norm-harmonic kernel G, as in (3.1).

Remark. The reader interested only in holomorphic-function algebras can deduce the result above without making use of the contents of [11]: if $D \subseteq C^n$ and $\mathscr{B}\mathscr{H}_D$ is the space of real-valued functions on D that are bounded and harmonic in the usual sense, then the constructions in the proof of (3.2) are valid, and the functions $z \to \mu_z$ are scalarly ("weak-star") harmonic in the sense that for every $f \in \mathscr{C}_R(X)$, $z \to \int f\,d\mu_z$ is a (bounded) harmonic function on D. Using the Poisson kernel for spheres, the reader can easily establish (in the way in which he would use the Cauchy kernel to show that scalarly holomorphic functions are norm-holomorphic) that $z \to \mu_z$ is norm-harmonic, and therefore norm-real-analytic. If D is connected, he can similarly establish the existence of an $\underline{L}^\infty(\mu_{z_0})$-valued function $z \to g_z$ for which $\mu_z \equiv g_z \cdot \mu_{z_0}$; the range of the function $z \to g_z$ is easily seen to be a separable subset of $L^\infty(\mu_{z_0})$, and a lifting argument will produce a function $z \to G(z, \cdot)$ for which $G(z, \cdot)$ is a representative of g_z in $\mathscr{L}^\infty(\mu_{z_0})$. G will be bounded (and so forth), as in (3.1); however, the "infinite series" representation given for G in (3.1) depends on nuclearity.

4. It is not clear what direction the extension of our arguments might take. Of course, any improvement in techniques for producing maximal scalar measures could probably be adapted to this setting, and (as [6] indicates) the harmonic-function axioms of Brelot that we used in Section 3 may well be unnecessarily restrictive. But an order-complete vector lattice in which our linear transformations take their values seems indispensable. One only needs to know that upward-directed majorized sets possess suprema and that L_e is complete in its norm in order to prove propositions like (2.3) and (2.6) (and even monotone completeness suffices for the latter), and one knows this, for example, for the space of bounded pluriharmonic functions on an open set in C^n. However, such propositions are not very useful without a theorem like (2.1). Moreover, an ordered vector space with a monotone completeness property has to be fairly far from being a lattice in order not to be one; in fact, Schaefer [9, Theorem (13.2), p. 123] has shown that the Riesz decomposition property implies lattice-ordering in the presence of monotone completeness. Also, if the dimension n of C^n (in the setting of [2]) is larger than 1, there is no reason for expecting the constant c_N of (3.1) above to be the "part" constant for A(D); therefore, the question raised in [2] concerning whether that constant could be tied to the part metric on the maximal ideal space of the algebra remains unanswered.

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