TWO THEOREMS ABOUT PERIODIC TRANSFORMATIONS OF THE 3-SPHERE

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To R. L. Wilder on his seventieth birthday.

Let T denote a transformation of period $\nu > 1$ of the 3-sphere Σ, and let Λ denote the set of fixed points of T. It is generally believed that *if* Λ *is a tame simple closed curve it must be unknotted,* but this conjecture of P. A. Smith has not yet been proved. In his thesis [4], C. H. Giffen proved that Λ can not be a torus knot. In this note I shall give a new and elementary proof of Giffen's theorem (Giffen's proof makes heavy use of fiber space theory), and then establish a condition that must be satisfied by the group Γ of Λ.

Suppose that Λ is a tame simple closed curve, and denote the orbit space Σ/T by S. Denote the collapsing map $\Sigma \to S$ by e and the image of Λ under e by L. It is known [10] that S is a closed 3-manifold, that L is a tame simple closed curve in S, and that e: $(\Sigma, \Lambda) \to (S, L)$ is a ν-fold cyclic covering, branched over L.

Since $e \mid \Sigma - \Lambda: \Sigma - \Lambda \to S - L$ is an unbranched ν-fold cyclic covering, the group $\Gamma = \pi(\Sigma - \Lambda)$ must be a normal subgroup of index ν of the group $G = \pi(S - L)$. It is known [6] that *the* 3-*manifold* S *must be simply connected.* Since this last statement has appeared in several places [2], [3], [4], [7], [8] without reference, I digress to give a brief proof of it.

Let m be a meridian of L—that is, an element of G represented by a meridian curve on the boundary of a tubular neighborhood of L. Since Λ is the branch curve of the ν-fold cyclic covering e, the element $\mu = m^{\nu}$ of Γ is a meridian of Λ. Since filling in the knot Λ maps Γ onto $\pi(\Sigma) = 1$, the consequence $\langle \mu \rangle$ of the element μ must be all of Γ, and hence $\langle m \rangle$, the consequence in G of m, must contain Γ. Since the elements $1, m, \cdots, m^{\nu-1}$ represent the ν cosets of Γ in G, it follows that $\langle m \rangle$ must be all of G. Now, filling in the knot L maps G onto $\pi(S)$. Since m is thereby mapped into 1, we see that $\pi(S) \approx G/\langle m \rangle = 1$. ∎

Let $\mathscr{A}(\Gamma)$ and $\mathscr{I}(\Gamma)$ denote the group of automorphisms and the group of inner automorphisms, respectively, and denote by $\mathscr{B}(\Gamma)$ the group of those automorphisms of Γ that induce the identity automorphism of Γ/Γ'. Thus an automorphism B of Γ belongs to $\mathscr{B}(\Gamma)$ if and only if each element γ of Γ has the same linking number with Λ as does its image $B(\gamma)$. Of course, $\mathscr{I}(\Gamma) \subset \mathscr{B}(\Gamma) \subset \mathscr{A}(\Gamma)$. The inner automorphism $D_m: g \to mgm^{-1}$ of G maps the normal subgroup Γ onto itself, and so it induces an automorphism Δ_m of Γ (which may or may not be an inner automorphism). Since the inner automorphism D_m^{ν} of G induces the inner automorphism $\Delta_{\mu}: \gamma \to \mu\gamma\mu^{-1}$, we see that $\Delta_m^{\nu} = \Delta_{\mu} \in \mathscr{I}(\Gamma)$. Since a loop in $\Sigma - \Lambda$ has linking number 0 with Λ if and only if the loop in $S - L$ into which it is projected by e has linking number 0 with L, we see that $\Gamma' = G'$; since D_m induces the identity automorphism of G/G', it follows that Δ_m induces the identity automorphism of $\Gamma/\Gamma' = \Gamma/G'$. Thus Δ_m belongs to the group $\mathscr{B}(\Gamma)$ of homologically faithful automorphisms.

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THEOREM 1. Λ *can not be a torus knot.* (Proved in [3] for the special case $(\nu, ab) = 1$, and in [4] in complete generality.)

Suppose that Λ is a torus knot of type a, b, where, of course, a and b are relatively prime integers and $|ab| > 1$. Then Γ has [9] a presentation $(\phi, \psi: \phi^a = \psi^b)$. Schreier has shown [9] that the group of outer automorphisms is cyclic of order two, the generator being $\phi \to \phi^{-1}$, $\psi \to \psi^{-1}$. Since the automorphism of Γ/Γ' that this outer automorphism induces is not the identity, $\mathscr{I}(\Gamma)$ must be all of $\mathscr{B}(\Gamma)$. Thus Δ_m must be an inner automorphism, say $\gamma \to \omega\gamma\omega^{-1}$, where ω is some element of Γ. Since G is an extension of Γ by $G/\Gamma = |m: m^\nu = 1|$, G has a presentation

$$(\phi, \psi, m: \phi^a = \psi^b, \Delta_m(\phi) = m\phi m^{-1}, \Delta_m(\psi) = m\psi m^{-1}, m^\nu = \mu) .$$

Clearly the element $M = \omega^{-1} m$ commutes with both ϕ and ψ and hence with ω; consequently ω and m commute. Thus

$$G = |\phi, \psi, M: \phi^a = \psi^b, [M, \phi] = [M, \psi] = 1, M^\nu = z| ,$$

where z is the element $(\omega^{-1} m)^\nu = \omega^{-\nu} m^\nu = \omega^{-\nu}\mu$ of Γ. Since M belongs to the center of G, the element z of Γ must belong to the center of Γ. Schreier has shown [9] that the center of Γ is the cyclic group generated by the element $\phi^a = \psi^b$. Thus $z = \phi^{ac} = \psi^{bc}$ for some integer c.

Now it is known [5] that $\mu = \psi^\alpha \phi^{-\beta}$, where α and β are positive integers that satisfy the condition $\alpha a - \beta b = 1$. Since Γ/Γ' is the infinite cyclic group generated by μ, and $\mu^b \equiv \phi$, $\mu^a \equiv \psi$ (mod Γ'), it follows from the relation $\omega^{-\nu}\mu = z$ between the elements ω, μ, and z of Γ that abc must be relatively prime to ν. Thus there must be integers σ_1, τ_1, σ_2, τ_2 such that

$$\sigma_1 ac + \tau_1 \nu = 1 \quad \text{and} \quad \sigma_2 bc + \tau_2 \nu = 1 .$$

Define $f = M^{\sigma_1} \phi^{\tau_1}$ and $g = M^{\sigma_2} \psi^{\tau_2}$. Then

$$f^\nu = \phi, \quad f^{ac} = M, \quad g^\nu = \psi, \quad g^{bc} = M .$$

Thus $G = |f, g: f^{ac} = g^{bc}, f^{a\nu} = g^{b\nu}|$. Since c and ν are relatively prime, it follows that

$$G = |f, g: f^a = g^b| .$$

Consider the group $F = |f: f^a = 1| * |g: g^b = 1|$. We have the consistent diagram

$$\begin{array}{ccc} G & \longrightarrow & G/G' \\ \downarrow & & \downarrow \\ F & \longrightarrow & F/F', \end{array}$$

where the horizontal arrows denote abelianization, and the homomorphisms indicated by the vertical arrows are induced by adjoining the relations $f^a = 1$ and $g^b = 1$. Of course, G/G' is the infinite cyclic group $|t:|$, where the abelianization $G \to G/G'$ maps f into t^b and g into t^a. Thus $F/F' = |t: t^{ab} = 1|$, and m is mapped into the element t of F/F'.

In the group F, each element can be written [8] in the unique normal form $f^{\delta_1} g^{\varepsilon_1} \cdots f^{\delta_\ell} g^{\varepsilon_\ell}$, where $0 \leq \delta_1 < a$, $0 < \varepsilon_1 < b$, $0 < \delta_2 < a$, $\cdots$, $0 < \delta_\ell < a$, $0 \leq \varepsilon_\ell < b$. The *length* of this normal form is

$$2\ell \text{ if } \delta_1 \varepsilon_\ell \neq 0,$$

$$2\ell - 1 \text{ if } \delta_1 = 0 \text{ and } \varepsilon_\ell \neq 0 \text{ or } \delta_1 \neq 0 \text{ and } \varepsilon_\ell = 0,$$

$$2\ell - 2 \text{ if } \delta_1 = \varepsilon_\ell = 0.$$

Replacing m by a suitable conjugate if necessary, we may assume that the normal form of m is one of the following:

$$f^{\delta_1} g^{\varepsilon_1} \cdots f^{\delta_\ell} g^{\varepsilon_\ell}, \text{ where } 0 < \delta_i < a,\ 0 < \varepsilon_i < b \text{ for } i = 1, 2, \cdots, \ell \text{ and } \ell \geq 1,$$

$$f^{\delta}, \text{ where } 0 \leq \delta < a,$$

$$g^{\varepsilon}, \text{ where } 0 \leq \varepsilon < b,$$

In the first case, m^ν has normal form of length $2\ell\nu$, and this cannot be shortened by conjugation. But m^ν is conjugate to μ, and as was noted above, $\mu = \psi^\alpha \phi^{-\beta}$. Hence m^ν is conjugate to $\psi^\alpha \phi^{-\beta} = g^{\alpha\nu} f^{-\beta\nu}$, where the exponents are supposed to be reduced mod a and mod b, respectively. Since $g^{\alpha\nu} f^{-\beta\nu}$ is a normal form of length 2, it follows from the fact that $\nu > 1$ that this case can not occur. Hence m must be represented either by f^δ, with $0 < \delta < a$, or by g^ε, with $0 < \varepsilon < b$. Hence in F/F' either $t = t^{\delta b}$ or $t = t^{\varepsilon a}$. Consequently either $\delta b \equiv 1 \pmod{ab}$ or $\varepsilon a \equiv 1 \pmod{ab}$. But neither of these congruences is possible, because $|ab| > 1$. ■

THEOREM 2. *The group $\mathscr{B}(\Gamma)/\mathscr{I}(\Gamma)$ of homologically faithful outer automorphisms must contain an element of order ν.* (This sharpens an observation made by Neuwirth [8, p. 68].)

We shall show that the outer automorphism determined by Δ_m has the required property. We know that $\Delta_m \in \mathscr{B}(\Gamma)$; let d be the smallest positive integer for which $\Delta_m^d \in \mathscr{I}(\Gamma)$. Of course d divides ν, say $\nu = dh$. Let $(\lambda_1, \cdots, \lambda_n : \rho_1, \cdots, \rho_{n-1})$ be a presentation of Γ. Then, since G is an extension of Γ by $G/\Gamma = |m: m^\nu = 1|$, a presentation of G is

$$(\lambda_1, \cdots, \lambda_n, m: \rho_1, \cdots, \rho_{n-1}, \Delta_m(\lambda_1) = m\lambda_1 m^{-1}, \cdots, \Delta_m(\lambda_n) = m\lambda_n m^{-1}, m^\nu = \mu).$$

Since $\Delta_m^d \in \mathscr{I}(\Gamma)$, there is an element ξ of Γ such that $\Delta_m^d(\gamma) = \xi\gamma\xi^{-1}$. Thus the element $Q = \xi^{-1} m^d$ of G commutes with each λ_j and hence also with ξ. Since $\mu = m^{dh} = (\xi Q)^h = \xi^h Q^h$, we see that Q^h must belong to Γ. Since Q^h commutes with each λ_j, it must belong to the center of Γ. By Theorem 1 we know that Λ is not a torus knot. Hence [1] the center of Γ is trivial, and therefore $Q^h = 1$. Since G/G' is infinite cyclic, we see that Q must belong to G'. But, as shown above, $G' = \Gamma'$, hence Q belongs to Γ. Since ξ is an element of Γ, it follows from the equation $Q = \xi^{-1} m^d$ that m^d must belong to Γ. Consequently d must be equal to ν. ■

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