SMOOTH APPROXIMATIONS TO POLYHEDRA

Stewart S. Cairns

1. INTRODUCTION

This paper is concerned with relations between combinatorial and differential topology. In studying such relations, we work with piecewise smooth homeomorphisms restricted to subcomplexes of differentiable triangulations of euclidean n-space. This class of maps and spaces, although it is not a category, is appropriate because it contains the relevant piecewise linear and smooth categories, together with other spaces and maps useful in passing from one of these categories to the other.

1.1. *Vertical bars and carriers.* If V is a set of point sets, $|V|$ will denote the union of the elements of V. If these elements are disjoint, the *carrier* (*in* V) of a point $x \in |V|$ will mean the element of V containing x.

1.2. *Omitted modifiers.* In the foregoing and subsequent definitions, parentheses around a modifier indicate that it will sometimes be omitted for brevity.

We use K with or without indices to denote finite linear simplicial complexes in euclidean n-space E^n. Thus $|K|$ denotes a polyhedron. Since we use open simplexes, each point of $|K|$ has a carrier in K.

Dimensions are indicated, where relevant, by superscripts. The empty set $\emptyset$ has dimension -1. Except when we write K^{-1}, for the trivial complex $\{\emptyset\}$, K denotes an m-complex ($m \geq 0$).

A topological manifold is said to be *closed* if it is compact and has no boundary. A closed (n - 1)-manifold in E^n *surrounds* each subset of its interior.

Closed, half-open, and open directed line segments are denoted by [qp], [qp), (qp], and (qp); [qp] is a *vector*.

The terms *smooth*, *differentiable*, and *of class* C^∞ are used synonymously.

We depart from the piecewise smooth class for the sake of a result involving an (n - 1)-manifold of differentiability class C^1.

THEOREM I. *For each polyhedron* $|K| \subset E^n$ $(n > 1)$, *there exist a closed* C^1-(n - 1)-*manifold* M^{n-1} *and a set* J *of vectors such that*

(a) M^{n-1} *surrounds* $|K|$,

(b) *if* $[qp] \in J$, *then* $p \in M^{n-1}$, $q \in |K|$, *and* [pq] *is normal to* M^{n-1} *at* p,

(c) *the interior* N *of* M^{n-1} *satisfies the conditions*

$$\overline{N} = |J| \quad \textit{and} \quad \overline{N} - |K| = |\{[pq): [pq] \in J\}|$$

(the horizontal bar denotes closure),

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(d) *the half-open segments* $\{[pq): [pq] \in J\}$ *are disjoint,*

(e) *if* $r \in [pq)$ *and* $[pq] \in J$, *then* p *is the unique nearest point of* M^{n-1} *to* r.

The theorems stated in this introductory section will be proved later.

THEOREM II. *For each* m-*complex* $K = K^m \subset E^n$ $(0 \leq m \leq n)$, *there exists a smooth real function* ρ_K *on* E^n *with the following properties:*

(a) ρ_K *is determined by* K *and* $m + 1$ *positive parameters;*

(b) *for a certain pair of neighborhoods* $N_0(K)$ *and* $N(K)$,

$$\rho_K(x) = 0 \qquad (x \in \overline{N}_0(K)),$$

$$0 < \rho_K(x) < 1 \qquad (x \in N(K) - \overline{N}_0(K)),$$

$$\rho_K(x) = 1 \qquad (x \in E^n - N(K));$$

(c) *each set* $\Sigma_t = \{x \in E^n: \rho_K(x) = t\}$ $(0 < t < 1)$ *is a* C^∞-(n - 1)-*manifold surrounding* $|K|$;

(d) *for* $0 < \tau < t < u < 1$, *there exists a smooth isotopy of* E^n *onto itself that takes* Σ_t *onto* Σ_τ *and is the identity on* $|K|$ *and outside* Σ_u.

The closure of the interior of Σ_t can be retracted onto $|K|$ by a homotopy that is an isotopy up to (but not including) the end. The isotopy can be interpreted as a shrinking during which the family $\{\Sigma_s: 0 < s < u\}$ is preserved. Furthermore, the isotopy can be made smooth, except for an arbitrarily small interval near the end; but we believe a final interval of nonsmoothness to be unavoidable.

An alternate known procedure leading to smooth surrounding manifolds of $|K|$ is as follows. On a regular neighborhood N of $|K|$, one can define a polyhedral surrounding manifold P^{n-1}, with a transverse field of line segments covering $N - |K|$. There exists (see [1], [4]) a C^∞-(n - 1)-manifold approximating P^{n-1} and surrounding $|K|$. However, this manifold depends on a triangulation of a neighborhood of $|K|$, and the proof of its existence is not constructive. Our smooth surrounding manifolds $\{\Sigma_t\}$, which depend only on K and on $(1 + \dim K)$ parameters, are defined by a specified function ρ_K. This fact permits simple proofs of Theorem II (d), of the statements in the preceding paragraph, and of the following result.

THEOREM III. *Each special isotopy* (see Section 11) *taking a complex* K_0 *onto a complex* K_1 *can be extended to an isotopy taking a family of surrounding* C^∞-(n - 1)-*manifolds* $\{\Sigma_{0t}: 0 < t < 1\}$ *for* K_0 (see Theorem II) *into such a family* $\{\Sigma_{1t}: 0 < t < 1\}$ *for* K_1 *with* Σ_{0t} *going onto* Σ_{1t}.

We note the related (but only partly developed) work of Thom [3], who defined his surrounding manifolds (*variétés tapissantes*) as equipotentials of a distribution on $|K|$.

Let Σ be one of the manifolds Σ_t, and let N denote its interior. Then $\overline{N}$ and Σ can be built up by a smooth handlebody construction and a sequence of smooth surgeries, respectively. If K is constructed simplex by simplex through a sequence of subcomplexes, then a handlebody addition for $\overline{N}$ and a surgery for Σ correspond to each adjunction of a simplex. In Section 9, we point out an isomorphism between the construction of $\overline{N}$ and the building up of a block-bundle structure [2] in the piecewise linear category.

A block-bundle structure, whether smooth (as in this paper) or piecewise linear, may be the best substitute for a disk bundle in the case of a combinatorial manifold so imbedded in a euclidean space as to admit no disk bundle.

Marston Morse and the writer are preparing a joint paper in which some of the present work will be used to study relations between combinatorial and differential Schoenflies problems.

2. SHORTEST SEGMENTS AND ν-VECTORS

All geometric objects throughout this paper are in E^n.

2.1. *Definitions.* Let ρ denote the euclidean metric. For each point p and each set C, a *nearest point* of C to p is a point $q \in C$ such that $\rho(p, q) = \rho(p, C)$. If q is such a point and $q \neq p$, we call [pq], [pq), (pq], (pq) *shortest segments* from p to C, and we call [qp] a *ν-vector (from* C *at* q) [see 1.2; the symbol ν is suggested by the word *normal*]. If C is a closed smooth manifold, then the ν-vectors from C are the sufficiently short normal vectors.

The symbols $B^n(p, \varepsilon)$ and $S^{n-1}(p, \varepsilon)$ are used for the open euclidean n-ball and the (n - 1)-sphere with center p and radius ε. We also use the notation defined by

$$S_q^{n-1}(p) = S^{n-1}(p, \rho(p, q)) \qquad \text{(center p, through q)},$$

$$B_q^n(p) = B^n(p, \rho(p, q)).$$

For each subset C of E^n and each $\varepsilon > 0$, let

$$\text{(2.1)} \qquad \begin{aligned} &\text{(a) } N(C, \varepsilon) = \{x \in E^n: \rho(C, x) < \varepsilon\}, \\ &\text{(b) } \Sigma(C, \varepsilon) = \{x \in E^n: \rho(C, x) = \varepsilon\}, \end{aligned}$$

with the convention $\Sigma(\emptyset, \varepsilon) = N(\emptyset, \varepsilon) = \emptyset$.

2.2. *The radii* $J(C, \varepsilon)$. If $C \neq \emptyset$, then each vector [qp], of length ε and with $q \in \overline{C}$ and $p \in \Sigma(C, \varepsilon)$ is called a *radius (vector)* of $\Sigma(C, \varepsilon)$. We denote by $J(C, \varepsilon)$ the set of all radius vectors of $\Sigma(C, \varepsilon)$, and by $J^0(C, \varepsilon)$ the set of *open radii* $\{(qp): [qp] \in J(C, \varepsilon)\}$.

2.3. *Characterization of nearest points.* (a) If $p \in C$, then p is the only nearest point of C to p. (b) If $p \in \overline{C} - C$, there is no nearest point of C to p. (c) Suppose $p \in E^n - \overline{C}$. If $\varepsilon > 0$ exists such that $C \cap S^{n-1}(p, \varepsilon) \neq \emptyset$ and $C \cap B^n(p, \varepsilon) = \emptyset$, then $C \cap S^{n-1}(p, \varepsilon)$ is the set of nearest points of C to p. Otherwise, no such nearest points exist. Obviously, $\varepsilon = \rho(C, p)$, if there are nearest points.

LEMMA A. *If* [pq] *is a shortest segment from* p *to* C *and if* $x \in (pq]$, *then* q *is the only nearest point of* C *to* x.

This follows from 2.3 (c) and the relation $\overline{B}_x^n(q) - B_p^n(q) = q$.

LEMMA B. *If* $[qp] \in J(C, \varepsilon)$, *then* (a) [qp] *is a* ν*-vector from* $\overline{C}$ *and a shortest segment from* q *to* $\Sigma(C, \varepsilon)$; *and* (b) [pq] *is a* ν*-vector from* $\Sigma(C, \varepsilon)$ *and a shortest segment from* p *to* $\overline{C}$.

Proof. Since all points of $\Sigma(C, \varepsilon)$ are at distance ε from $\overline{C}$,

$$\text{(2.4)} \qquad B^n(q, \varepsilon) \cap \Sigma(C, \varepsilon) = \emptyset = B^n(p, \varepsilon) \cap \overline{C}.$$

From 2.3 (c), it follows that p is a nearest point of $\Sigma(C, \varepsilon)$ to q and q is a nearest point of $\overline{C}$ to p. The lemma now follows from Definitions 2.1.

COROLLARY B1. *If* $[qp] \in J(C, \varepsilon)$, *then* (a) $\overline{C} \cap [qp] = q$ *and* $\Sigma(C, \varepsilon) \cap [qp] = p$, *and* (b) *the open radii* $J^0(C, \varepsilon)$ *are disjoint.*

Proof. Part (a) follows from (2.4). If (b) were false, there would exist two unequal radii (qp) and (sr) in $J^0(C, \varepsilon)$ with a common point x. Lemmas A and B would imply that $p = r$ and $q = s$, so that $(qp) = (sr)$.

COROLLARY B2. *If* $J(C, \varepsilon)$ *covers* $N(C, \varepsilon) - \overline{C}$, *then* $|J^0(C, \varepsilon)| = N(C, \varepsilon) - \overline{C}$, *and each point of* $N(C, \varepsilon) - \overline{C}$ *has a carrier in* $J^0(C, \varepsilon)$.

Proof. This corollary follows from Corollary B1 and Section 1.1.

3. SURROUNDING MANIFOLDS

3.1. *The function* ϕ *and the angle function* α. Let J' denote a set of disjoint segments (closed, open, or half-closed), and let S^{n-1} denote the unit sphere in E^n. The function *associated with* J' will mean the function $\phi\colon |J'| \to S^{n-1}$ such that, for each $x \in N(C, \varepsilon) - \overline{C}$, $\phi(x)$ is the unit vector parallel to the carrier of x in $|J'|$. We shall denote by α the angle function for segments and lines, with the stipulation that $0 \leq \alpha \leq \pi$ if both arguments are directed and $0 \leq \alpha \leq \pi/2$ otherwise. Segments are directed by definition, but lines need not be.

THEOREM IV. *If* $J^0(C, \varepsilon)$ *covers* $N(C, \varepsilon) - \overline{C}$, *then the function associated with* $J^0(C, \varepsilon)$ *is defined and is a map* $\phi\colon N(C, \varepsilon) - \overline{C} \to S^{n-1}$.

Proof. From Corollaries B1 and B2 it follows that ϕ is defined and is a function $N(C, \varepsilon) - \overline{C} \to S^{n-1}$. We shall deduce its continuity from the next lemma.

LEMMA C. *Under the hypotheses of the theorem, let* (qp) *denote the carrier in* $J^0(C, \varepsilon)$ *of a point* $x \in N(C, \varepsilon) - \overline{C}$. *Then, for each* $\varepsilon' > 0$, *there exists a* $\delta > 0$ *so small that each element of* $J^0(C, \varepsilon)$ *intersecting* $B^n(x, \delta)$ *also intersects* $B^n(q, \varepsilon')$.

Proof. Let δ be so small that $B^n(x, \delta) \subset B^n_q(p) \cap [N(C, \varepsilon) - \overline{C}]$. If $y \in B^n(x, \delta)$ and (sr) is the carrier of y in $J^0(C, \varepsilon)$ [see Corollary B2], then, by 2.3 (c), $s \in S^{n-1}_s(y) - B^n_q(p)$. We can arbitrarily restrict the angle $\alpha([px], [py])$ by restricting the upper bound δ on $\rho(x, y)$. By a sufficient restriction on that angle, we can ensure that the part $S^{n-1}_s(y) - B^n_q(p)$ of $S^{n-1}_s(y)$ which "bulges out" of $B^n_q(p)$ is entirely within distance ε' of q. This proves Lemma C.

COROLLARY C1. *For each* $\theta > 0$ *there exists a* $\delta > 0$ *such that* $\alpha((qp), (sr)) < \theta$ *if* $(sr) \in J^0(C, \varepsilon)$ *and* (sr) *intersects* $B^n(x, \delta)$.

Proof. Let δ and ε' be so small that each line meeting both $B^n(q, \varepsilon')$ and $B^n(x, \delta)$ makes an angle less than θ with (qx), where ε' and δ are related as in Lemma C. The corollary follows because $(qx) \subset (qp)$ and $(yz) \subset (sr)$, so that $\alpha((qx), (yz)) = \alpha((qp), (sr))$.

Corollary C1 completes the proof of Theorem IV.

LEMMA D. *If* $J^0(C, \varepsilon)$ *covers* $N(C, \varepsilon) - \overline{C}$, *then* (a) $\Sigma(C, t\varepsilon)$ *is a* C^1-(n - 1)-*manifold for each* t $(0 < t < 1)$, *and* (b) $N(C, \varepsilon) - \overline{C}$ *is the disjoint union of the manifolds* $\Sigma(C, t\varepsilon)$ $(0 < t < 1)$. *If* C *is of finite diameter, each* $\Sigma(C, t\varepsilon)$ *surrounds* C. *The hypothesis of this lemma is fulfilled for all* $\varepsilon > 0$ *if* C *is convex, and for no* ε *if* C *is a triangle, for example.*

Proof. The second sentence of Lemma D is easily verified.

Let x be a point of $N(C, \varepsilon) - \overline{C}$. Then, for some t $(0 < t < 1)$, $\rho(C, x) = t\varepsilon$, so that $x \in \Sigma(C, t\varepsilon)$. Let (qp) denote the carrier of x in $J^0(C, \varepsilon)$ [Corollary B2]. Then x is a nearest point of $\Sigma(C, t\varepsilon)$ to p and to q.

Hence, the spheres $S_x^{n-1}(q)$ and $S_x^{n-1}(p)$, which are externally tangent at x, enclose no points of $\Sigma(C, t\varepsilon)$ [2.3 (c)]. Since $\rho(q, y) < t\varepsilon$ for $y \in B_x^n(q)$ and $\rho(q, z) > t\varepsilon$ for $z \in B_x^n(p)$, it follows from Theorem IV that a radius $(sr) \in J^0(C, \varepsilon)$ that intersects a sufficiently small neighborhood of x must meet $S_x^{n-1}(q)$, $\Sigma(C, t\varepsilon)$, and $S_x^{n-1}(p)$ in one point each, in the order named. It follows that $\Sigma(C, t\varepsilon)$ is an $(n - 1)$-manifold and that the common tangent plane of $S_x^{n-1}(p)$ and $S_x^{n-1}(q)$ is tangent to $\Sigma(C, t\varepsilon)$. The remainder of the proof and the following corollaries offer no difficulty.

COROLLARY D1. *Under the hypothesis of Lemma* D, $J^0(C, \varepsilon)$ *is a field of open line segments orthogonal to the family of manifolds* $\{\Sigma(C, t\varepsilon)\colon 0 < t < 1\}$.

COROLLARY D2. *If the hypothesis of Lemma* D *is satisfied and if either* C *or* $E^n - C$ *is of finite diameter, then the map* ϕ *is onto.*

4. ADMISSIBLE SETS

Notation. Hereafter, with occasional exceptions, if a capital letter denotes a simplex, the corresponding lower case letter will denote its dimension, so that $A = A^a$, $B = B^b$, and so on. It will be understood that each simplex mentioned is in K^m.

Let K^m be an m-complex $(0 \le m < n)$, and let $\zeta = (\zeta_0, \cdots, \zeta_m)$ be an ordered set of $m + 1$ positive numbers. We shall use the notation

$$(4.1)\quad \begin{aligned} &\text{(a)}\ N(K^m, \zeta) = \bigcup \{N(A, \zeta_a)\colon A \in K^m\} \qquad \text{[see (2.1)]},\\ &\text{(b)}\ \Sigma(K^m, \zeta) = \overline{N}(K^m, \zeta) - N(K^m, \zeta). \end{aligned}$$

Note that $N(K^m, \zeta)$ is a neighborhood of $|K^m|$. Together with $\Sigma(K^m, \zeta)$, it is determined by K^m and ζ.

Admissibility conditions. We call $\zeta = (\zeta_0, \cdots, \zeta_m)$, $N(K, \zeta)$, and $\Sigma(K, \zeta)$ *admissible* (*for a subcomplex* K of K^m) [see 1.2] provided the following three conditions are fulfilled: (a) $0 < \zeta_{a+1} < \zeta_a/2$ $(a = 0, \cdots, m - 1)$; (b) if K^a is the a-skeleton of K and $0 \le a < \dim K$, then ζ_a is less than the distance from $|K^a|$ to the set of barycenters of the simplexes $K - K^a$; (c) if $A, B \in K$ and

$$(4.2)\qquad \overline{A} \cap \overline{B} = \overline{C} \qquad (-1 \le c < \min(a, b)),$$

then

$$(4.3)\qquad \overline{N}(A, \zeta_a) \cap \overline{N}(B, \zeta_b) \subset N(C, \zeta_c/2) \qquad (\zeta_c = 1 \text{ if } c = -1).$$

LEMMA E. *If* $X \subset E^n$, $Y \subset E^n$, *and* $\overline{X}$ *is compact; then, for each* $\varepsilon > 0$, *there exists a number* $\gamma > 0$ *such that* $\overline{N}(X, \gamma) \cap \overline{N}(Y, \gamma) \subset N(\overline{X} \cap \overline{Y}, \varepsilon)$. *The latter condition implies*

$$\overline{N}(X, \gamma') \cap \overline{N}(Y, \gamma'') \subset N(\overline{X} \cap \overline{Y}, \varepsilon) \qquad (0 < \gamma' \le \gamma,\ 0 < \gamma'' \le \gamma).$$

The proof of Lemma E is straightforward.

THEOREM V. *For each finite complex* $K^m \subset E^n$ $(0 \leq m < n)$, *admissible sets exist.*

Our proof will be inductive.

Basic step. There exists a set $\underline{\zeta}$ satisfying (1) admissibility conditions (a) and (b) and (2) the condition $\overline{N}(A, \zeta_0) \cap \overline{N}(B, \zeta_0) = \emptyset$ for each pair $\{A, B\} \subset K$ such that $\overline{A} \cap \overline{B} = \emptyset$. This follows from Lemma E with $\overline{X} \cap \overline{Y} = \emptyset$.

Hypothesis. For some $h \in (1, \cdots, m-1)$, there exists a set ζ satisfying (1) admissibility conditions (a) and (b), and (2) the condition

$$\overline{N}(A^a, \zeta_{c+1}) \cap \overline{N}(B^b, \zeta_{c+1}) \subset N(C^c, \zeta_c/2) \tag{4.4}$$

for each triple $\{A^a, B^b, C^c\} \subset K^m$ such that $\overline{A}^a \cap \overline{B}^b = \overline{C}^c$ $(-1 \leq c < h-1)$.

The basic step above verifies the hypothesis for the case $h = 1$. By Lemma E, if ζ_h is sufficiently small, then

$$\overline{N}(A^a, \zeta_h) \cap \overline{N}(B^b, \zeta_h) \subset N(A^{h-1}, \zeta_{h-1}/2)$$

for each triple $\{A^a, B^b, A^{h-1}\}$ in K^m satisfying (4.2) with $c = h-1$. With ζ_h thus restricted, admissibility condition (a) merely imposes new upper bounds on $(\zeta_{h+1}, \cdots, \zeta_m)$, and admissibility condition (b) continues to hold.

We have proved that ζ can be chosen so as to satisfy admissibility conditions (a) and (b) and condition (4.4) for each triple $\{A, B, C\} \subset K$ such that (4.2) holds. Since $a > c$ and $b > c$, admissibility condition (a) implies that $\zeta_a \leq \zeta_{c+1}$ and $\zeta_b \leq \zeta_{c+1}$. Condition (4.4) therefore implies (4.3), by the last sentence in Lemma E. This completes our proof of Theorem V.

5. GEOMETRIC LEMMAS AND DEFINITIONS

Notation. Linear k-dimensional subspaces of E^n will be denoted by E^k $(0 \leq k < n)$. A formula $\pi: E^n \to E^k$ will mean that π is the projection (orthogonal if $k > 0$) of E^n onto E^k.

Definitions. The sets $\Sigma(E^k, \varepsilon)$ and $N(E^k, \varepsilon)$ [see (2.1)] will be called the *cylindrical* (n - 1)-*manifold* and *region,* respectively, with *axis* E^k and *radius* ε $(0 \leq k < n)$. Note that each $\Sigma(E^0, \varepsilon)$ is a sphere and each $\Sigma(E^{n-1}, \varepsilon)$ is a pair of parallel (n - 1)-planes.

LEMMA F. *Let* X *be a subset of a* k-*plane* E^k $(0 < k < n)$. *A point* q *is a nearest point of* X *to a point* $p \in E^n$ *if and only if* q *is a nearest point of* X *to* πp $(\pi: E^n \to E^k)$.

This is a consequence of the following obvious generalization of the Pythagorean Theorem:

$$\rho^2(X, p) = \rho^2(X, \pi p) + \rho^2(p, \pi p) \qquad (p \in E^n,\ X \subset E^k). \tag{5.1}$$

The following results are similarly established.

LEMMA G. *Let* X *be a subset of* E^k $(0 < k < n)$, *and let* $Y = \Sigma(E^k, \varepsilon) \cap \pi^{-1}X$ *for some* $\varepsilon > 0$ $(\pi: E^n \to E^k)$. *A point* p *is a nearest point of* Y *to a point of* E^k *if and only if* πp *is a nearest point of* X *to* q.

LEMMA H. *Let* X *be an open set relative to* E^k $(0 < k < n)$. *Then* (a) $\Sigma(X, \varepsilon)$ *intersects* $\Sigma(E^k, \varepsilon)$ *in the set*

$$\Sigma(E^k, \varepsilon) \cap \pi^{-1}\overline{X} \qquad (\pi\colon E^n \to E^k)$$

(b) *the part of* $\Sigma(X, \varepsilon)$ *not on* $\Sigma(E^k, \varepsilon)$ *is on* $N(E^k, \varepsilon) \cap \Sigma(\overline{X} - X, \varepsilon)$, *and*
(c) $\overline{N}(X, \varepsilon) - \pi^{-1}X \subset \overline{N}(\overline{X} - X, \varepsilon)$.

6. CONSTRUCTION OF ADMISSIBLE NEIGHBORHOODS AND BOUNDARIES

Notation. Hereafter, K^m denotes an arbitrary, fixed m-complex in E^n $(0 \leq m < n)$; K^a is its a-skeleton $(-1 \leq a \leq m)$; K denotes a subcomplex of K^m; and $\zeta = (\zeta_0, \cdots, \zeta_m)$ denotes an arbitrary, fixed admissible set for K^m. The boundary complex of a simplex A is denoted by $\beta A = \{B\colon B < A\}$, $\beta A^0 = K^{-1} = \{\emptyset\}$, where $<$ means *"is a boundary face of."*

Definition of a general step. Assume that a concept, process, or property Q has been defined or proved for all subcomplexes of K - A, where A is a principal simplex of some complex $K \subset K^m$ (that is, A is not a face of any other simplex of K). A *general step* (*in defining or proving* Q for K^m) consists in giving a definition or proof of Q for K based on the assumption of Q for K - A. We shall sometimes assume that dim K = dim A, implying a construction of K^m by successive additions of simplexes in order of nondecreasing dimensions.

LEMMA J. *The admissibility of* ζ *implies that* $\overline{N}(K^0, \zeta)$ *and* $\Sigma(K^0, \zeta)$ *are the disjoint unions of the closed* n*-balls* $B^n(A^0, \zeta_0)$ *and of the* (n - 1)*-spheres* $S^{n-1}(A^0, \zeta_0)$, *respectively* $(A^0 \in K^0)$.

This lemma, a direct consequence of the definitions, will be basic to several inductive proofs.

We symbolize the general steps in defining $\overline{N}(K^m, \zeta)$ and $\Sigma(K^m, \zeta)$ as a pair of *transitions*

$$(6.1)\qquad \begin{aligned} &(a)\ \overline{N}(K - A, \zeta) \twoheadrightarrow \overline{N}(K, \zeta) \qquad (a > 0),\\ &(b)\ \Sigma(K - A, \zeta) \twoheadrightarrow \Sigma(K, \zeta). \end{aligned}$$

Lemma J takes care of the cases a = 0.

The relation

$$(6.2)\qquad \overline{N}(|\beta A|, \zeta_a) \subset N(\beta A, \zeta/2) \subset N(\beta A, \zeta)$$

is a consequence of admissibility conditions (c) and (a) (with βA in place of K^m) in Section 4. It implies the following result.

LEMMA K. *If* $B \in K - \beta A$, *then* $\overline{N}(A, \zeta_a) \cap \overline{N}(B, \zeta_b) \subset N(\beta A, \zeta/2)$.

LEMMA L. *Let*

$$(6.3)\qquad \begin{aligned} &(a)\ \overline{\triangle}(A, \zeta) = \overline{N}(A, \zeta_a) - N(\beta A, \zeta),\\ &(b)\ \overline{\sigma}(A, \zeta) = \Sigma(\beta A, \zeta) \cap \overline{N}(A, \zeta_a),\\ &(c)\ \overline{\Gamma}(A, \zeta) = \Sigma(A, \zeta_a) - N(\beta A, \zeta). \end{aligned}$$

Then

$$(6.4)\qquad \begin{aligned}&(a)\ \overline{N}(K, \zeta) = \overline{N}(K - A, \zeta) \cup \overline{\Delta}(A, \zeta),\\ &(b)\ \Sigma(K, \zeta) = [\Sigma(K - A, \zeta) - \overline{\sigma}(A, \zeta)] \cup \overline{\Gamma}(A, \zeta).\end{aligned}$$

Proof. The transition (6.1) (a) can obviously be made by adjoining $\overline{N}(A, \zeta_a) - N(K - A, \zeta)$ to $\overline{N}(K - A, \zeta)$. But, by Lemma K,

$$\overline{\Delta}(A, \zeta) = \overline{N}(A, \zeta_a) - N(K - A, \zeta).$$

In fact, equations (6.3) all hold with K - A substituted for βA. These equations, after the substitution, imply equations (6.4).

Definitions. The set $\overline{\sigma}(A, \zeta)$ is the *deleted part* of $\Sigma(K - A, \zeta)$. The sets $\overline{\Delta}(A, \zeta)$ and $\overline{\Gamma}(A, \zeta)$ are the *adjoined parts* of $\overline{N}(K, \zeta)$ and of $\Sigma(K, \zeta)$, respectively.

COROLLARY L1. *The transition* $\overline{N}(K - A, \zeta) \to\to \overline{N}(K, \zeta)$ *can be effected by adjoining* $\overline{\Delta}(A, \zeta)$ *to* $\overline{N}(K - A, \zeta)$. *The transition* $\Sigma(K - A, \zeta) \to\to \Sigma(K, \zeta)$ *can be effected by deleting* $\overline{\sigma}(A, \zeta)$ *from* $\Sigma(K - A, \zeta)$, *then adjoining* $\overline{\Gamma}(A, \zeta)$ *to the resulting set. The deleted and adjoined parts depend only on* $A \cup \beta A$ *and* ζ.

We have stated Corollary L1, which follows directly from Lemma L, so as to prepare for a later interpretation of the transition $\overline{N}(K - A, \zeta) \to\to N(K, \zeta)$ as the addition of a handle and of $\Sigma(K - A, \zeta) \to\to \Sigma(K, \zeta)$ as a surgery.

Let $\eta = (\eta_0, \cdots, \eta_a)$ be an ordered set of $a + 1$ positive numbers. The *η-collar* and the *η-core* of A are defined as follows [see (4.1)]:

$$(6.5)\qquad \begin{aligned}&(a)\ c(A, \eta) = A \cap N(\beta A, \eta) \qquad \text{(the } \eta\text{-collar)},\\ &(b)\ \gamma(A, \eta) = A - \bar{c}(A, \eta) \qquad \text{(the } \eta\text{-core)}.\end{aligned}$$

The *closed* η-collar and η-core are $\bar{c}(A, \eta)$ and $\bar{\gamma}(A, \eta)$, respectively.

LEMMA M. *The adjoined part* $\overline{\Gamma}(A, \zeta)$ *of* $\Sigma(K, \zeta)$ *is given by the formula*

$$\overline{\Gamma}(A, \zeta) = \Sigma(E, \zeta) \cap \pi^{-1}\bar{\gamma}(A, \eta) \qquad (\pi\colon E^n \to E^a),$$

where $E^a \supset A$ *and*

$$(6.6)\qquad \eta_c = (\zeta_c^2 - \zeta_a^2)^{1/2} \qquad (c = 0, \cdots, a - 1).$$

Proof. The relation

$$\Sigma(A, \zeta_a) - \Sigma(E^a, \zeta_a) \cap \pi^{-1}\overline{A} \subset \Sigma(|\beta A|, \zeta_a) \subset N(\beta A, \zeta)$$

follows from Lemma H and relation (6.1). Hence, by (6.3) (c), $\overline{\Gamma}(A, \zeta)$ is the closure of the part of $\Sigma(E^a, \zeta_a) \cap \pi^{-1}\overline{A}$ outside all the manifolds $\{\Sigma(C, \zeta_c)\colon C \in \beta A\}$. That is, $p \in \overline{\Gamma}(A, \zeta)$ if and only if

$$\pi p \in \overline{A}, \qquad \rho(p, \pi p) = \zeta_a, \qquad \rho(C, p) \geq \zeta_c.$$

Applying relations (5.1) with X = C, we conclude that $p \in \overline{\Gamma}(A, \zeta)$ if and only if

$$p \in \Sigma(E^a, \zeta_a) \cap \pi^{-1}\overline{A} \quad \text{and} \quad \rho(C, \pi p) \geq (\zeta_c^2 - \zeta_a^2)^{1/2} = \eta_c \qquad (C \in \beta A).$$

The lemma follows.

For each $q \in E^a$, the set $\Sigma(E^a, \zeta_a) \cap \pi^{-1} q$ is the $(n - a - 1)$-sphere with center q and radius ζ_a in the normal $(n - a)$-plane to E^a through q. This yields the following result.

COROLLARY M1. *The adjoined set* $\overline{\Gamma}(A, \zeta)$ *is homeomorphic to* $\bar{\gamma}(A, \eta) \times S^{n-a-1}$.

7. THE RADII. PROOF OF THEOREM I

Definition. A *radius* of $\Sigma(K, \zeta)$ will mean a shortest segment from a point of $|K|$ to the set $\Sigma(K, \zeta)$. The set of all such radii will be denoted by $J(K, \zeta)$. If $[qp] \in J(K, \zeta)$, then (qp) will be called an *open radius*. Let

$$J^0(K, \zeta) = \{(qp) | [qp] \in J(K, \zeta)\} .$$

The sets of negatives of the radii of $\Sigma(K, \zeta)$ will be denoted by

$$J_*(K, \zeta) = \{[pq]: [qp] \in J(K, \zeta)\} ,$$

$$J_*^0(K, \zeta) = \{(pq): [pq] \in J_*(K, \zeta)\} .$$

They are ν-vectors from $\Sigma(K, \zeta)$ [Definitions 2.1].

THEOREM VI. (a) $\overline{N}(K^m, \zeta) = |J(K^m, \zeta)|$ *and* $N(K^m, \zeta) - |K| = |J^0(K^m, \zeta)|$. (b) *The radii* $J^0(K^m, \zeta)$ *are disjoint.* (c) *The function*

$$\psi: N(K^m, \zeta) - |K^m| \to S^{n-1} \text{ (the unit sphere)}$$

that maps each point $x \in (qp) \in J^0(K^m, \zeta)$ *onto the unit vector parallel to* (qp) *is continuous and onto.*

Hypothesis. Theorem VI holds for some complex $K - A \subset K^m$ $(a > 0)$, where A is a principal simplex of K.

LEMMA N. *The transition* $J(K - A, \zeta) \to\to J(K, \zeta)$ *can be made as follows:*

(a) *From* $J(K - A, \zeta)$ *delete the set* J_δ *of radii terminating on* $N(A, \zeta_a)$.

(b) *To* $J(K - A, \zeta) - J_\delta$ *adjoin the radii* J_γ *of* $\Sigma(E^a, \zeta_a)$ $(E^a \supset A)$ *terminating on* $\overline{\Gamma}(A, \zeta)$ [Lemma L].

(c) *Finally, adjoin the radii* J_c *of* $\Sigma(K, \zeta)$ *from points of the* η*-collar* $c(A, \eta)$.

Proof. (a) *The deletions.* If $[qp] \in J(K - A, \zeta)$, then p is a nearest point of $\Sigma(K - A, \zeta)$ to q. If $p' \in \Sigma(K, \zeta) - \Sigma(K - A, \zeta)$, then p' is outside (and q is inside) $\Sigma(K - A, \zeta)$. Therefore,

$$\rho(q, p') > \rho(q, \Sigma(K - A, \zeta)) = \rho(q, p) .$$

It follows that $[qp] \in J(K, \zeta)$ unless, of course, $p \in N(A, \zeta_a) \subset N(K, \zeta)$.

(b) *The addition of* J_γ. The radii of $\Sigma(E^a, \zeta_a)$ are of length ζ_a, and $\overline{\Gamma}(A, \zeta)$ is the part of $\Sigma(K, \zeta)$ at distance ζ_a from A.

(c) *The addition of* J_c. As a consequence of the hypothesis, the terminal points of the radii $(J(K - A, \zeta) - J_\delta) \cup J_\gamma$ cover $\Sigma(K, \zeta)$. The set $J(K - A, \zeta) - J_\delta$ contains the radii from $|\beta A|$ to $\Sigma(K, \zeta)$. Hence, by the preceding two paragraphs and Lemma M, only the radii from $c(\gamma, \eta)$ remain to be introduced.

We complete the proof of Theorem VI by showing that

$$|J_c| = \overline{N}(K, \zeta) - |(J(K - A, \zeta) - J_\delta) \cup J_\gamma| = N_c \quad \text{(introducing } N_c\text{)}.$$

From the proof of Lemma N, it follows that the deleted radii begin on $|\beta A|$ and terminate on $\Sigma(K - A, \zeta) \cap N(A, \zeta_a)$. As in the proof of Lemma M, one can verify that the projection $\pi: E^n \to E^a \supset \overline{A}$ maps $|J_\delta|$ into $\overline{A}$. One can also verify the relation

$$N_c \subset N(A, \zeta_a) \cap \pi^{-1} c(\gamma, \eta).$$

Let $[qp]$ be a radius from a point $q \in |\beta A|$ to a point $p \in \overline{\Gamma}(A, \zeta)$. Then, by the proof of Lemma N, p is also the terminal point of a radius $[q'p] \in J_\gamma$, where $q' \in \bar{\gamma}(A, \eta) - \gamma(A, \eta)$. From Lemmas F and G it follows that q' is a nearest point of $\bar{\gamma}(A, \eta)$ to q and that q is a nearest point of $\overline{\Gamma}(A, \zeta)$ to q'' for each $q'' \in [qq']$. Since a radius from q'' cannot intersect (qp) or $(q'p)$ [Lemma A], $[q''p] \in J_c$. Thus J_c covers the closure $[qq'p]$ of the interior of the triangle $qq'p$. The union of the closed triangular regions $[qq'p]$ as $[qp]$ ranges over the radii from points of $|\beta A|$ to points on $\overline{\Gamma}(A, \zeta) \cap \Sigma(K - A, \zeta)$ is easily shown to be $\overline{N}_c$. Parts (b) and (c) of Theorem VI are proved by the methods used to prove Corollary B1 (b) and Theorem IV.

Proof of Theorem I. Let $C = E^n - N(K, \zeta)$, and let $\varepsilon = \zeta_a$, where $a = \dim K$. The hypotheses of Lemma D are satisfied, with $|J^0(C, \varepsilon)| \subset |J_*^0(K, \zeta)|$. Therefore $M^{n-1} = \Sigma(C, t\varepsilon)$ satisfies the requirements of Theorem I for each t $(0 < t < 1)$, where J is the set of closed subvectors of $J_*(K, \zeta)$ from M^{n-1} to $|K|$. The manifold M^{n-1} can be confined to an arbitrary neighborhood of $|K|$ by an upper bound on ζ_0. This is consistent with the proof of Theorem V. We remark that the lengths of the inner normals from $\Sigma(C, t\varepsilon)$ take on values from $(1 - t)\zeta_a$ to $\zeta_0 - t\zeta_a$, inclusively.

8. SMOOTH SURROUNDING MANIFOLDS

Let μ be the function defined on the reals by

$$\mu(t) = \begin{cases} 0 & (t^2 - t \geq 0), \\ \exp (t^2 - t)^{-1} & (t^2 - t < 0), \end{cases}$$

and let ω be defined by

$$\omega(t) = k \int_{-\infty}^{t} \mu(\tau) d\tau ,$$

where k is chosen so that $\omega(t) = 1$ $(t \geq 1)$. For each pair of parameters u and v $(u < v)$, let $\omega_{u,v}$ be defined by

$$\omega_{u,v}(t) = \omega\left(\frac{t - u}{v - u}\right) .$$

Then $\omega_{u,v}$ is smooth, and

$$\omega_{u,v}(t) = \begin{cases} 0 & (t \leq u), \\ 1 & (t \geq v), \end{cases}$$

$$\omega'_{u,v}(t) > 0 \qquad (u < t < v).$$

For each $A \in K^m$, let a function ρ_A be defined on E^n by $\rho_A(x) = \omega_{u,v}(\rho(A, x))$ $(u = \zeta_a/2,\ v = \zeta_a)$. Then it follows directly from the definitions that

$$\text{(8.1)}\quad \begin{aligned} &\text{(a) } \overline{N}(A, \zeta_a/2) = \{x: \rho_A(x) = 0\}, \\ &\text{(b) } N(A, \zeta_a) - \overline{N}(A, \zeta_a/2) = \{x: 0 < \rho_A(x) < 1\}, \\ &\text{(c) } E^n - N(A, \zeta_a) = \{x: \rho_A(x) = 1\}. \end{aligned}$$

For $K \subset K^m$, let ρ_K be defined by

$$\rho_K(x) = \prod_{A \in K} \rho_A(x).$$

Relations (8.1) then imply that

$$\text{(8.2)}\quad \begin{aligned} &\text{(a) } \overline{N}(K, \zeta/2) = \{x: \rho_K(x) = 0\}, \\ &\text{(b) } N(K, \zeta) - \overline{N}(K, \zeta/2) = \{x: 0 < \rho_K(x) < 1\}, \\ &\text{(c) } E^n - N(K, \zeta) = \{x: \rho_K(x) = 1\}. \end{aligned}$$

LEMMA O. *The function* ρ_K *is smooth on* E^n.

Proof. If $\dim K = 0$, this is obvious.

Hypothesis. For some complex K with principal simplex A $(a > 0)$, ρ_{K-A} is smooth.

On $\pi^{-1}A$ $(\pi: E^n \to E^a \supset A)$, ρ_A is smooth, being a smooth function of the distance from E^a. For $x \in E^n - \overline{N}(A, \zeta_a)$, $\rho_A(x) = 1$. The complement of $\pi^{-1}A \cup (E^n - \overline{N}(A, \zeta_a))$ is on $N(\dot{A} - A, \zeta_a)$, which is a subset of

$$N(\beta A, \zeta/2) \subset N(K, \zeta/2),$$

by admissibility condition (a) [Section 4] and the definitions. Therefore, by (8.2) (a), ρ_A is smooth wherever $\rho_{K-A} \neq 0$. Hence $\rho_K = \rho_{K-A}\rho_A$ is smooth.

THEOREM VII. *For each* τ $(0 < \tau < 1)$, *the set* $\Sigma^*_\tau(K, \zeta) = \{x: \rho_K(x) = \tau\}$ *is a smooth manifold surrounding* K. *It is on* $N(K, \zeta) - \overline{N}(K, \zeta/2)$, *and* $J^0(K, \zeta)$ *is a field of transverse open segments with* Σ^*_τ *as a section.*

Proof. On each $[p_0 p_1] \in J(K, \zeta)$, introduce a linear parameter z that increases from 0 at p_0 to 1 at p_1. Then, by definition,

$$N(K, \zeta) - \overline{N}(K, \zeta/2) = |\{(p_{1/2} p_1): [p_0 p_1] \in J(K, \zeta)\}|.$$

If $N(A, \zeta_a)$ intersects $(p_{1/2}p_1) \subset [p_0 p_1] \in J(K, \zeta)$, then $(p_{1/2}p_1) \cap N(A, \zeta_a) \subset \pi^{-1}(A)$ (π: $E^n \to E^a \supset A$). For

$$N(A, \zeta_a) - \pi^{-1}(A) \subset N(\beta A, \zeta/2) \subset N(K, \zeta/2),$$

as consequence of admissibility condition (a). Therefore, π maps $(p_{1/2}p_1)$ into A, which implies that $\rho(A, p_t)$ increases linearly with t on $(p_{1/2}p_1)$. It follows that the restriction of ρ_K to $(p_{1/2}p_1)$ is a product of factors each of which, at a point $p_z \in (p_{1/2}p_1)$, is either equal to 1 and has derivative zero with respect to z, or is between 1/2 and 1 and has a positive derivative. Theorem VII now follows, the smoothness being a consequence of Lemma O.

Theorem II is now easy to prove.

9. HANDLEBODIES AND BLOCK-BUNDLES

The results in this section are simple consequences of our previous results and constructions. We give brief indications of proofs.

THEOREM VIII. *Let* $N^*_\tau(K, \zeta)$ *be the neighborhood of* $|K|$ *bounded by* $\Sigma^*_\tau(K, \zeta)$ [Theorem VII], *and consider the transitions*

(a) $\overline{N}(K - A, \zeta) \to\to \overline{N}(K, \zeta)$,

(b) $\overline{N}^*_\tau(K - A, \zeta) \to\to \overline{N}^*_\tau(K, \zeta)$,

(c) $\Sigma(K - A, \zeta) \to\to \Sigma(K, \zeta)$,

(d) $\Sigma^*_\tau(K - A, \zeta) \to\to \Sigma^*_\tau(K, \zeta)$.

Each of steps (a) *and* (b) *is a piecewise smooth handlebody addition of type* a. *Steps* (c) *and* (d) *are, respectively, a piecewise smooth and a smooth surgery, in which a* $D^a \times S^{n-a-1}$ *replaces an* $S^{a-1} \times D^{n-a}$ [D is a topological disk, or closed ball].

Proof. In Lemma L, the adjoined part of $\overline{N}(K, \zeta)$, $\overline{\Delta}(A) = \overline{N}(A, \zeta_a) - N(\beta A, \zeta)$, can be isotopically deformed onto $\pi^{-1}\bar{\gamma}(A, \eta) \cap \overline{N}(A, \zeta_a)$ [see (6.5) and Lemma M]. The adjoined part $\overline{\Gamma}(A, \zeta)$ of $\Sigma(K, \zeta)$, which is on the boundary of $\overline{\Delta}(A)$, can be kept fixed during the isotopy. The isotopy can be defined with the aid of radii $J(K, \zeta')$, where ζ' is an admissible set slightly larger than and proportional to ζ. It follows that $\bar{\gamma}(A, \zeta)$ is a D^a, $\overline{\sigma}(A, \zeta) = \overline{\Delta}(A, \zeta) \cap \overline{N}(K - A, \zeta)$ is an $S^{a-1} \times D^{n-a}$, and $\overline{\Gamma}(A, \zeta)$ is a $D^a \times S^{n-a}$. The remaining details are straightforward.

In connection with the smooth cases (b) and (d), it is interesting to note that the handlebody additions correspond to multiplying the left sides of the following defining relations by $\rho_A(x)$, thus:

$$\overline{N}_\tau(K - A, \zeta)\colon \rho_{K-A}(x) \le \tau \to\to \overline{N}_\tau(K, \zeta)\colon \rho_{K-A}(x)\rho_A(x) \le \tau,$$

$$\Sigma^*_\tau(K - A, \zeta)\colon \rho_{K-A}(x) = \tau \to\to \Sigma^*_\tau(K, \zeta)\colon \rho_{K-A}(x)\rho_A(x) = \tau.$$

The neighborhoods in Theorem VIII are easily seen to be closely related to the block-bundle structures of Rourke and Sanderson [2]. The latter are in the piecewise linear category; while $\overline{N}(K, \zeta)$ is a smooth handlebody built up around a polyhedron $|K|$.

10. FURTHER ISOTOPY THEOREMS

THEOREM IX. *The space* $Z(K^m)$ *of admissible sets for* K^m *is connected.*

LEMMA P. *Let* $\zeta = (\zeta_0, \cdots, \zeta_m)$ *and* $\zeta' = (\zeta'_0, \cdots, \zeta'_m)$ *be admissible for* K^m, *where* $\zeta'_i \leq \zeta_i$ $(i = 0, \cdots, m)$. *Then* ζ *can be deformed into* ζ' *in* $Z(K^m)$ *by a sequence of steps in each of which all but one of the* ζ_i *are held fixed, while that one, say* ζ_j, *is reduced to the value* ζ'_j.

Proof. Admissibility conditions (a) and (b) are obviously preserved during such deformations. We turn to condition (c).

Assume that $(\zeta_0, \cdots, \zeta_m)$ has been deformed into $(\zeta_0, \cdots, \zeta_j, \zeta'_{j+1}, \cdots, \zeta'_m)$ in $Z(K^m)$. We shall show that the latter set can be deformed into

$$(\zeta_0, \cdots, \zeta_{j-1}, \zeta'_j, \cdots, \zeta'_m)$$

in $Z(K^m)$ through the family

$$(\zeta_0, \cdots, \zeta_{j-1}, (\zeta'_j - \zeta_j)t + \zeta_j, \zeta'_{j+1}, \cdots, \zeta'_m) \qquad (0 \leq t \leq 1).$$

Let $\{A, B, C\} \subset K^m$ be such that $\overline{A} \cap \overline{B} = \overline{C}$ $(c < \min(a, b))$.

Case 1 $(c \geq j,\ a > j,\ b > j)$. The relation

$$\overline{N}(A, \zeta'_a) \cap \overline{N}(B, \zeta'_b) \subset N(C, \zeta'_c/2)$$

holds because ζ' is admissible. It is preserved when $\zeta'_c = \zeta'_j$ is replaced by the larger values $(\zeta'_j - \zeta_j)t + \zeta_j$ $(0 < t \leq 1)$.

Case 2 $(c < j,\ a = j,\ b \geq j)$. If $b > j$, the relation

$$\overline{N}(A, \zeta_a) \cap \overline{N}(B, \zeta'_b) \subset N(C, \zeta_c/2)$$

holds by assumption. It becomes apparent when $\zeta_a = \zeta_j$ is replaced by the smaller values $(\zeta'_j - \zeta_j)t + \zeta_j$. If $b = j$, ζ_b is also replaced by these smaller values.

All other cases are covered by the assumption.

LEMMA Q. *For each* $\varepsilon > 0$, *there exists an admissible set* $(\zeta''_0, \cdots, \zeta''_m)$ *for* K^m *such that* $\zeta''_i < \varepsilon$ $(i = 0, \cdots, m)$.

Proof. The condition $\zeta''_0 < \varepsilon$, which implies $\zeta''_i < \varepsilon$, is consistent with the proof of Theorem V.

Given two admissible sets ζ and ζ', let ζ'' be smaller than both of them. Then ζ can be deformed into ζ'' by the process of Lemma P, then into ζ' by a reverse of that process. Theorem IX follows.

THEOREM X. *If* $\zeta = (\zeta_0, \cdots, \zeta_m)$ *and* $\zeta' = (\zeta'_0, \cdots, \zeta'_m)$ *are admissible, then*

$$[(\zeta'_0 - \zeta_0)t_0 + \zeta_0, (\zeta'_1 - \zeta_1)t_1 + \zeta_1, \cdots, (\zeta'_m - \zeta_m)t_m + \zeta_m]$$

is admissible $(0 \leq t_0 \leq \cdots \leq t_m \leq 1)$. *In particular,*

$$[(\zeta'_0 - \zeta_0)t + \zeta_0, (\zeta'_1 - \zeta_1)t + \zeta_1, \cdots, (\zeta'_m - \zeta_m)t + \zeta_m]$$

is admissible $(0 \leq t \leq 1)$.

The proof is like that of Lemma P, with

$$[\zeta_0, \cdots, \zeta_j, (\zeta'_{j+1} - \zeta_{j+1})t_{j+1} + \zeta_{j+1}, \cdots, (\zeta'_m - \zeta_m)t_m + \zeta_m]$$

in the inductive assumption.

THEOREM XI. *Two piecewise smooth admissible neighborhoods* $\overline{N}(K, \zeta)$ *and* $\overline{N}(K, \zeta')$ *can be isotopically deformed into one another, and so can the corresponding smooth neighborhoods* $\overline{N}^*_\tau(K, \zeta)$ *and* $\overline{N}^*_\tau(K, \zeta')$. *The deformations can be extended to isotopic deformations of* E^n *onto itself that are the identity on some neighborhoods of* $|K|$ *and outside some neighborhood of* $\overline{N}(K, \zeta) \cup \overline{N}(K, \zeta')$.

This follows easily from Theorem X.

Proof of Theorem III. Although a number of other isotopy theorems could be proved, we confine ourselves here to a special class of isotopies.

Let K_0 be deformed into a complex K_1 by a *continuous isotopy* defined as follows. Each vertex v follows a path v(t) $(0 \leq t \leq 1)$, and $v_{i_0}(t) \cdots v_{i_k}(t)$ is a simplex of K_t $(0 \leq t \leq 1)$ if and only if it is a simplex of K_0 for $t = 0$. A set ζ can be chosen so as to be admissible for each K_t $(0 \leq t \leq 1)$. Then $\overline{N}(K_t, \zeta)$ and $\overline{N}^*_\tau(K_t, \zeta)$ $(0 \leq t \leq 1)$ define deformations of the admissible neighborhoods and of the corresponding smooth neighborhoods of K_0 into those of K_1 within the respective families of such neighborhoods. If K_t is a continuous isotopy, then it can be extended to an ambient isotopy [5, Chapter 5], and so can the corresponding isotopies of $\overline{N}(K_t, \zeta)$ and $\overline{N}^*_\tau(K_t, \zeta)$.

Another type of isotopy can be associated with subdivisions. Let K' be a subdivision of K; let ζ be admissible for K, and let ζ' be admissible for both K and K', where $\zeta'_i \leq \zeta_i$ $(i = 0, \cdots, m)$. We outline a deformation of $\overline{N}(K, \zeta)$ into $\overline{N}(K', \zeta')$.

Let A^0 be a vertex of K' not in K. Consider the neighborhood $\overline{N}(K, \zeta') \cup \overline{N}(A^0, t\zeta'_0)$ as t increases from 0 to 1. After a certain value (it is ζ'_d/ζ'_0, if d is the dimension of the carrier of A^0 in K'), the growing n-ball $\overline{N}(A^0, t\zeta'_0)$ bulges out of the neighborhood $\overline{N}(K, \zeta')$ and continues to grow until it becomes a part $\overline{N}(A^0, \zeta'_0)$ of $\overline{N}(K', \zeta')$. Let such deformations be carried out simultaneously for all the vertices of K' not in K. Then let similar deformations be effected for the 1-simplexes of K' not on 1-simplexes of K, with A', ζ'_1 replacing A^0, ζ'_0. Proceeding inductively in order of nondecreasing dimensions of j-simplexes of K' not on j-simplexes of K, we arrive at an isotopic deformation of $\overline{N}(K, \zeta')$ into $\overline{N}(K', \zeta')$.

With the aid of parametrized factors of the form $\rho_A(A, x)$ [see Section 8], we can define deformations of $\overline{N}^*(K, \zeta')$ onto $\overline{N}(K', \zeta')$.

We can apply Theorems IX, X, and XI to arrive at the following result.

THEOREM III (extended). *Let* K_0 *be deformed into* K'_1 *by a finite sequence of continuous isotopies and subdivisions. Then there exist corresponding isotopies of the admissible neighborhoods* $\overline{N}(K_0, \zeta)$ *and the smooth neighborhoods* $\overline{N}^*_\tau(K_0, \zeta)$ *into the corresponding neighborhoods* $\overline{N}(K'_1, \zeta')$ *and* $\overline{N}^*_\tau(K'_1, \zeta')$ *within their respective classes. Restrictions of these isotopies apply to the bounding manifolds* $\Sigma(K, \zeta)$ *and* $\Sigma^*_\tau(K, \zeta)$. *Such isotopies can all be extended into ambient isotopies* [5].

The statements in the paragraph following Theorem II (Section 1) can be proved with the aid of Theorem X and Lemma Q.

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The University of Illinois