LOOSELY CLOSED SETS AND PARTIALLY CONTINUOUS FUNCTIONS

Gordon T. Whyburn

Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

In this paper we study certain pseudo-closedness properties of sets, similar to semi-closedness [11], in relation to partial continuity of functions. We characterize peripheral continuity of multifunctions and functions in terms of countervariance of such properties. The monotone-light factorization theorem [1], [2], [9], [10], [12] is presented in a new and improved form; in this form, it includes in one simple statement not only all previous versions for continuous functions known to the author, but also Hagan's recent extensions to peripherally continuous and connectivity functions [3]. By using cohesion properties closely related to unicoherence, we display relations between these two types of functions in a new setting. Our new theorems include as special cases recent results of Hagan [4], Long [7], and the author [15]. These are extended to multifunctions, and they assert approximately that peripherally continuous functions are connectivity functions. Also, we give a new and simplified proof for the reverse implication as established by Hamilton [6] and Stallings [8].

By X, Y, $\cdots$ we denote topological spaces with the usual open-set topology, additional restrictions being clearly stated when they are imposed. Functions $f: X \to Y$ are always single-valued, except when they are specifically called multifunctions. The double arrow $f: X \Rightarrow Y$ means that the relation is from X *onto* Y. A *mapping* is always a *continuous* function. A function $f: X \to Y$ is *monotone* provided each point-inverse $f^{-1}(y)$ is a continuum, that is, a compact connected set; and f is *light* provided each $f^{-1}(y)$ $(y \in Y)$ is totally disconnected.

A set M is *totally separated* provided there exists a separation of M between any two of its points, or, equivalently, provided each point of M is a quasi-component of M. A space X is *completely normal* provided any two separated sets in X lie in disjoint open sets in X.

A connected set is *cyclic* provided it has no cut-point. A *region* in a space X is a connected open set in X. The *boundary* of an open set U will be denoted by Fr(U).

2. LOOSELY CLOSED AND RELATED SETS

A set S in a topological space X is semi-closed (see [11, p. 131]) provided each of its components is closed and each convergent sequence of its components whose limit set intersects X - S converges to a single point of X - S. We shall define and use two related but stronger properties.

A point p is called an *adhesion point* of a set M provided there exists a point q in $\overline{M}$ - p that is not separated from p in M + p + q. A set is *loosely closed*

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provided it contains all of its adhesion points. A set E in X has *exterior* (or *external*) *dimension* 0 provided the Menger-Urysohn dimension of $E + p$ at each point p of $X - E$ is 0.

Clearly, every set of exterior dimension 0 in a Hausdorff space is loosely closed, and every loosely closed set is semi-closed. In general, the reverse implications do not hold. However, we shall show below that in a completely normal T_1-space X a set with a locally compact closure is loosely closed if and only if it has exterior dimension 0. Thus in particular, in a locally compact completely normal T_1-space, "loose closedness" and "exterior dimension 0" are entirely equivalent properties.

Since every totally disconnected set is semi-closed, semi-closedness is a weaker property than the two just mentioned, even in compact metric spaces. It is interesting to note, however, that in a space X that is a hereditarily locally connected continuum, all three properties are equivalent; and indeed each is equivalent to the simple property of having closed components. (See [11, page 93].)

Note. We see at once that the intersection of an arbitrary collection of loosely closed sets (sets of exterior dimension 0) is a loosely closed set (a set of exterior dimension 0). However, the union of two such sets may fail to inherit the property in question. For example, both the set R of rationals and the set I of irrationals on the open real interval $J = (0, 1)$ are loosely closed and of exterior dimension 0, but their union is J.

(2.1) THEOREM. *Let* E *be a loosely closed set with a locally compact closure in a completely normal* T_1*-space* X. *Let* Q_1 *be a compact subset of a quasi-component* Q *of* E, *and suppose that* Q_1 *is open in* Q *and that* U *is an open set about* Q_1. *Then there exists an open set* V *such that* $Q_1 \subset V \subset U$, $Fr(V) \cdot E = \emptyset$, *and* $\overline{V} \cdot \overline{E}$ *is compact.*

Remark. In case E itself is locally compact, we need only assume X to be a Hausdorff space to get the same conclusions. In particular, by standard methods of argument in a Hausdorff space, the following is readily shown: *If* K *is a compact component of a locally compact subset* E *of a Hausdorff space* X, *then for each open set* U *about* K *there exists an open set* V *such that* $K \subset V \subset U$, $Fr(V) \cdot E = \emptyset$, *and* $E \cdot \overline{V}$ *is compact.*

Proof. Using compactness of Q_1, local compactness of $\overline{E}$, and normality of X, we can readily show that U contains an open set W about Q_1 such that $\overline{E} \cdot \overline{W}$ is compact and does not intersect $Q - Q_1$. Let $x \in \overline{E} \cdot Fr(W)$. If $x \in E$, then since x is not in Q, there is a separation of E between x and Q, and thus by complete normality there exist disjoint open sets V_x and V_q containing x and Q, respectively, with $E \subset V_x + V_q$. If x is not in E, it is not an adhesion point of E, and thus there is a separation of $E + x$ between x and Q_1; and again, x and Q_1 are contained in disjoint open sets V_x and V_q whose union contains E. Since $\overline{E} \cdot Fr(W)$ is compact, some finite union R of the V_x contains $\overline{E} \cdot Fr(W)$, and the corresponding finite intersection S of the V_q contains Q_1. Since R and S are open and disjoint, the set $V = S \cdot W$ is open and contains Q_1, $V \subset W \subset U$, and $Fr(V) \cdot E = \emptyset$. The last relation holds because $R + S \supset E \cdot \overline{W}$. Thus V meets all our requirements.

(2.11) COROLLARY. *Every compact quasi-component of* E *is connected and thus is a component of* E.

(2.12) COROLLARY. *If the quasi-components of* E *are all compact, they are identical with the components of* E.

(2.13) COROLLARY. *If* Q *is any compact quasi-component of* E *and* U *is any open set about* Q, *there exists an open set* V *such that* $\overline{V} \cdot \overline{E}$ *is compact,* $Q \subset V \subset U$, *and* $\mathrm{Fr}(V) \cdot E = \emptyset$.

(2.14) COROLLARY. *If* X *is compact, the components and quasi-components of any loosely closed set in* X *are identical.*

Now we can readily see that if E is any loosely closed set in a T_1-space X, then *for each* $p \in X - E$, $E + p$ *is loosely closed and* p *is a compact quasi-component of* $E + p$. For let $E' = E + p$, let $q \in X - E'$, and take any $r \in \overline{E}' - q$. Then, if $r = p$, we have the separation

$$E' + r + q = (E + p) + r + q = E + p + q = E_p + E_q$$

between r and q. If $r \neq p$, the separations

$$E + p + r = E_p + E_r^1 \quad \text{and} \quad E + q + r = E_q + E_r^2$$

between p and r and between q and r, respectively, yield the separation

$$E' + r + q = (E + p) + r + q = (E_p + E_q) + E_r^1 \cdot E_r^2$$

between r and q. Thus q is not an adhesion point of E'. Thus, using (2.13), we get at once the following theorem.

(2.2) THEOREM. *In a completely normal* T_1*-space* X, *a set* E *with a locally compact closure is loosely closed if and only if it has external dimension* 0.

Thus in a metric space X, a totally separated set with locally compact closure in X is loosely closed if and only if it is 0-dimensional.

(2.21) COROLLARY. *In a locally compact, completely normal* T_1*-space, a set is loosely closed if and only if it has external dimension* 0. *In a locally compact metric space a totally separated set is loosely closed if and only if it is* 0*-dimensional.*

(2.22) COROLLARY. *If* E *is a loosely closed set with compact quasi-components and locally compact closure in a completely normal* T_1*-space* X, *the decomposition* G *of* X *into components of* E *and single points of* $X - E$ *is upper-semicontinuous.*

To see this, note first that by (2.12) the elements of G are the same as the quasi-components of E and the single points of $X - E$. Now let U be any open set in X, and let K be an element of G lying in U. Then if K is a component of E, (2.13) yields an open set V in U about K whose boundary does not intersect E, so that every element of V intersecting V is contained in V. On the other hand, if K is a single point $p \in X - E$, then since $E + p$ is loosely closed and has p as a quasi-component, again the same reasoning gives an open set V of the same sort about p. Thus each element of G lying in U in interior to the union of all elements of G lying in U, and thus G is upper-semicontinuous.

Next we show that the property of being loosely closed is invariant under the action of certain types of monotone functions. To that end, we first prove a lemma.

(2.3) LEMMA. *Let* P *and* Q *be disjoint compact sets in a Hausdorff space* X, *and let* $R = X - P - Q$. *If for each* $p \in P$ *and* $q \in Q$, $R + p + q$ *is separated between* p *and* q, *then* X *is separated between* P *and* Q.

Proof. Fix $p \in P$. Then for each $q \in Q$ we have a separation

$$R + p + q = R_p + R_q$$

between p and q. Let A_p and A_q be disjoint open sets about p and q, respectively, chosen so that $A_p \cdot (R_q + Q) = \emptyset = A_q \cdot (R_p + P)$, and let

$$R'_p = A_p + (R_p - p), \qquad R'_q = A_q + (R_q - q).$$

These sets are open, since $R_p - p$ and $R_q - q$ are open. Hence some finite union U_q of the R'_q sets contains Q. Let U_p be the intersection of the corresponding finite collection of sets R_p. Then

$$p \in U_p, \qquad Q \subset U_q, \qquad U_p + U_q \supset R + p + Q, \qquad U_p \cdot U_q = \emptyset.$$

We now choose such sets U_p and U_q for each $p \in P$. Some finite union V_p of the sets U_p contains P, and we let V_q be the intersection of the corresponding collection of sets U_q. Then V_p and V_q are open and contain P and Q, respectively. Also,

$$X = R + P + Q = V_p + V_q$$

and $V_p \cdot V_q = \emptyset$, so that this is a separation of X between P and Q.

(2.4) THEOREM. *If* $f: X \Rightarrow Y$ *is a monotone function that is quasi-compact on every inverse set in the Hausdorff space* X, *then the image* H *of any loosely closed inverse set* K *in* X *is loosely closed.*

Proof. Let $p \in \overline{H} - H$, $q \in \overline{H} - p$. Then for each $p' \in P = f^{-1}(p)$, $q' \in Q = f^{-1}(q)$, the set $K + p' + q'$ is separated between p' and q'. Thus, since $X' = K + P + Q$ is a Hausdorff space with relative topology in X, there exists by (2.3) a separation

$$K + P + Q = K_p + K_q$$

with $P \subset K_p$, $Q \subset K_q$.

Now, since f is monotone, K_p and K_q are inverse sets such that if $H_p = f(K_p)$ and $H_q = f(K_q)$, then $H_p \cdot H_q = \emptyset$; and H_p and H_q are both open (and closed) relative to $H + p + q$, since $f \mid X'$ is quasi-compact. Thus $H + p + q = H_p + H_q$ is a separation between p and q, and H is loosely closed.

(2.41) COROLLARY. *The conclusion of* (2.4) *holds in case* (a) f *is closed and monotone, or* (b) f *is open and monotone, or* (c) Y *is a weakly separable Hausdorff space and* f *is monotone, quasi-compact, and continuous.*

To see this, note that in each of these cases f is necessarily quasi-compact on each inverse set (see [13]).

(2.5) THEOREM. *Let* X *be a* T_1*-space, and let* $\phi: X \Rightarrow Y$ *be a monotone closed mapping. If the inverse* $D = \phi^{-1}(E)$ *of a subset* E *of* Y *is of external dimension* 0, *then so is* E.

Let $p \in Y - E$, and let V be any open set in Y about p. Then $P = \phi^{-1}(p)$ is compact and lies in the open set $S = \phi^{-1}(V)$. Since $P \subset X - D$, each $x \in P$ lies in an open set U_x with $U_x \subset S$ and $\mathrm{Fr}(U_x) \cdot D = \emptyset$. Thus P lies in a finite union U of the U_x-sets, and $\mathrm{Fr}(U) \cdot D = \emptyset$. Then, if $W = Y - \phi(X - U)$, we see that $p \in W \subset \phi(U) \subset V$ and W is open. Also, $\mathrm{Fr}(W) \cdot E = \emptyset$, because each point q of

$Fr(W)$ is in both $\phi(X - U)$ and $\overline{\phi(U)} = \phi(\overline{U})$, so that $\phi^{-1}(q)$ intersects both $\overline{U}$ and $X - U$. Thus $\phi^{-1}(q)$ intersects $Fr(U)$, since ϕ is monotone. Hence q is not in E.

Note. Complete normality is invariant under closed onto mappings $f: X \Rightarrow Y$. For let α and β be separated sets in Y. Then $A = f^{-1}(\alpha)$ and $B = f^{-1}(\beta)$ are separated in X and thus lie in disjoint open sets U_a and U_b, respectively. Therefore $V_\alpha = Y - f(X - U_a)$ and $V_\beta = Y - f(X - U_b)$ are open and disjoint and contain α and β, respectively.

(2.6) THEOREM. *If* $f: X \Rightarrow Y$ *is a closed mapping and* E' *is a set of external dimension* 0 *in* Y *such that* f^{-1} *is single-valued on* $Y - E'$, *then* $E = f^{-1}(E')$ *has external dimension* 0.

Let $p \in \overline{E} - E$, and let U be any open set about p. If $p' = f(p)$, then $E' + p'$ is 0-dimensional at p'. Accordingly, since $Y - f(X - U)$ is open and contains p', there exists an open set V about p' with

$$V \subset Y - f(X - U) \subset f(U) \quad \text{and} \quad Fr(V) \cdot E' = \emptyset.$$

Hence, if $W = f^{-1}(V)$, W is an open set about p lying in U and satisfying $Fr(W) \cdot E = \emptyset$.

(2.7) THEOREM (Converse). *Let* E *be a loosely closed set with compact quasi-components and locally compact closure in a completely normal* T_1*-space* X. *Then there exists a closed monotone mapping* $\phi: X \Rightarrow Q$ *of* X *such that* ϕ *is one-to-one on* $X - E$ *and carries* E *onto a totally separated set* E' *of external dimension* 0, *and such that the nondegenerate components of* E *are exactly the nondegenerate point-inverses for* ϕ.

We have only to take ϕ as the natural mapping of the decomposition G of X into components of E together with single points of $X - E$. Since G is upper-semicontinuous, $\phi: X \Rightarrow Q$ is closed and monotone and Q is a completely normal T_1-space. Thus, since E is a loosely closed inverse set for ϕ, it follows by (2.4) and (2.41) that $E' = \phi(E)$ is loosely closed. Also, E' is totally separated, $\overline{E}'$ is locally compact, and E' has external dimension 0 by (2.2).

3. PERIPHERAL CONTINUITY OF MULTIFUNCTIONS

A multiple-valued function from X to Y, that is, a relation that associates with each point $x \in X$ a subset f(x) of Y, is called a *multifunction* (see [5], [14]). In case f(x) is a single point for each $x \in X$, f(x) is an ordinary *function*, of course. All concepts and results developed here for multifunctions f apply to the special case in which f is a (single-valued) function.

A multifunction $f: X \to Y$ is said to be *peripherally continuous* provided that for each $x \in X$ and each pair of open sets U and V containing x and f(x), respectively, there is an open set W such that $x \in W \subset U$ and $f[Fr(W)] \subset V$. (Compare with [6], [8]).

(3.1) THEOREM. *A multifunction* $f: X \to Y$ *is peripherally continuous if and only if the inverse of each closed set in* Y *is of external dimension* 0.

Suppose first that f is peripherally continuous. Let C be any closed set in Y, take any point $p \in X - f^{-1}(C)$ and any open set U about p. Then, if V denotes the open set $Y - C$, we have the inclusion relation $f(p) = P \subset V$. Thus by peripheral continuity there exists an open set W in X with $p \in W \subset U$ and $f[Fr(W)] \subset V$. This gives $\mathrm{Fr}(W) \cdot f^{-1}(C) = \emptyset$, so that $f^{-1}(C) + p$ has dimension 0 at p. Thus $f^{-1}(C)$ is of external dimension 0.

Now suppose the inverse of each closed set in Y is of external dimension 0. Let $p \in X$ and $P = f(p)$, and let U and V be open sets about p and P, respectively. Define $C = Y - V$. Then, since $p \in X - f^{-1}(C)$ and C is closed (so that $f^{-1}(C) + p$ has dimension 0 at p), there exists an open set W such that $p \in W \subset U$ and $\mathrm{Fr}(W) \cdot f^{-1}(C) = \emptyset$. This gives the relation

$$f[\mathrm{Fr}(W)] \subset Y - C = V.$$

Thus f is peripherally continuous at p.

(3.11) COROLLARY. *Any upper-semicontinuous multifunction is peripherally continuous. Thus the inverse of any closed multifunction (or function) is peripherally continuous.*

In view of the characterizations in Section 2, particularly (2.21), we have the following immediate consequence.

(3.2) THEOREM. *If* X *is a locally compact, completely normal* T_1*-space, a multifunction* $f: X \to Y$ *is peripherally continuous if and only if the inverse of every closed set in* Y *is loosely closed.*

4. THE COMPONENT DECOMPOSITION. FACTORIZATION

For any function $f: X \to Y$, the decomposition of X into the collection G of components of point inverses for f, that is, components of the sets $f^{-1}(y)$ $(y \in Y)$, is called the *component decomposition* of X generated by f. The elements of G are disjoint, since f is single-valued. It will be recalled that a collection G of sets is upper-semicontinuous provided that for any open set U in X the union U_0 of all elements of G lying wholly in U is open. Throughout this section, the symbols f, ϕ, and ℓ refer to (single-valued) functions. A function $f: X \to Y$ is *locally closed* provided that for each $p \in X$ there exists an open set W about p such that $f \mid \overline{W}: \overline{W} \to Y$ is closed.

(4.1) THEOREM. *Let* X *be a Hausdorff space, let the peripherally continuous function* $f: X \Rightarrow Y$ *have point-inverses with compact quasi-components and locally compact closures, and let* G *be the component decomposition for* f. *If for each* $K \in G$ *and each open set* U *about* K *there exists an open set* V *about* K *such that*

$$V \subset U \quad \textit{and} \quad f(K) = p \in Y - \overline{f[\mathrm{Fr}(V)]},$$

then G *is upper-semicontinuous. In particular, for a completely normal* X, *this holds in case* (a) f *is locally closed or* (b) X *is locally compact and* Y *is a Hausdorff space.*

Proof. Let U be any open set in X, and let K be any element of G contained in U. Let V be an open set about K $(V \subset U)$ such that $\overline{f[\mathrm{Fr}(V)]} = C$ does not contain $p = f(K)$. By peripheral continuity, each $x \in K$ lies in an open set $W_x \subset V$ such that $f[\mathrm{Fr}(W_x)] \subset Y - C$. Thus K lies in a finite union W of these sets W_x; moreover, $W \subset U$ and $f[\mathrm{Fr}(W)] \subset Y - C$, so that $f[\mathrm{Fr}(W)] \cdot f[\mathrm{Fr}(V)] = \emptyset$. Now, if H is any element of G intersecting W, it must lie wholly in U, since it cannot intersect both $\mathrm{Fr}(W)$ and $\mathrm{Fr}(V)$. Thus $K \subset W \subset U_0 \subset U$, so that U_0 is open. (Here U_0 denotes the union of all elements of G lying wholly in U.)

To verify (a), suppose that f is locally closed, and take any element K of G and any open set U about K. Since K is compact, there exists an open set W about K ($W \subset U$) such that $f \mid \overline{W}$ is a closed mapping. Now by (3.1) and the fact that X is a Hausdorff space, the set $E = f^{-1}[f(K)]$ is of external dimension 0 and hence is loosely closed. By (2.1) there exists an open set V such that $K \subset V \subset W$ and $\mathrm{Fr}(V) \cdot E = \emptyset$. Since $f \mid \overline{W}$ is closed, this implies that

$$f[\mathrm{Fr}(V)] \cdot f(K) = \emptyset \quad \text{and} \quad f[\mathrm{Fr}(V)] \text{ is closed.}$$

To verify (b), let X be locally compact, let Y be a Hausdorff space, and again take any element K of G and any open set U about K. We may assume U to be chosen so that $\overline{U}$ is compact. Taking $E = f^{-1}(p)$, where $p = f(K)$, we again deduce from (2.1) the existence of an open set S such that $K \subset S \subset U$ and $\mathrm{Fr}(S) \cdot E = \emptyset$. Now, since $\mathrm{Fr}(S)$ is compact and $f[\mathrm{Fr}(S)] \subset Y - p$, the peripheral continuity of f readily implies the existence of an open set W about $\mathrm{Fr}(S)$ such that $\overline{f[\mathrm{Fr}(W)]} \subset Y - p$. Indeed, since Y is a Hausdorff space, we can find an open set W_x about each point $x \in \mathrm{Fr}(S)$, such that p does not belong to $\overline{f[\mathrm{Fr}(W_x)]}$; some finite union W of these W_x covers $\mathrm{Fr}(S)$ and meets our requirements.

Now let $V = S - \overline{W}$. Then $K \subset V \subset S \subset U$; the inclusion $\mathrm{Fr}(V) \subset \mathrm{Fr}(W)$ implies that $\overline{f[\mathrm{Fr}(V)]} \subset \overline{f[\mathrm{Fr}(W)]} \subset Y - p$.

Since each compact component of a closed locally compact set E in a Hausdorff space is also a quasi-component of E, and in view of the remark following the statement of (2.1), above, we have the following simplified form of the results for continuous functions.

(4.11) COROLLARY. *If* X *is a Hausdorff space and the locally closed mapping* $f: X \Rightarrow Y$ *has locally compact point-inverses with compact components, then the component decomposition of* f *is upper-semicontinuous.*

(4.12) COROLLARY. *If* X *is locally compact and both* X *and* Y *are Hausdorff spaces, then each mapping* $f: X \Rightarrow Y$ *with compact components of point-inverses generates an upper-semicontinuous component decomposition.*

(4.2) FACTORIZATION THEOREM. *Let* X *be a Hausdorff space, and let* $f: X \Rightarrow Y$ *be peripherally continuous and have compact components of point-inverses. If the component decomposition* G *is upper-semicontinuous, then* f *factors uniquely into the form* $f = \ell\phi$, *where* $\phi: X \Rightarrow X'$ *is a closed monotone mapping,* $\ell: X' \Rightarrow Y$ *is light and peripherally continuous, and where* X' *is a Hausdorff space. Also,* ℓ *is continuous if and only if* f *is continuous.*

Proof. Let $\phi: X \Rightarrow X'$ be the natural mapping for G. Then since G is upper-semicontinuous, X' is a Hausdorff space and ϕ is closed and monotone. Thus if we define $\ell(x') = f\phi^{-1}(x')$ for $x' \in X'$, then ℓ is single-valued. It is also light. For if Q is a connected subset of $\ell^{-1}(y)$ for some $y \in Y$, then $\phi^{-1}(Q)$ is connected, since ϕ is closed and monotone. Thus $\phi^{-1}(Q)$ lies in a component K of $f^{-1}(y)$; and since $K \in G$, $Q = \phi\phi^{-1}(Q) \subset \phi(K) \in X'$, so that Q is a single point.

Finally, ℓ is peripherally continuous. Indeed, let L be any closed set in Y. Then $f^{-1}(L)$ has external dimension 0 by (3.1). By (2.5) $\phi f^{-1}(L)$ also has external dimension 0, because $f^{-1}(L) = \phi^{-1}\phi f^{-1}(L)$, so that $f^{-1}(L)$ is an inverse set for ϕ. Hence, by (3.1), ℓ is peripherally continuous, because $\ell^{-1}(L) = \phi f^{-1}(L)$.

To show that the factorization $f = \ell\phi$ is topologically unique, let $f = \ell_1 \phi_1$ be any other factorization, where $\phi_1: X \Rightarrow X_1$ is a closed monotone mapping and $\ell_1: X_1 \Rightarrow Y$ is a light function. Take any $x_1 \in X_1$, and let $y = \ell_1(x_1)$. Then, since

$\phi_1^{-1}(x_1)$ is connected and contained in $f^{-1}(y)$, it lies in a single component K of $f^{-1}(y)$. However, since $\phi_1(K)$ is connected and lies in $\ell_1^{-1}(y)$ and ℓ_1 is light, $\phi_1(K)$ must be the single point x_1. Hence $\phi_1^{-1}(x_1) = K$, and ϕ_1 generates precisely the same decomposition of X as does ϕ, namely, the component decomposition of f.

Accordingly, if we define $h(x) = \phi_1 \phi^{-1}(x)$ for $x \in X'$, then $h: X' \Rightarrow X_1$ is one-to-one. It is also continuous and closed, and thus topological, since ϕ and ϕ_1 are continuous and closed. Since $\phi_1 \equiv h\phi$ and $\ell_1 \equiv \ell h^{-1}$, the mappings ϕ_1 and ℓ_1 are topologically equivalent to ϕ and ℓ, respectively.

Finally, to verify the last statement in (4.2), we need only note that if f is continuous, then $\ell^{-1}(C) = \phi f^{-1}(C)$ is closed when $C \subset Y$ is closed.

We remark also that our uniqueness proof for the factorization depends only on the continuity, quasi-compactness (closedness), and monotoneity of the monotone factor and the lightness of the second factor. Thus no form of continuity of f or of ℓ is needed for uniqueness.

(4.21) COROLLARY. *If* X *is a completely normal Hausdorff space, then a peripherally continuous function* $f: X \Rightarrow Y$ *with point-inverses having compact quasi-components and locally compact closures has a unique factorization* $f = \ell\phi$ *as in* (4.2), *in case* (a) f *is locally closed or* (b) X *is locally compact and* Y *is a Hausdorff space. The middle space* $X' = \phi(X)$ *is completely normal, and in case* (b) *it is locally compact. In either case, if* X *is separable and metric, so is* X'.

Again, since compact components of a locally compact Hausdorff space are also quasi-components, (4.11) implies the following result for continuous functions.

(4.3) THEOREM. *If* X *is a Hausdorff space, each locally closed mapping* $f: X \to Y$ *that has locally compact point inverses with compact components has a unique factorization* $f = \ell\phi$, *where* $\phi: X \Rightarrow X'$ *is a closed monotone mapping,* $\ell: X' \to Y$ *is a light mapping, and* X' *is a Hausdorff space.*

Since each of the properties involved is invariant under monotone closed mappings, we have the following result.

(4.31) COROLLARY. *If the space* X *is normal, completely normal, perfectly separable, separable and metric, or locally compact, then* X' *has the corresponding property.*

Also, since any mapping from a locally compact Hausdorff space to a Hausdorff space is locally closed and has locally compact point-inverses, we have the following corollary.

(4.32) COROLLARY. *If* X *and* Y *are Hausdorff spaces and* X *is locally compact, then each mapping* $f: X \to Y$ *with compact components of point-inverses has a unique factorization* $f = \ell\phi$, *as in* (4.3).

5. COHESIVE SPACES

A connected space or set M is said to be *unicoherent*, or *cohesive, between disjoint connected subsets* (or points) A *and* B *of* M provided $H_a \cdot H_b$ is connected for every representation $M = H_a + H_b$, where H_a and H_b are closed and connected and contain A and B, respectively, in their interiors relative to M. In case M is locally connected, unicoherence is equivalent to each of the following:

(a) Every subset of M that separates A and B in M contains a closed connected subset also separating A and B in M.

(b) Every subset of M that separates A and B irreducibly in M is connected.

Also, it is clear that a connected set is unicoherent if and only if it is unicoherent between every pair of its points. Likewise a connected Hausdorff space M is unicoherent between two of its compact connected disjoint sets A and B if and only if $\phi(M)$ is unicoherent between $\phi(A)$ and $\phi(B)$, where $\phi: M \Rightarrow M'$ is the natural mapping of the decomposition of M into A, B, and single points of M - A - B.

A connected space M is *locally cohesive* provided that for each $p \in M$ and each open set U about p there exists a region R in M that contains p, lies in U, has a connected boundary, and is such that $\bar{R}$ is cohesive between p and Fr(R). Such a region R will be called a *canonical region* about p in U. Note that a locally cohesive space is always locally connected and locally peripherally connected.

(5.1) THEOREM. *Every unicoherent, cyclic, connected, locally connected, and locally compact Hausdorff space* M *is locally cohesive.*

Suppose that $p \in M$ and U is an open set about p. Since M is cyclic, there exists an open set V such that $p \in V \subset U$ and $M - V = N$ is connected. If R is the component of M - N containing p, then R lies in U, and Fr(R) is connected by unicoherence of M. Thus R is a canonical region for p in U.

The same conclusion holds in case M is assumed to be *locally unicoherent* and without local cut points instead of being cyclic and unicoherent. Local unicoherence of M means that each $p \in M$ lies interior to some connected set in M that is unicoherent.

We remarked earlier that each unicoherent space is unicoherent between every pair of its points, and conversely. Remarkably enough, *a cyclic, locally connected continuum* M *is necessarily unicoherent if it is unicoherent between one pair of distinct points.* This is a special case of the next theorem. (By a *continuum* we mean a compact connected metric space.)

(5.2) THEOREM. *A locally connected continuum* M *is unicoherent between two of its points* a *and* b *if and only if the cyclic chain* C(a, b) (see [11, p. 71]) *is unicoherent.*

If C(a, b) is unicoherent, it is unicoherent between a and b. Thus M also is unicoherent between a and b, because a set X in M separates a and b in M if and only if $X \cdot C(a, b)$ separates them in C(a, b).

Suppose, on the other hand, that M is unicoherent between a and b. To prove that C(a, b) is unicoherent, it suffices to show that each cyclic element E of M in C(a, b) is unicoherent, because unicoherence is cyclicly extensible. Now E contains exactly two points p and q of $E(a, b) + a + b$, where E(a, b) denotes the set of all points separating a and b in M. Further, E is unicoherent between p and q, because any subset of E separating p and q in E irreducibly also separates a and b in M irreducibly and is therefore connected.

Hence it remains only to show that E is unicoherent. We suppose the contrary. Then there exists a nonalternating open retraction $r: E \Rightarrow J$ of E onto a simple closed curve J in E (see [11, p. 216]). In case $r(p) \neq r(q)$, let x and y be points separating r(p) and r(q) on J. Then the set $K = r^{-1}(x) + r^{-1}(y)$ separates p and q in E. However, neither $r^{-1}(x)$ nor $r^{-1}(y)$ separates p and q in E, because neither of these sets separates E at all. Thus *any* subset of K that separates p and q meets both of these sets and hence is disconnected. This contradicts the fact that E is cohesive between p and q.

There remains the case where $r(p) = r(q)$. Here we choose distinct points a_1 and b_1 on J, each different from $r(p)$, such that if $A = r^{-1}(a_1)$ and $B = r^{-1}(b_1)$, then $E - (A + B)$ consists of two regions R and S, each bounded by $A + B$ and each constituting the r-inverse of one of the open arcs of J from a_1 to b_1 (see [11, p. 215]). Thus R or S, say R, contains $p + q$.

Now let $\phi(\overline{R}) = N$ be the natural mapping of the decomposition of $\overline{R}$ into the sets A and B and single points of R. Since E is cyclic, it follows at once that N is the cyclic chain $C(a', b')$ in N, where $a' = \phi(A)$ and $b' = \phi(B)$. Now if $p' = \phi(p)$ and $q' = \phi(q)$ lie together in a cyclic element C of N, then C contains disjoint arcs joining p' and q' to the two points of C that separate a' and b' in N. Thus there exist disjoint arcs α and β in N joining a' and b', respectively, to the set $p' + q'$. (In case some point x separates p' and q' in N, we get such arcs α and β at once, since then x also separates a' and b' in N.) To be definite, suppose $\alpha = a'p'$ and $\beta = b'q'$.

Let K' be any compact set separating α and β in N. Then $K = \phi^{-1}(K')$ separates $A + \phi^{-1}(\alpha)$ and $B + \phi^{-1}(\beta)$ in $\overline{R}$. Now, if $H = r^{-1}(x)$ for some fixed x on the open arc of J in S, then $H + K$ separates p and q in E. However, neither H nor K separates p and q in E. (Note that K does not separate p and q because $\overline{S} + \phi^{-1}(\alpha) + \phi^{-1}(\beta)$ is connected.) Thus any subset of $H + K$ that separates p and q in E meets both H and K and is therefore disconnected. Again, this contradicts the fact that E is cohesive between p and q, and our proof is complete.

(5.21) COROLLARY. *A cyclic, locally connected continuum is unicoherent provided it is cohesive between some pair of its points.*

(5.22) COROLLARY. *If the locally connected continuum* M *is unicoherent between two of its points* a *and* b, *and if* $\phi(M) = N$ *is a monotone closed mapping, then* N *is unicoherent between* $\phi(a)$ *and* $\phi(b)$.

This corollary is easily proved directly for arbitrary connected spaces, and independently of any requirement that M is compact or locally connected.

We can now assert the following converse to (5.1).

(5.3) THEOREM. *Let the locally connected continuum* M *be locally cohesive at* $x \in M$. *Let* R *be a canonical region* R *in* M *about* x, *and let* $\phi: M \to M'$ *be the natural mapping for the decomposition of* M *into* Fr(R) *and single points of* M - Fr(R). *Then* $\phi(\overline{R})$ *and* M *are unicoherent between* $p = \phi(x)$ *and* $q = \phi[\mathrm{Fr}(R)]$. *Thus the cyclic chain* C(p, q) *in* M' *is unicoherent, and if* M *is cyclic, then* C(p, q) *is cyclic and is identical with* $\phi(\overline{R})$.

Also, it is now clear that a cyclic, locally connected continuum (or generalized continuum) M is locally cohesive at $x \in M$ if and only if M is *unicoherent modulo the complement of some region* R *in* M *about* x, in other words, if and only if M maps onto a unicoherent space under the decomposition (identification) mapping that sends M - R into a single point and is topological otherwise.

6. PERIPHERAL CONTINUITY AND PRESERVATION OF CONNECTEDNESS

Let X be a connected, locally cohesive regular T_1-space, and let Y be a completely normal T_1-space.

(6.1) THEOREM. *Under each peripherally continuous multifunction* f: X → Y *with connected point values, the image of every connected set is connected.* (See [4], [15].)

To prove this theorem, we first establish a lemma.

LEMMA. *For any* $x \in X$ *and any open sets* U *and* V *containing* x *and* $f(x)$, *respectively, there exists a canonical region* Q *in* X *about* x, *with* $\overline{Q} \subset U$ *and* $f[Fr(Q)] \subset V$.

Let R be a canonical region about x with $\overline{R} \subset U$. Then, by peripheral continuity, R contains the closure of an open set G about x such that $f(K) \subset V$, where $K = Fr(G)$. Let Q_1 be the component of G containing x, let S be the component of $\overline{R} - \overline{Q}_1$ containing $Fr(R)$, and let Q be the component of $\overline{R} - \overline{S}$ containing Q_1. Then $C = Fr(Q) \subset K$, and unicoherence of $\overline{R}$ between x and $Fr(R)$ implies at once that C is connected. Hence, $f(C) \subset f(K) \subset V$, and Q has all the desired properties.

Now suppose, contrary to our theorem, that for some connected set E in X there is a separation

$$f(E) = E_1 + E_2.$$

Let a and b be points of E, with $f(a) \in E_1$ and $f(b) \in E_2$. By complete normality of Y, there exist disjoint open sets U_1 and U_2 in Y containing E_1 and E_2, respectively. Now, using the lemma, let Q_a and Q_b be disjoint regions about a and b, respectively, having connected boundaries, and such that

$$(*) \qquad f[Fr(Q_a)] \subset U_1, \qquad f[Fr(Q_b)] \subset U_2.$$

For each $x \in E - Q_a - Q_b$, there exists by the lemma a region Q_x about x such that $\overline{Q}_x$ does not contain a or b and has a connected boundary C_x, with $f(C_x) \subset U_i$ if $f(x) \subset E_i$ $(i = 1, 2)$.

Since E is connected and is covered by the regions Q_a, Q_b, and $[Q_x]$, there exists a simple chain of these regions

$$a \in Q_a = Q_1, Q_2, \cdots, Q_{n-1}, Q_n = Q_b \supset b$$

from a to b. Let $C_1, C_2, \cdots, C_n$ be the boundaries of $Q_1, Q_2, \cdots, Q_n$, respectively. Since Q_2 intersects Q_1 but does not contain a, C_2 intersects Q_1; and since Q_2 also contains a point of Q_3, but not all of Q_3, and Q_3 is connected, C_2 also intersects Q_3. Thus C_2 intersects both C_1 and C_3, since C_2 is connected. Accordingly, $f(C_2) \subset U_1$, since $f(C_1) \subset U_1$; and $f(C_3) \subset U_1$, since $f(C_2) \subset U_1$. Now, reasoning similarly with C_3, we see that, for $n > 3$, C_3 intersects Q_4, since Q_3 intersects but does not contain all of Q_4. Thus, $C_3 \cdot C_4 \neq \emptyset$, since C_3 is connected and also intersects Q_2. Hence, $f(C_4) \subset U_1$, because $f(C_3) \subset U_1$. Also, $C_4 \cdot Q_3 \neq \emptyset$, since Q_3 intersects Q_4 but is not contained in Q_4. Thus, for $n > 4$ again $C_4 \cdot C_5 \neq \emptyset$, so that $f(C_5) \subset U_1$, and so on. However, continuation of this reasoning leads to the conclusion that $f(C_n) \subset U_1$, contrary to the fact that Q_n is the same as Q_b and the boundary of Q_b maps into U_2 under f, by (*). This contradiction completes the proof.

(6.11) COROLLARY. *If* X *and* Y *satisfy the conditions in the first sentence of this section and, in addition,* $X \times Y$ *is completely normal, then every peripherally continuous multifunction* $f\colon X \to Y$ *having compact connected point values is a connectivity multifunction.*

We define the graph of a set E in X as the set of points (p, q) in $X \times Y$ with $p \in E$, $q \in f(p)$. The corollary asserts that the graph of every connected set E in X is connected. This follows at once because the induced multifunction $g\colon X \to X \times Y$

defined by $g(p) = p \times f(p)$ is peripherally continuous when f is peripherally continuous and has compact point values.

We next give a new proof and extension of the converse of this corollary for the case of functions, as proved originally by Hamilton [6] and Stallings [8]. The converse for multifunctions is not valid even in the most restricted setting. Let X be the unit square plus its interior in a plane, and let X be represented as the union $X = X_1 + X_2$ of two disjoint, totally imperfect sets X_1 and X_2. If I_1 and I_2 denote the intervals $0 \leq t \leq 1$ and $1 \leq t \leq 2$, and if $f(x) = I_n$ for each $x \in X_n$ (n = 1, 2), then the f-image of *every set* in X is connected. However, the boundary of any sufficiently small neighborhood of a point in X intersects both X_1 and X_2, so that f is not peripherally continuous.

To establish the converse theorem for functions, we shall need the restriction that the domain space X is connected, locally cohesive, locally compact, and metric.

Remarks (1). If M is connected, locally connected, and unicoherent, then for any totally disconnected set D of noncut points of M, the set M - D is connected, since otherwise some subset of D would separate M irreducibly between some two points, and this set would have to be connected and would thus reduce to a single point by unicoherence.

(2). If X is compact and $f: X \to Y$ is a connectivity function, then for any closed set C in Y, $f^{-1}(C)$ is semi-closed, by Hagan's Theorem 3.1 in [3]. Thus, if X is compact, the decomposition of X into components of $f^{-1}(C)$ and single points of $X - f^{-1}(C)$ is upper-semicontinuous. (See [11, pp. 131-132].)

(6.2) THEOREM (Hamilton and Stallings). *Let* X *and* Y *be separable and metric, and suppose that* X *is locally compact and locally cohesive. Then every connectivity function* $f: X \to Y$ *is peripherally continuous.*

Proof. For any $x \in X$, let U and V be open sets about x and f(x), respectively. We may suppose that U is chosen to be a canonical region, so that it is connected and has a connected boundary B and a compact closure unicoherent between x and Fr(U). Now let U_1 and V_1 be open sets about x and f(x), respectively, with $\overline{U}_1 \subset U$ and $\overline{V}_1 \subset V$. Let $D = \overline{U}_1 \cdot f^{-1}(\overline{V}_1)$.

If either x is interior to the set A consisting of the component A_0 of D containing x together with the union of all components of $\overline{U} - A_0$ except the one containing B, or x is separated in $\overline{U}$ from B by a component H of D, we get an open set $W \subset U$ about x with $Fr(W) \subset A_0$ or $Fr(W) \subset H$. We take W = int A in the first case, W = component of $\overline{U} - H$ containing x in the second. In either case, $f[Fr(W)] \subset V$. Thus we assume that x is not interior to A and is not separated in $\overline{U}$ from B by any single component of D, and we show that this leads to a contradiction.

It follows by Remark (2) that the decomposition of $\overline{U}$ into the sets A and B, components of D not contained in A, and single points of U - A - D is upper-semicontinuous. Thus if $\phi(\overline{U}) = M$ is the natural mapping of this decomposition, ϕ is monotone and closed, and M is a locally connected continuum. Further, if $a = \phi(A)$, $b = \phi(B)$, and the cyclic chain $C(a, b) = N$ is taken in M, then no point of $\phi(D) \cdot N$ is a cut point of N, since no such point can separate a and b in M or in N. Now, since $\overline{U}$ is unicoherent between x and Fr(U), it follows by (5.2) that N is unicoherent. Thus, since $\phi(D)$ is totally disconnected, the set $R = N - \phi(D)$ is connected, by Remark (1), and $\overline{R} \supset a$. Hence $\phi^{-1}(R) = Q$ is connected, since ϕ is monotone and closed. Also $\overline{Q} \supset x$, since any region S in U_1 about x must intersect Q. To see this, note that S is not in A and a is not a cut point of M, so that $\phi(S) \cdot N$ is non-degenerate and connected and thus is not contained in $\phi(D)$. However, $Q + x$ is then connected, whereas $[x, f(x)]$ is an isolated point of the graph of $f \mid (Q + x)$, because $[q, f(q)]$ is never in $U_1 \times V_1$ for $q \in Q$ since $f(q) \in Y - V_1$ for all $q \in Q$. This contradiction completes the proof.

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University of Virginia
Charlottesville, Virginia