VALUE DISTRIBUTION AND POWER SERIES WITH MODERATE GAPS

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1. INTRODUCTION

For entire functions it is known (see [1], [3], and [5]) that certain assumptions on the gaps in the power series expansion of the function about zero imply that the function has not only one zero (or a-value) but infinitely many. To obtain corresponding results for functions analytic in the unit disk, it is necessary to link the gap assumption with a growth assumption on the function.

THEOREM 1 (Nevanlinna's notation [4, pp. 4 and 18]). *Let*

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \tag{1}$$

be analytic in $|z| < 1$, *with* $n_0 = 0$. *Let* $N^0(t)$ *be the number of* n_k *not greater than* t. *If for some fixed* β $(0 < \beta < 1)$

$$N^0(t) = O(t^{1-\beta}) \qquad (t \to \infty), \tag{2}$$

and if

$$n(r, 1/f) = O((1 - r)^{-\lambda}) \qquad (r \to 1),$$

then

$$\log M(r) = O((1 - r)^{-\alpha}) \qquad (r \to 1)$$

for every α *with*

$$\alpha > \max\left(\lambda, \frac{1-\beta}{\beta}\right). \tag{3}$$

COROLLARY. *Let* f(z) *be analytic in* $|z| < 1$ *and of the form* (1), *with* $n_0 = 0$ *and the* n_k *satisfying* (2). *If*

$$\limsup_{r \to 1} \frac{\log \log M(r)}{-\log (1 - r)} \geq \alpha \qquad \left(\alpha > \frac{1-\beta}{\beta}\right),$$

then

$$\limsup_{r \to 1} \frac{\log n(r, 1/f)}{-\log (1 - r)} \geq \alpha.$$

Theorem 1, which has an elementary proof, extends a theorem stated by F. Sunyer I. Balaguer [7]. Related problems in the disk have recently been investigated for larger gaps—Hadamard gaps—by G. and M. Weiss [8] and Ch. Pommerenke [6].

Received June 13, 1966.

A function satisfying the gap condition of Theorem 1 is the theta-function

$$1+\sum_{n=1}^{\infty} 2z^{n^2} = \prod_{n=1}^{\infty} (1+z^{2n-1})^2 \cdot (1-z^{2n}).$$

It has $M(r) = O((1-r)^{-1/2})$.

2. STATEMENTS OF PRELIMINARY LEMMAS

The following two lemmas form the basis of the proof of Theorem 1; we shall prove them in Section 4.

LEMMA 1. *If* $0<\varepsilon<1$ *and* $\phi(x)$ *is a continuous, increasing function in* $0<x<1$ *such that*

$$\phi(x_0) > \frac{4}{\varepsilon(1-x_0)} \tag{4}$$

for some x_0 $(0<x_0<1)$, *then there exists an* x' $(x_0 \le x' < (1+x_0)/2)$ *such that*

$$\phi\left(x'+\frac{1}{\phi(x')}\right) < (1+\varepsilon)\,\phi(x'). \tag{5}$$

LEMMA 2. *Let* $\{n_k\}$ *be a strictly increasing sequence of integers satisfying* (2). *For each sequence* $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ *with* $\varepsilon_k = \pm 1$, *there exists a real-valued function* $g(t)$ *in* $0 \le t \le 1$ *such that*

(i) $\varepsilon_k \int_0^1 g(t)\,t^{n_k}\,dt = A_k > 0,$

(ii) $\int_0^1 |g(t)|\,dt \le 1/2,$

(iii) $\inf_k A_k \rho^{-n_k} > \exp\left(-K(1-\rho)^{-(1-\beta)/\beta}\log^{1/\beta}(1/(1-\rho))\right)$

for $R_0<\rho<1$, *where* K *and* R_0 *are independent of the sequence* $\{\varepsilon_k\}$.

Lemma 1 goes back essentially to Borel.

3. PROOF OF THEOREM 1

First we estimate

$$U(r) = \sup_{0\le u\le r} |f(u)|$$

from below. Let

$$\varepsilon_k = \operatorname{sign}(\Re\, c_k),$$

where

$$\operatorname{sign} v = \begin{cases} 1 & \text{if } v \geq 0\,, \\ -1 & \text{if } v < 0\,. \end{cases}$$

Construct the function $g(t)$ of Lemma 2 for this choice of $\{\varepsilon_k\}$. Then, for $R_0 < \rho < 1$

$$\begin{aligned} \exp\left(-K(1-\rho)^{-(1-\beta)/\beta}\log^{1/\beta}(1/(1-\rho))\right)\cdot \sum |\Re\, c_k(\rho r)^{n_k}| &\\ \leq \{\inf_k A_k \rho^{-n_k}\}\cdot \sum |\Re\, c_k(\rho r)^{n_k}| &\\ \leq \sum |\Re\, c_k(\rho r)^{n_k}|\, A_k \rho^{-n_k}. \end{aligned}$$

The right-hand side is equal to

$$\begin{aligned} \sum \Re\, c_k r^{n_k} \int_0^1 t^{n_k} g(t)\,dt &= \Re \int_0^1 f(rt)\,g(t)\,dt \\ &\leq \int_0^1 |f(rt)|\,|g(t)|\,dt \leq U(r) \int_0^1 |g(t)|\,dt \leq \tfrac{1}{2}\,U(r)\,. \end{aligned}$$

By the same argument, for $R_0 < \rho < 1$,

$$\exp\left(-K(1-\rho)^{-(1-\beta)/\beta}\log^{1/\beta}(1/(1-\rho))\right)\cdot \sum |\Im\, c_k(\rho r)^{n_k}| \leq \tfrac{1}{2}\,U(r)\,. \tag{6}$$

But

$$\sum |\Re\, c_k(\rho r)^{n_k}| + \sum |\Im\, c_k(\rho r)^{n_k}| \geq \sum |c_k(\rho r)^{n_k}| \geq M(\rho r)\,.$$

Therefore, adding (6) and the corresponding inequality for the real parts of the c_k, we obtain the inequality

$$U(r) \geq \exp\left(-K(1-\rho)^{-(1-\beta)/\beta}\log^{1/\beta}(1/(1-\rho))\right)\cdot M(\rho r)\,. \tag{7}$$

By considering $f(ue^{i\theta})$ in place of $f(u)$ and noting in (iii) of Lemma 2 that K and R_0 are independent of the sequence $\{\varepsilon_k\}$, we see that if $\varepsilon' > 0$, then (7) implies that there exists a value ρ_0, independent of θ, such that the inequalities

$$\begin{aligned} \sup_{u \leq r} |f(ue^{i\theta})| &\geq \exp\left(-K(1-\rho)^{-(1-\beta)/\beta}\log^{1/\beta}(1/(1-\rho))\right)\cdot M(\rho r) \\ &\geq \exp\left(-K(1-\rho)^{-(1-\beta)/\beta - \varepsilon'}\right)\cdot M(\rho r) \end{aligned} \tag{8}$$

are valid when $\rho_0 < \rho < 1$.

Suppose the theorem is false. Then for each ε $(0 < \varepsilon < 1)$, there is a sequence of values of r_0 approaching 1 with

$$(\log M(r_0))^\gamma > \frac{4}{\varepsilon(1-r_0)} \qquad \left(\gamma < \frac{\beta}{1-\beta}\right).$$

By Lemma 1, there exists for each r_0 an r' such that

$$\log M\left(r' + \frac{1}{(\log M(r'))^\gamma}\right) < (1+\varepsilon)^{1/\gamma}\log M(r') \le (1+\eta)\log M(r'). \tag{9}$$

Let

$$r = r' + \frac{1}{(\log M(r'))^\gamma} \quad \text{and} \quad \rho = \frac{r'}{r}.$$

Since

$$\frac{1}{(\log M(r'))^\gamma} \le \frac{1}{(\log M(r_0))^\gamma} < \frac{\varepsilon(1-r_0)}{4},$$

we find with the aid of Lemma 1 that

$$r_0 < r < r' + \frac{\varepsilon(1-r_0)}{4} < \frac{1+r_0}{2} + \frac{\varepsilon(1-r_0)}{4} < r_0 + \frac{3}{4}(1-r_0).$$

Consequently

$$(\log M(r))^\gamma > (\log M(r_0))^\gamma > \frac{4}{\varepsilon(1-r_0)} \ge \frac{1/\varepsilon}{1-\left(r_0+\frac{3}{4}(1-r_0)\right)} > \frac{1}{\varepsilon(1-r)}. \tag{10}$$

Also,

$$(1-\rho) > r - r' = \frac{1}{(\log M(r'))^\gamma} > \frac{1}{(\log M(r))^\gamma}.$$

Therefore, by (8) and (9),

$$\sup_{0\le u\le r} |f(ue^{i\theta})| \ge \exp(-K(\log M(r))^{\gamma((1-\beta)/\beta+\varepsilon')})\cdot(M(r))^{1/(1+\eta)}.$$

Choose ε' so small that $E = \gamma((1-\beta)/\beta+\varepsilon') < 1$. Then

$$\begin{aligned}\sup_{0\le u\le r} |f(ue^{i\theta})| &\ge \exp\left(\frac{\log M(r)}{1+\eta} - K(\log M(r))^E\right)\\ &\ge \exp((1-\varepsilon_0)\log M(r)) = M(r)^{1-\varepsilon_0},\end{aligned} \tag{11}$$

where $\varepsilon_0 < \frac{1}{8}$, say, for r sufficiently near 1.

For these values of r, we shall see (using an argument that originated with Pólya) that the region of $|z| \le r$ in which

$$|f(z)| \ge (M(r))^{1-2\varepsilon_0}$$

encircles the origin. If it did not, there would exist a curve c from the origin to $|z| = r$ along which $|f(z)| < M(r)^{1-2\varepsilon_0}$. Let z_0 be the first point at which c

intersects $|z| = r$, and consider the radius R joining the origin to z_0. Let $\bar{c}$ be the reflection of c across R. Since $|f(z)| \leq M(r)$ on $\bar{c}$, the inequality

$$|f(z')| < M(r)^{1/2} \cdot M(r)^{(1-2\varepsilon_0)/2} = M(r)^{1-\varepsilon_0}$$

holds for all z' on R. This contradicts (11). Hence $|f(z)| \geq M(r)^{1-2\varepsilon_0}$ on some curve K encircling the origin.

Let

$$P(z) = \prod_1^n (1 - z/z_k),$$

where $\{z_k\}$ is the set of zeros of f(z) in $|z| < r$, and set

$$e^{\phi(z)} = \frac{f(z)}{P(z)}. \tag{12}$$

Clearly, for z on K,

$$\log|P(z)| = \sum_1^n \log\left|1 - \frac{z}{z_k}\right| \leq \sum_1^n \log\left(1 + \left|\frac{z}{z_k}\right|\right) \leq K_0 \frac{1}{(1-r)^\lambda}, \tag{13}$$

where K_0 is a positive constant independent of r. We note that

$$K_0 \frac{1}{(1-r)^\lambda} = \frac{K}{\varepsilon^{1/\gamma}} \frac{(1-r)^{1/\gamma-\lambda}}{(1-r)^{1/\gamma}},$$

where K_1 is a positive constant independent of r. Taking values of r_0 (and thus of the associated r) nearer 1, if necessary, we see that (10) and (13) together with the fact that $1/\gamma > \lambda$ now yield

$$\log|P(z)| \leq \frac{1}{2} \cdot \frac{1}{(\varepsilon(1-r))^{1/\gamma}} < \frac{1}{2} \cdot \log M(r)$$

for z on K. Hence, with the aid of (12), the minimum modulus theorem, and the definition of K, we have a sequence of values of r approaching 1 for which we know the inequality

$$e^{\phi(0)} > M(r)^{1/4}.$$

But this is clearly impossible.

4. PROOFS OF THE LEMMAS

Proof of Lemma 1. Suppose (5) is false for x_0. Define x_1 to be the lower bound of all $x > x_0 + 1/\phi(x_0)$ for which the inequality is false. Inductively, define x_n to be the lower bound of all $x > x_{n-1} + 1/\phi(x_{n-1})$ for which the inequality is false. Continuity ensures the falseness of the inequality for the x_i also. Then the total measure of the set of x in $[x_0, 1)$ for which (5) is false is majorized by

$$\sum_{n=0}^{\infty} \frac{1}{\phi(x_n)} < \frac{1}{\phi(x_0)} \sum_{n=0}^{\infty} \left(\frac{1}{1+\varepsilon}\right)^n = \frac{1}{\phi(x_0)} \left(\frac{1+\varepsilon}{\varepsilon}\right).$$

From (4) we obtain the inequality

$$\frac{1}{\phi(x_0)} \left(\frac{1+\varepsilon}{\varepsilon}\right) < \frac{1 - x_0}{2};$$

this completes the proof.

Proof of Lemma 2. Consider the function

$$G(z) = \frac{\varepsilon_0}{(z+1)^2} \prod_{k=0}^{\infty} \frac{m_k + 1 - z}{m_k + 1 + z},$$

where the m_i are the midpoints of the segments (n_k, n_{k+1}) for which ε_k and ε_{k+1} are distinct. By (2), we see that $\sum 1/n_k < \infty$, which implies $\sum 1/m_k < \infty$. Hence $G(z)$ defines an analytic function in $\Re z > -1$.

A Laplace inversion theorem (see Churchill [2, p. 178]) implies that $G(z)$ is the Laplace transform of the function

$$g(e^{-s}) = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} e^{zs} G(z)\, dz,$$

where c_0 is any real number greater than -1. Therefore the function

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iy \log t} G(iy)\, dy$$

satisfies (i), with $A_k = |G(n_k + 1)|$.

(ii) follows easily, because

$$\int_0^1 |g(t)|\, dt \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(iy)|\, dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{1}{1+y^2}\right) \prod_{k=0}^{\infty} \left|\frac{m_k + 1 - iy}{m_k + 1 + iy}\right| dy \leq \frac{1}{2}.$$

To obtain (iii), we proceed to estimate $|G(n_q + 1)|$ for fixed $q > 0$. We note that for positive z

$$|G(z)| \geq \frac{1}{|z+1|^2} \left\{ \prod_{k=0}^{\infty} \left| \frac{\frac{1}{2}(n_k + n_{k+1}) + 1 - z}{\frac{1}{2}(n_k + n_{k+1}) + 1 + z} \right| \right\},$$

so that

$$|G(n_q + 1)| \geq \frac{1}{(n_q + 2)^2} \prod_{k=0}^{\infty} \left| \frac{n_k + n_{k+1} - 2n_q}{n_k + n_{k+1} + 4 + 2n_q} \right|.$$

Setting $\mu_k = n_k + n_{k+1}$, we estimate separately the terms of the products Π_1, Π_2, Π_3 with $\mu_k > 4n_q$, $2n_q < \mu_k \le 4n_q$, and $\mu_k < 2n_q$, respectively.

We have the inequalities

$$
\begin{aligned}
\Pi_1 &= \prod_{\mu_k > 4n_q} \left(\frac{1 - \frac{2n_q}{\mu_k}}{1 + \frac{2(n_q+2)}{\mu_k}} \right) \geq \prod_{\mu_k > 4n_q} \exp\left(- \frac{4n_q}{\mu_k} - \frac{2(n_q+2)}{\mu_k} \right) \\
&\geq \exp\left(-(6n_q + 4) \sum_{S'} \frac{1}{\mu_k} \right),
\end{aligned} \tag{14}
$$

where $S' = \{k \mid \mu_k > 4n_q\}$. But since $\mu_k = n_k + n_{k+1} < 2n_{k+1}$, S' is a subset of $S = \{k \mid n_{k+1} > 2n_q\}$, so that

$$
2 \sum_{S'} \frac{1}{\mu_k} \leq \sum_{S} \frac{1}{n_k} \leq \int_{n_q}^{\infty} \frac{dN^0(u)}{u} = -\frac{N^0(n_q)}{n_q} + \int_{n_q}^{\infty} \frac{N^0(u)}{u^2}\, du. \tag{15}
$$

Therefore, since (2) implies $N^0(u) < K_1 u^{1-\beta}$, we find by combining (14) and (15) that

$$
\Pi_1 \geq \exp(-K_2 n_q^{1-\beta}) \tag{16}
$$

(the K_i denote constants).

Next we observe that

$$
\Pi_2 \geq (6n_q + 4)^{-(K_1 (2n_q)^{1-\beta} - 1 - (q-1))}, \tag{17}
$$

because $n_{k+1} \leq 2n_q$ implies $k + 1 < K_1 (2n_q)^{1-\beta}$, by (2).

Finally,

$$
\begin{aligned}
\Pi_3 &= \prod_{k=0}^{q-1} \left(\frac{2n_q - (n_k + n_{k+1})}{2n_q + 4 + (n_k + n_{k+1})} \right) \geq \left\{ \prod_{k=0}^{q-2} \left(\frac{2n_q - 2n_{k+1}}{2n_q + 4 + 2n_{k+1}} \right) \right\} \cdot \frac{n_q - n_{q-1}}{3n_q + n_{q-1} + 4} \\
&\geq \left\{ \prod_{k=0}^{q-2} \left(\frac{n_q - n_{k+1}}{n_q + 2 + n_{k+1}} \right) \right\} \cdot \frac{n_q - n_{q-1}}{3n_q + n_{q-1} + 4} \geq (2n_q + 2)^{-(q-1)} \cdot (4n_q + 4)^{-1}.
\end{aligned} \tag{18}
$$

From (16), (17), and (18) it follows that

$$
|G(n_q + 1)| \geq (10n_q)^{-(K_1 2^{1-\beta} + 2) n_q^{1-\beta}} \cdot \exp(-K_2 n_q^{1-\beta}) \geq (K_3 n_q)^{-K_4 n_q^{1-\beta}}.
$$

Let

$$
h(u) = (K_3 u)^{-K_4 u^{1-\beta}} \cdot \frac{1}{\rho^u} \qquad (u > 0);
$$

then

$$\frac{h'(u)}{h(u)} = -(1-\beta)u^{-\beta}K_4 \log(K_3 u) - K_4 u^{-\beta} + \log\frac{1}{\rho}.$$

Therefore, for $u > u_0$, we find that $\inf h(u)$ occurs when

$$\log\frac{1}{\rho} \approx (1-\beta)K_4 u^{-\beta}(\log u).$$

That is, setting $\log(1/\rho) = s$, we have the estimate

$$u \approx K_5 s^{-1/\beta}\left(\log\frac{1}{s}\right)^{1/\beta}.$$

Returning then to $h(u)$, we see that

$$\inf_k A_k \rho^{-n_k} > \exp\left(-K s^{-(1-\beta)/\beta} \log^{1/\beta}(1/s)\right)$$

$$> \exp\left(-K(1-\rho)^{-(1-\beta)/\beta} \log^{1/\beta}(1/(1-\rho))\right).$$

Remarks. The method of proof above yields similar results for other assumptions on gaps and corresponding growth. In addition, the proof shows the nonexistence of finite asymptotic paths for such functions.

The theorem of Sunyer I. Balaguer previously mentioned can be stated by replacing (3) in Theorem 1 by the condition

$$\alpha > \max\left(\lambda, \frac{(1-\beta)^{1/2} + (1-\beta)}{\beta}\right).$$

The method of proof of Sunyer I. Balaguer's assertion is considerably more complicated than that of Theorem 1.

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