ON AN UNKNOWN RAMSEY NUMBER

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1. INTRODUCTION

A complete graph on n vertices will be called an *n-clique*. Suppose that each of the $\binom{n}{2}$ edges of the n-clique is painted either red or blue. For $p, q \geq 2$, a (p, q)-*colouring* of the n-clique is a colouring that contains no red p-clique (no set of p points interjoined entirely by red edges) and no blue q-clique. A theorem of Ramsey [7] implies the existence of a least integer M(p, q) such that for $n \geq M$ no (p, q)-colouring of the n-clique exists.

It is obvious that M(p, q) = M(q, p), and M(2, q) = q. Lemmas 1 and 2 were given by Erdös and Szekeres [1]. Lemma 2 provides an inductive upper bound for M(p, q).

LEMMA 1. *Each point in a* (p, q)-*colouring is the endpoint of at most* M(p - 1, q) - 1 *red edges and at most* M(p, q - 1) - 1 *blue edges.*

LEMMA 2. $M(p, q) \leq M(p - 1, q) + M(p, q - 1)$, *with strict inequality if both terms on the right are even.*

Greenwood and Gleason [2] have proved that

$$M(3, 3) = 6, \quad M(3, 4) = 9, \quad M(3, 5) = 14, \quad M(4, 4) = 18.$$

For these cases, Lemma 2 gives the least upper bound, and there exist "extremal" (p, q)-colourings of a remarkably symmetrical nature. However, for $q > 5$ the evaluation of M(3, q) is much more difficult. Kéry [6] and Kalbfleisch [4] have proved that M(3, 6) = 18. In this case, the upper bound given by Lemma 2 is not best, and this seems to be the case in general. Several of these upper bounds are reduced in [5]. Furthermore, all (3, 6)-colourings of the 17-clique are irregular, and this may also be the case in general.

M(3, 7) is the smallest unknown Ramsey number. Lemma 2 yields an upper bound of 25 for M(3, 7), but it is proved in [5] that in fact $M(3, 7) \leq 24$. The best lower bound given previously is $M(3, 7) \geq 22$ (see [3] and [6]). In this paper, we construct a (3, 7)-colouring of the 22-clique, and the uncertainty is thus reduced to the inequalities

$$23 \leq M(3, 7) \leq 24.$$

In addition, we discuss the structure of (3, 7)-colourings of the 22-clique. We are unable to establish whether or not there exists a (3, 7)-colouring of the 23-clique, but we make some progress toward solving this problem.

2. PRELIMINARY RESULTS

Two colourings G and H will be called *isomorphic* if there exists a one-to-one mapping f of the vertices of G onto the vertices of H such that for each vertex pair

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XY in G, edge f(X)f(Y) in H has the same colour as edge XY in G. A (p, q)-colouring G of the n-clique will be called *unique* if every other (p, q)-colouring of the n-clique is isomorphic to G. If a vertex is the endpoint of m red (blue) edges, it will be called m-valent in red (blue).

Several uniqueness theorems for (p, q)-colourings were proved in [4] and [6]. Only two of these will be needed here. *Note that in all diagrams only the red edges are drawn.*

LEMMA 3. *Of the twelve non-isomorphic* (3, 5)*-colourings of the* 12*-clique, only one* (*given by the subgraph of lettered vertices in Figure* 2(i)) *contains* 20 *red edges.*

LEMMA 4. *Of the seven non-isomorphic* (3, 6)*-colourings of the* 17*-clique, only one* (*given in Figure* 2(i)) *has both the following properties:*

(a) *It contains* 40 *red edges:*

(b) *One of its points* (*the point* 1 *in Figure* 2(i)) *is* 4*-valent in red, and each of the four points* (2, 3, 4, 5) *to which it is joined by red edges is* 5*-valent in red.*

Lemma 3 was proved by Kéry [6], who also constructed the (3, 6)-colouring referred to in Lemma 4. A simpler proof of Lemma 3 was given in [4], where a complete list of all (3, 5)-colourings of the 12-clique will be found. Lemma 4 is proved in [4], and a complete list of all (3, 6)-colourings of the 17-clique is given there.

The following lemma on the possible structures of (3, 6)-colourings of the 17-clique was also proved in [4], where it was used in the reduction of the upper bound for M(3, 7) to 24, and in the proof that in a (3, 7)-colouring of the 23-clique, each point has red valency 6. It is this lemma which suggests the construction to be attempted in Section 4.

LEMMA 5. *In a* (3, 6)*-colouring of the* 17*-clique, each point is* 4*-valent or* 5*-valent in red, and the number of* 4*-valent points is* 1, 3, *or* 5. *There are* 40, 41, *or* 42 *red edges.*

3. SOME STRUCTURAL PROPERTIES OF (3, 7)-COLOURINGS

By Lemma 1, in a (3, 7)-colouring, each point has red valency at most M(2, 7) - 1 = 6 and blue valency at most M(3, 6) - 1 = 17. Since in the 22-clique there are 21 edges from each point, only three types of points may occur in a (3, 7)-colouring of the 22-clique:

A — endpoints of 4 red edges and 17 blue edges,

B — endpoints of 5 red edges and 16 blue edges,

C — endpoints of 6 red edges and 15 blue edges.

LEMMA 6. *In a* (3, 7)*-colouring of the* 22*-clique, no point can be joined by red edges to more than one point of type* A.

Proof. If some point 1 were joined by red edges to two points 2 and 3 of type A, then 2 - 3 would be blue, and 3 would be at most 3-valent in red within the 17-clique joined to 2 by blue edges. By Lemma 5, this 17-clique contains a red triangle or a blue 6-clique. But a blue 6-clique joined by blue edges to 2 is a blue 7-clique. Thus the 22-clique could not be (3, 7)-coloured.

Now consider a (3, 7)-colouring of the 22-clique containing a point 1 of type A. Let $\mathscr{X}$ denote the subgraph formed by the 4 points joined to 1 by red edges, and $\mathscr{Y}$ the subgraph formed by the 17 points joined to 1 by blue edges. Let $\mathscr{X}$ contain x_A, x_B, x_C points of types A, B, C, respectively. Then by Lemma 6,

$$x_A + x_B + x_C = 4, \quad x_A \leq 1. \tag{1}$$

Now $\mathscr{Y}$ must be (3, 6)-coloured, and we can apply Lemma 5 to $\mathscr{Y}$. A point of type A in $\mathscr{Y}$ is joined by red edges to no points in $\mathscr{X}$; let there be y_0^A such points. A point of type B in $\mathscr{Y}$ is joined by red edges to no points in $\mathscr{X}$ or to one point in $\mathscr{X}$; let there be y_0^B and y_1^B such points. Finally, a point of type C in $\mathscr{Y}$ is joined by red edges to 1 or 2 points in $\mathscr{X}$; let there be y_1^C and y_2^C such points. Then, by Lemma 5,

$$y_0^A + y_0^B + y_1^B + y_1^C + y_2^C = 17, \tag{2}$$

$$y_0^A + y_1^B + y_2^C = 1, 3, \text{ or } 5. \tag{3}$$

By counting the number of red edges from $\mathscr{X}$ to $\mathscr{Y}$ and using (1), we find that

$$y_1^B + y_1^C + 2y_2^C = 20 - X_A - X_B. \tag{4}$$

Also, $\mathscr{X}$ contains no red edges, and thus there can be no blue triangle joined to $\mathscr{X}$ by blue edges. The number of points joined by blue edges to all points in $\mathscr{X}$ is at most M(3, 3) - 1 = 5. Thus

$$y_0^A + y_0^B \leq 5. \tag{5}$$

If there were four points joined by red edges to a particular point of $\mathscr{X}$ and by blue edges to the other three points of $\mathscr{X}$, there would be a blue 7-clique. Therefore

$$y_1^B + y_1^C \leq 3 \cdot 4 = 12. \tag{6}$$

For each set (X_A, X_B, X_C) satisfying (1), we can write down all the numbers y satisfying conditions (2) to (6). There are 68 possibilities, but most of these may be eliminated. It soon becomes apparent that the best prospects for (3, 7)-colourings lie with cases having a large number of red edges from $\mathscr{X}$ to $\mathscr{Y}$. Accordingly, we consider the case $x_A = x_B = 0$, $x_C = 4$. Of the three sets of y's corresponding to these x-values, we take the simplest,

$$y_0^A = 1, \quad y_0^B = y_1^B = 0, \quad y_1^C = 12, \quad y_2^C = 4.$$

In the following section, we shall construct a (3, 7)-colouring of the 22-clique with these parameters.

It would be interesting to know for what sets of parameters (3, 7)-colourings exist. This information would be of considerable help in establishing the value of M(3, 8). Although some results have been obtained (for example, there can be at most two points of type A), an examination of all cases has not been completed.

4. CONSTRUCTION OF A (3, 7)-COLOURING OF THE 22-CLIQUE

We shall now construct a (3, 7)-colouring of the 22-clique with the parameters set out in Section 3. A point 1 of type A is joined by red edges to points 2, 3, 4, 5 of type C. There is a point 6, also of type A, joined by blue edges to 1, 2, 3, 4, 5 ($y_0^A = 1$). All remaining points 7, 8, ⋯, 22 are of type C. Twelve of these ($y_1^C = 12$), say 7, 8, ⋯, 18, are joined by red edges to exactly one of the points 2, 3, 4, 5, while 19, 20, 21, 22 are joined by red edges to two of 2, 3, 4, 5. By the argument preceding condition (6), each of the points 2, 3, 4, 5 is joined by red edges to exactly three of the points 7, 8, ⋯, 18. Let 2-7, 2-11, 2-15; 3-8, 3-12, 3-16; 4-9, 4-13, 4-17; 5-10, 5-14, 5-18 be red (Figure 1(i)).

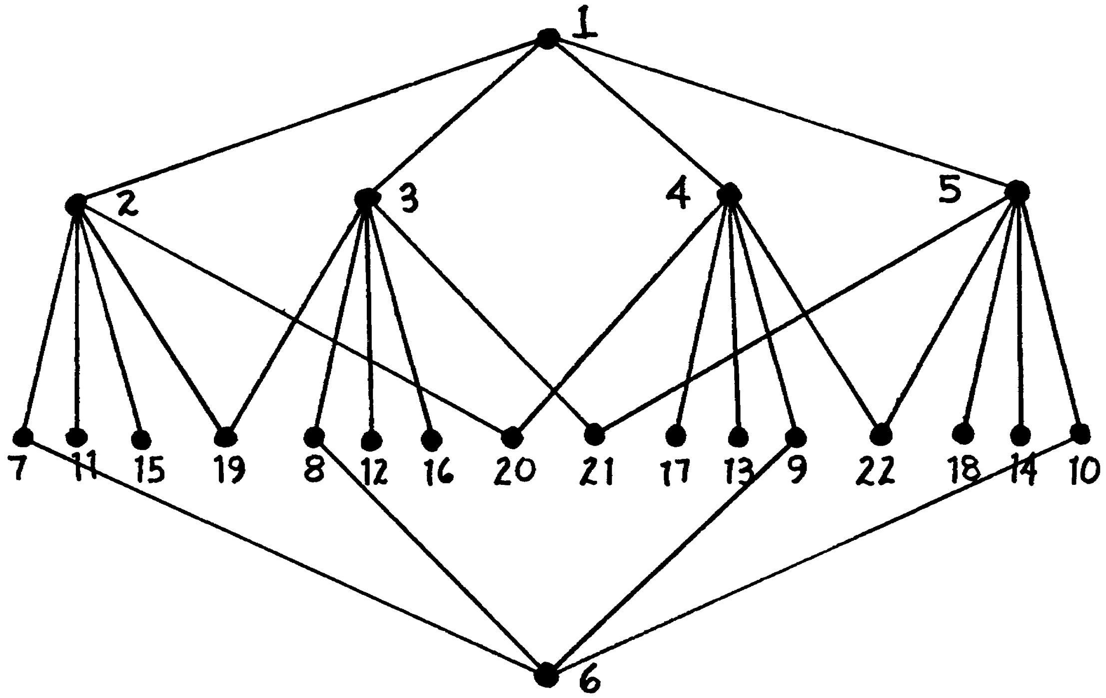

Figure 1(i).

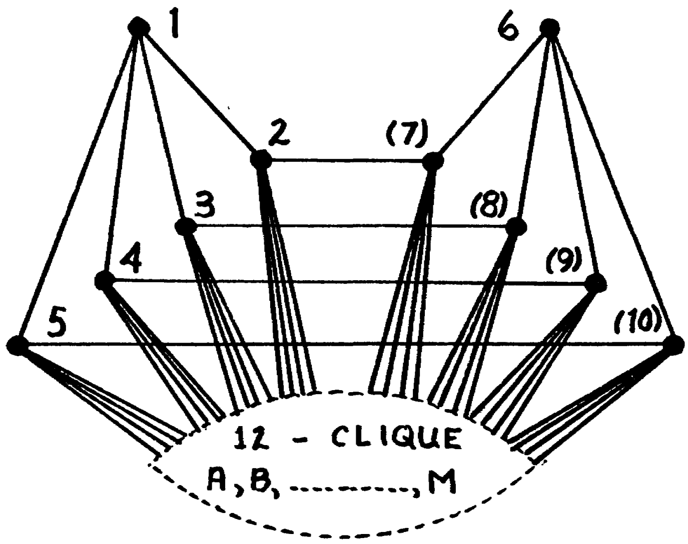

Figure 1(ii).

The point 2 is also joined by red edges to 19 and 20. By symmetry, we may take 3-19 to be red. If 3-20 were also red, {7, 11, 15, 19, 20, 4, 5} would be a blue 7-clique. Therefore 3-20 is blue and 4-20, say, is red. Now 5-21 and 5-22 must be red. By symmetry, take 3-21 and 4-22 to be red.

{6, 7, 11, 15, 3, 4, 5} is a blue 7-clique, unless 6 is joined by a red edge to 7, 11, or 15. Take 6-7 red, by symmetry. Similarly, 6-8, 6-9, and 6-10 are red. There are no more red edges from 6.

Thus in a (3, 7)-colouring of the type sought, the situation must be as illustrated in Figure 1(ii). Each of the points joined to 1 by a red edge is joined by a red edge to one of the points joined to 6 by a red edge. For convenience of notation, rename the remaining points A, B, ···, H, J, ···, M. This 12-clique is joined by blue edges to both 1 and 6, and it must be (3, 5)-coloured. All points of the graph other than 1 and 6 are 6-valent in red. There are $4 \cdot 8 = 32$ red lines from 2, 3, 4, 5, 7, 8, 9, 10 to this 12-clique, and the 12-clique thus contains $\frac{1}{2}(6 \cdot 12 - 32) = 20$ red lines. By Lemma 3, it is coloured as in Figure 2(i).

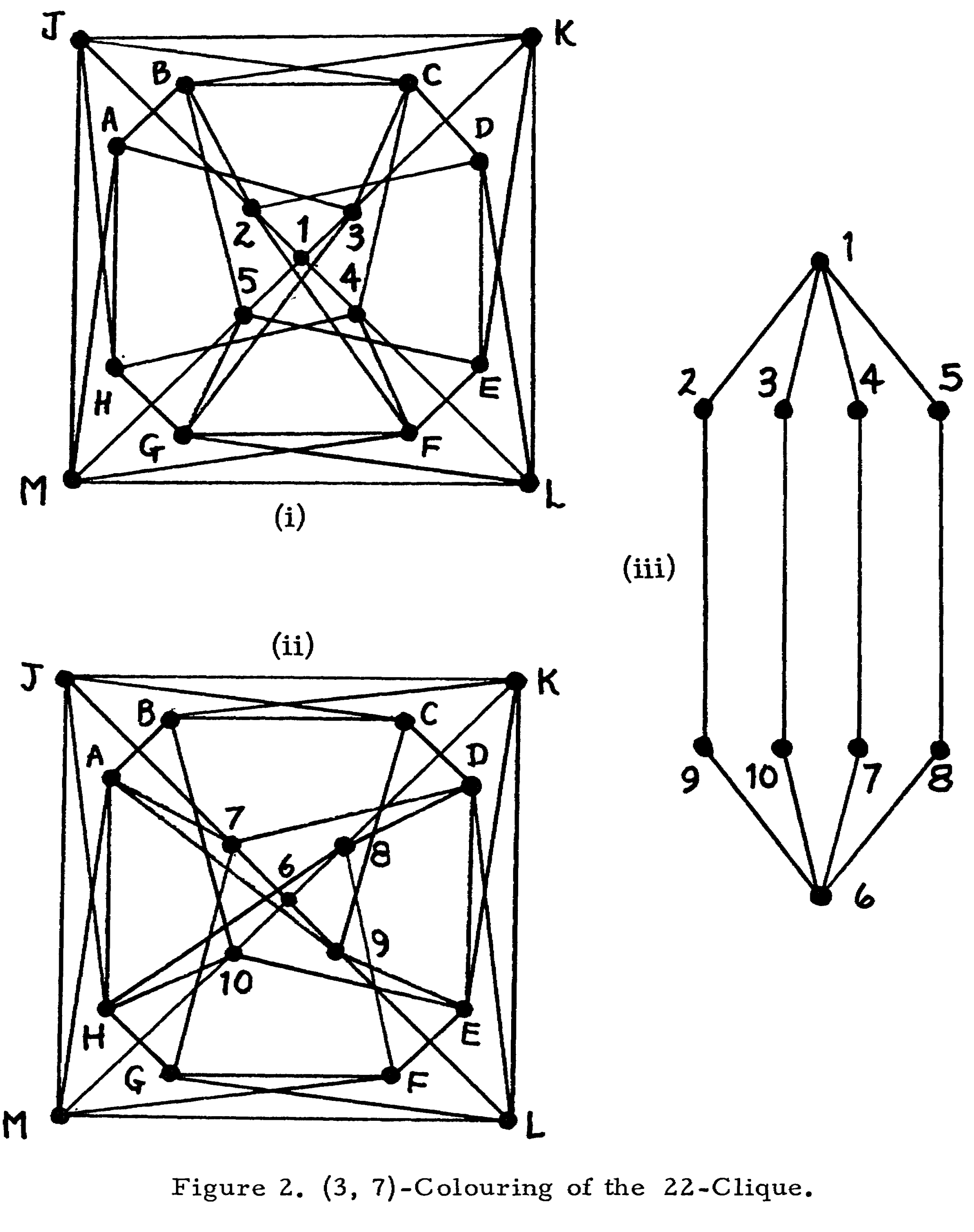

Figure 2. (3, 7)-Colouring of the 22-Clique.

The 17-clique $\{1, 2, 3, 4, 5, A, B, \cdots, M\}$ joined by blue edges to 6 must be (3, 6)-coloured. In this colouring, 1 is 4-valent in red and is joined by red edges to four points 2, 3, 4, 5 that are 5-valent in red in the 17-clique. The 17-clique contains $4 + 4 \cdot 4 + 20 = 40$ red edges. By Lemma 4, it may be coloured uniquely as in Figure 2(i).

The same comments apply to the 17-clique $\{6, 7, 8, 9, 10, A, \cdots, M\}$ joined by blue edges to 1, and it may also be coloured uniquely by Lemma 4. However, A, B, $\cdots$, M are common to the two 17-cliques, and each of these points must be 6-valent in red in the completed colouring. Because of the way in which the graphs have been drawn, it is easy to see how this may be accomplished. To get the colouring of the second 17-clique given in Figure 2(ii), merely "rotate" the picture in Figure 2(i) through a quarter turn clockwise. If Figures 2(i) and 2(ii) are taken together, each of A, B, $\cdots$, M is 6-valent in red.

Finally, each of 2, 3, 4, 5 must be joined by a red edge to one of 7, 8, 9, 10 in such a way that no red triangles are created. Now 2 is joined by red edges to 1, B, D, F, J, and 7 is joined by red edges to 6, A, D, G, J. If 2-7 were red, 2-7-D would be a red triangle. By similar arguments it may be shown that the only red edges that we may add without introducing any red triangles are 2-9, 3-10, 4-7, and 5-8 (Figure 2(iii)).

We shall now prove that Figures 2(i), (ii), (iii) together give the red edges in a (3, 7)-colouring of the 22-clique. First, any red triangle in the overall graph must have one vertex in 1, 2, 3, 4, 5 and one in 6, 7, 8, 9, 10, because Figures 5.2(i) and (ii) are (3, 6)-colourings. However, red edges from 2, 3, 4, 5 to 6, 7, 8, 9 were added in such a way that no red triangles were created.

No blue 7-clique contains 1, because the 17-clique joined by blue edges to 1 is (3, 6)-coloured (Figure 2(ii)). Similarly, no blue 7-clique contains 6. A blue 7-clique containing at most one of 2, 3, 4, 5 would imply a blue 6-clique lying in 6, 7, $\cdots$, 10, A, $\cdots$, M, contrary to the facts. Therefore, any blue 7-clique contains at least two of the points 2, 3, 4, 5 and at least two of 7, 8, 9, 10 as well. Since 2, 3, 4, 5 are paired by red edges with 7, 8, 9, 10, any blue 7-clique contains exactly two of 2, 3, 4, 5 and exactly two of 7, 8, 9, 10. There are $\binom{4}{2} = 6$ possibilities:

(i) $2 + 3 + 7 + 8 +$ blue triangle in ELM (LM red),

(ii) $2 + 4 + 8 + 10 +$ blue triangle in FH,

(iii) $2 + 5 + 7 + 10 +$ blue triangle in CKL (KL red),

(iv) $3 + 4 + 8 + 9 +$ blue triangle in BJM (MJ red),

(v) $3 + 5 + 7 + 9 +$ blue triangle in AG,

(vi) $4 + 5 + 9 + 10 +$ blue triangle in DJK (JK red).

None of these gives a blue 7-clique. Therefore, a (3, 7)-colouring of the 22-clique has been constructed, and $M(3, 7) \geq 23$.

5. (3, 7)-COLOURINGS OF THE 23-CLIQUE

It is possible to show, by means of Lemma 5, that if there exists a (3, 7)-colouring of the 23-clique, all points in it must be 6-valent in red. A proof is given in [5]. Thus, M(3, 7) is 24 or 23 according to whether a (3, 7)-colouring of the 23-clique with all points 6-valent in red exists or does not exist.

Suppose that such a colouring exists, and let 1 be any point in it. Denote by $\mathscr{X}$ the subgraph of 6 points joined to 1 by red edges, and by $\mathscr{Y}$ the subgraph of 16 points joined to 1 by blue edges. $\mathscr{X}$ can contain no red edges, and $\mathscr{Y}$ must be (3, 6)-coloured with $\frac{1}{2}(6 \cdot 16 - 5 \cdot 6) = 33$ red edges. Although it is demonstrated in [4] that such (3, 6)-colourings do exist, results as strong as Lemmas 4 and 5 are not available, and appear quite difficult to obtain. It will thus be more difficult to construct a colouring or to prove its nonexistence than it was for the 22-clique in Section 3.

Define y_i to be the number of points in $\mathscr{Y}$ that are joined by red edges to i points in $\mathscr{X}$ $(i = 0, 1, \cdots, 6)$. Lemma 1 implies that every point of $\mathscr{Y}$ is 2, 3, 4, or 5-valent in red in the (3, 6)-colouring of $\mathscr{Y}$, and is therefore joined by red edges to 4, 3, 2, or 1 points in $\mathscr{X}$. Thus

(a) $$y_1 + y_2 + y_3 + y_4 = 16.$$

By arguments similar to those used to obtain (4) and (6) in Section 3,

(b) $$y_1 + 2y_2 + 3y_3 + 4y_4 = 30,$$

(c) $$y_1 \leq 6.$$

We shall now prove that $y_4 = 0$. Let $\mathscr{X}$ consist of the points 2, 3, ⋯, 7, and $\mathscr{Y}$ of the points 8, 9, ⋯, 23. Suppose that $y_4 > 0$; some point 8 in $\mathscr{Y}$ is joined by red edges to four points 2, 3, 4, 5 in $\mathscr{X}$ and to 9, 10 in $\mathscr{Y}$ (Figure 3). Then 1-8 is blue, and 1, 8 are both joined by blue edges to the 13-clique $\{11, 12, \cdots, 23\}$. This 13-clique contains no red triangle, and if it contained a blue 5-clique, the whole graph would contain a blue 7-clique. The 13-clique must be (3, 5)-coloured, and by Lemma 1, all points of the 13-clique are 4-valent in red and 8-valent in blue within the 13-clique.

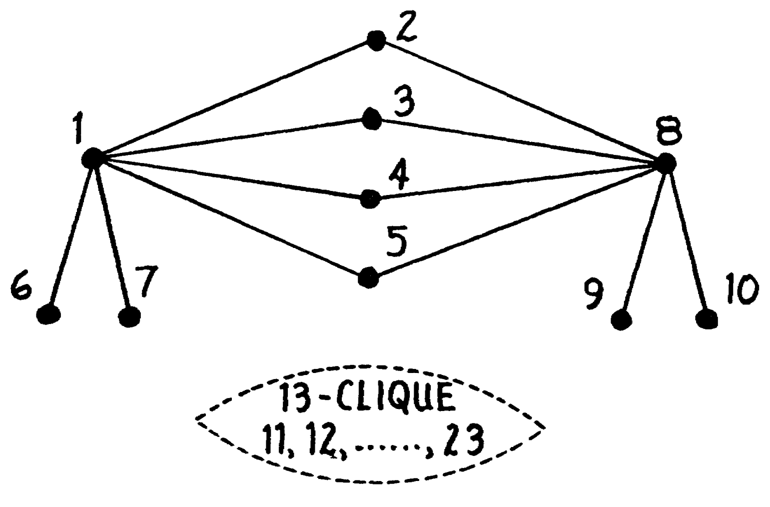

Figure 3.

But there are no red triangles, and therefore there are $4 \cdot 4 = 16$ red edges from 2, 3, 4, 5 and at least $3 \cdot 4 = 12$ red edges from 6, 7, 9, 10 to this 13-clique. The number of red edges within the 13-clique is at most $\frac{1}{2}(6 \cdot 13 - 16 - 12) = 25$. The 13-clique contains a point at most 3-valent in red—a contradiction. Thus $y_4 = 0$.

It is also proved in [4] that $y_3 > 0$. For every point 1 in a (3, 7)-colouring of the 23-clique, the parameters y_i defined above satisfy the relations

(a) $$y_1 + y_2 + y_3 = 16,$$

(b) $$y_1 + 2y_2 + 3y_3 = 30,$$

(c) $$y_1 \leq 6,$$

(d) $$y_3 \geq 1.$$

There exist five parameter sets satisfying (a) to (d). I think it likely that these possibilities can also be eliminated, and this would prove that M(3, 7) = 23. However I have not yet succeeded in doing this, and the question remains unresolved.

Added June 7th, 1966: At the conference in Combinatorial Mathematics held at the University of Waterloo on April 4 to 9, 1966, Professor Jack Graver of Dartmouth College announced that he and Dr. Jim Yackel had proved that there exist no (3, 7)-colourings of the 23-clique. This would establish M(3, 7) = 23.

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