BOUNDARY CORRESPONDENCE UNDER QUASICONFORMAL MAPPINGS

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1. INTRODUCTION

It is known that a quasiconformal mapping of $x^2 + y^2 < 1$ onto $u^2 + v^2 < 1$ can be extended to a homeomorphism between the closed discs [1]. In view of the conformal invariance of quasiconformal mappings, these closed domains can be mapped onto the half-planes $y \geq 0$ and $v \geq 0$, respectively, by Moebius transformations under which the points at infinity correspond. The boundary correspondence is then determined by a monotone continuous function u(x), in the sense that the point (x, 0) is mapped onto (u(x), 0) (it is sufficient to consider the case where u(x) is strictly increasing). By reflection in the real axes, we obtain a quasiconformal mapping of the extended plane onto itself. It follows from a result of A. Mori [9] that

$$\exp(-\pi K) \leq \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leq \exp(\pi K)$$

for all real x and all $t > 0$, where K is the maximal dilatation of the quasiconformal mapping. This condition indicates that u(x) possesses a degree of approximate symmetry, and for this reason we refer to u(x) as a *quasisymmetric function.*

A. Beurling and L. Ahlfors introduced these boundary functions and characterized them by means of a certain compactness criterion (see [4] and Section 7 of this paper). An analogous characterization can be given for quasiconformal mappings of the plane (or space [6]). Hence we can regard quasisymmetric functions as one-dimensional quasiconformal mappings, and we naturally expect them to have properties analogous to those of two-dimensional quasiconformal mappings. In the present paper we examine quasisymmetric functions, in the hope of attaining a better understanding of quasiconformal mappings. In Section 2 we define quasisymmetric functions and determine the one-dimensional conformal mappings. In Sections 3 to 8 we investigate properties of quasisymmetric functions. In Section 9 we show how these functions can be used to shed some light on a problem in quasiconformal mapping. In the final section we introduce the concept of local quasisymmetry, and we generalize a result of Beurling and Ahlfors.

2. QUASISYMMETRIC FUNCTIONS

Let u(x) be a strictly increasing, continuous, real-valued function of a real variable, defined on an interval (a, b) $(-\infty \leq a < b \leq \infty)$.

Definition 1. The function u(x) is k-*quasisymmetric* $(1 \leq k < \infty)$ if

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$$\frac{1}{k} \leq \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leq k \tag{1}$$

whenever $a < x - t < x < x + t < b$. A function is *quasisymmetric* if it is k-quasisymmetric for some k.

For convenience, we set

$$Q_u(x, t) = Q(x, t) = \frac{u(x+t) - u(x)}{u(x) - u(x-t)}; \tag{2}$$

where there is no confusion about the function that is under consideration, we shall omit the subscript. The *maximal dilatation* of u(x), denoted by k(u), is the infimum of all k for which (1) is satisfied.

Let us determine the 1-quasisymmetric functions. Certainly, linear functions are 1-quasisymmetric. On the other hand, if $a < x < y < b$, then (1) implies

$$u\left(\frac{x+y}{2}\right) = \frac{1}{2}[u(x) + u(y)].$$

By repeated application it follows that u(x) is identical with a linear function on a dense subset of each closed subinterval of (a, b). The continuity of u(x) implies that u(x) is itself linear on (a, b).

THEOREM 1. *A function* u(x) *is* 1-*quasisymmetric if and only if it is linear.*

It is easy to verify that the composition of a k-quasisymmetric function with a linear function yields a k-quasisymmetric function. Hence k-quasisymmetric functions are *"conformally" invariant.*

3. INVARIANCE OF DOMAIN TYPE

It is known that a plane quasiconformal mapping of a simply connected domain preserves the domain type [9]. On the line we distinguish four types of domains: the entire line, finite open intervals, open rays infinite to the left, and open rays infinite to the right.

THEOREM 2. *Quasisymmetric functions preserve domain type.*

Proof. Suppose u(x) is a quasisymmetric function defined on an interval (a, b). Since u(x) is monotone, its image is an interval (c, d). We shall prove that if a is finite, then c is finite (all other cases can be treated similarly). Assume to the contrary that $c = -\infty$. Fix $0 < t < (b - a)/3$, and consider Q(x, t) for

$$a + t < x < a + 2t.$$

The numerator of Q is continuous in the closure of this interval, and hence it is bounded. On the other hand, as $x \to a + t$, the denominator becomes unbounded, and hence $Q(x, t) \to 0$, which is impossible.

As an application of this theorem we note that a function quasisymmetric on a finite open interval is continuous and quasisymmetric on the closed interval.

4. REMOVABLE POINTS AND THE REFLECTION PRINCIPLE

Suppose $u(x)$ is continuous on (a, b) and quasisymmetric on (a, c) and (c, b) $(a < c < b)$. Then $u(x)$ need not be quasisymmetric on (a, b). For example, if $u(x) = \sqrt{x}$ $(x \geq 0)$ and $u(x) = -x^2$ $(x < 0)$, then $u(x)$ is quasisymmetric on $(-\infty, 0)$ and on $(0, \infty)$ [4]. However,

$$Q(0, t) = \frac{t^{1/2}}{t^2} = t^{-3/2},$$

and hence $u(x)$ is not quasisymmetric in any neighborhood of the origin.

In this section we derive necessary and sufficient conditions for the function $u(x)$ to be quasisymmetric on (a, b). There are two cases. If (a, b) is the entire line, then $u(x)$ is quasisymmetric on (a, b) if and only if $Q(c, t)$ is bounded away from zero and infinity, for all $t > 0$. If either a or b is finite, then $u(x)$ is quasisymmetric on (a, b) if and only if there exists a positive δ such that $Q(c, t)$ is bounded away from zero and from infinity, for $0 < t < \delta$.

THEOREM 3. *Suppose* $u(x)$ *is continuous on* $(-b, b)$ $(0 < b \leq \infty)$ *and* k*-quasisymmetric on* $(0, b)$ *and on* $(-b, 0)$. *Suppose further that there exists* $k_0 \geq 1$ *such that*

$$1/k_0 \leq Q(0, t) \leq k_0 \tag{3}$$

for all $0 < t < b$. *Then* $u(x)$ *is quasisymmetric on* $(-b, b)$, *and*

$$k(u) \leq k_0(1 + k + k^2). \tag{4}$$

Proof. Let $-b < x - t < x < x + t < b$. We must prove that $1/K \leq Q(x, t) \leq K$, where $K = k_0(1 + k + k^2)$. If $x + t \leq 0$, or $0 \leq x - t$, or $x = 0$, the result follows from the hypotheses of the theorem. We may therefore assume that $x - t < 0 < x$. The case $x < 0 < x + t$ will follow by symmetry.

Divide the interval $[-(x + t), (x + t)]$ into six equal parts by points a_i $(i = 0, \cdots, 6)$. Then $a_0 = -(x + t)$, $a_3 = 0$, and $a_6 = x + t$. Let

$$b_i = u(a_i) - u(a_{i-1}) \qquad (i = 1, \cdots, 6).$$

Then $1/k \leq b_i/b_{i-1} \leq k$ for $i = 2, 3, 5, 6$ and $1/k_0 \leq b_4/b_3 \leq k_0$. Now

$$u(x + t) - u(x) \leq b_4 + b_5 + b_6 \quad \text{and} \quad u(x) - u(x - t) \geq \min(b_3, b_4).$$

Therefore

$$Q(x, t) \leq \frac{b_4 + kb_4 + k^2 b_4}{b_4/k_0} = k_0(1 + k + k^2).$$

To obtain a lower bound greater than $1/K$, divide the interval $[-(x + t), (x + t)]$ into four equal parts. Using an analogous notation, we obtain the inequalities

$$\frac{1}{k} \leq \frac{b_2}{b_1}, \frac{b_4}{b_3} \leq k \quad \text{and} \quad \frac{1}{k_0} \leq \frac{b_3}{b_2}, \frac{b_3 + b_4}{b_1 + b_2} \leq k_0.$$

Also,

$$u(x+t) - u(x) \geq b_4, \qquad u(x) - u(x-t) \leq b_1 + b_2 + b_3.$$

Hence

$$Q(x, t) \geq \frac{b_4}{b_1 + b_2 + b_3}.$$

We consider four cases, depending on whether b_2/b_1 and b_4/b_3 lie between 1 and k or between $1/k$ and 1; in each case, $Q(x, t) \geq 1/K$. We omit the details of the proof.

If we merely assume that (3) holds for all sufficiently small values of t, Theorem 3 is no longer valid. For example, consider the function

$$u(x) = x \quad (x \leq 0), \qquad u(x) = x + x^2 \quad (x \geq 0),$$

which is quasisymmetric on $(-\infty, 0)$ and on $(0, \infty)$. Furthermore,

$$Q(0, t) = (t + t^2)/t = 1 + t,$$

and therefore $Q(0, t)$ is bounded away from zero and from infinity in each finite neighborhood of the origin. However, $u(x)$ is not quasisymmetric on the entire line, since $Q(0, t)$ is unbounded for $t > 0$.

COROLLARY 1. *Let* $b = \infty$, *and let* $u(x)$ *satisfy the hypotheses of Theorem* 3, *with the exception that* (3) *holds only for* $0 < t < \delta < \infty$. *Then* $u(x)$ *is quasisymmetric on each finite interval.*

Proof. Suppose $a < \infty$. By Theorem 3 and the hypotheses of the corollary, we need only show that $Q(0, t)$ is bounded away from zero and from infinity for $\delta \leq t < a$. But in this range,

$$\frac{u(\delta) - u(0)}{u(0) - u(-a)} \leq \frac{u(t) - u(0)}{u(0) - u(-t)} \leq \frac{u(a) - u(0)}{u(0) - u(-\delta)}.$$

If we set $k_0 = 1$, we obtain the *reflection principle* for quasisymmetric functions.

COROLLARY 2. *Suppose* $u(x)$ *is symmetric about the origin and* k-*quasisymmetric on* $(0, b)$. *Then* $u(x)$ *is quasisymmetric on* $(-b, b)$, *and*

$$k(u) \leq 1 + k + k^2. \tag{5}$$

In contrast to the reflection principle for quasiconformal mappings, the maximal dilatation of the reflected quasisymmetric function may increase. It is shown in [4] that if $u(x) = x^2$ $(x > 0)$, then $k(u) = 3$, whereas the maximal dilatation of the reflected function, $U(x) = (\operatorname{sgn} x)x^2$, is $2 + \sqrt{5}$.

THEOREM 4. *Suppose* $u(x)$ *is continuous on* $(-1, b)$ $(1 \leq b \leq \infty)$, *and* k-*quasisymmetric on* $(-1, 0)$ *and on* $(0, b)$. *Suppose further that there exist* k_0 *and* δ $(k_0 \geq 1,\ 0 < \delta \leq 1)$ *such that*

$$\frac{1}{k_0} \leq Q(0, t) \leq k_0$$

for $0 < t < \delta$. *Then* $u(x)$ *is quasisymmetric on* $(-1, b)$.

Proof. By the "conformal" invariance of quasisymmetric functions, we may assume $u(0) = 0$, $u(1) = 1$. Two applications of the method of Corollary 1 show that $k(u) = k_1 < \infty$ on $(-1, 3)$.

Let $-1 < x < b$. If $t \leq 1$, it is clear that

$$\frac{1}{\max(k_1, k)} \leq Q(x, t) \leq \max(k_1, k).$$

For $t \geq 1$, we need only consider the case where $-1 \leq x - t < 0$ and $1 \leq x$. Here $x < t \leq x + 1$, hence $2x < x + t \leq 2x + 1$ and

$$Q(x, t) = \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leq \frac{u(2x+1) - u(x)}{u(x)}$$

$$= \left(1 + \frac{u(1)}{u(x)}\right)\left(\frac{u(2x+1) - u(x)}{u(x) + u(1)}\right) \leq 2(1 + k + k^2)$$

(the last inequality follows from the reflection principle). Also,

$$\frac{1}{Q(x, t)} = \frac{u(x) - u(x-t)}{u(x+t) - u(x)} \leq \frac{u(x) - u(-1)}{u(2x) - u(x)}$$

$$\leq \frac{u(x) - u(0)}{u(2x) - u(x)} + (u(0) - u(-1))\left(\frac{u(x) - u(0)}{u(2x) - u(x)}\right) \leq k(1 - u(-1));$$

this completes the proof.

5. QUASISYMMETRIC CONTINUATION

Ahlfors [2] showed that a quasiconformal mapping of a Jordan domain onto a disc can be extended to a quasiconformal mapping of the entire plane if and only if the boundary of the domain possesses a certain geometric property (see also [10]). Suppose $u(x)$ is a quasisymmetric function that maps (a, b) onto (c, d). Since these intervals are always of the same type, $u(x)$ can be extended to a homeomorphism $U(x)$ of the entire line onto itself. If $U(x)$ is itself quasisymmetric, we say that $U(x)$ is a *quasisymmetric continuation* of $u(x)$. We allow $k(U) > k(u)$. In this section we prove that quasisymmetric continuation is always possible. In fact, it is always possible to find a quasisymmetric continuation that is continuously differentiable outside the closure of the original interval (see [7] for a simple proof based on the theory of quasiconformal mappings).

THEOREM 5. *Suppose* $u(x)$ *is a* k*-quasisymmetric function that maps* $(0, 1)$ *onto itself. Then there exists a quasisymmetric continuation* $U(x)$ *of* $u(x)$ *to the entire real line such that* $k(U) \leq 28k^4$.

Proof. Define $U(x)$ by repeated reflection of $u(x)$; that is, on $(-1, 0)$ let $U(x) = -u(-x)$, and then extend the definition by the rule $U(x+2) = U(x) + 2$.

By two applications of the reflection principle, we conclude that for each integer n, $U(x)$ is K-quasisymmetric on $(n, n+3)$, with

$$K = 1 + (1 + k + k^2) + (1 + k + k^2)^2 \leq 13k^4.$$

Hence, if $t \leq 1$, then $1/13k^4 \leq Q(x, t) \leq 13k^4$. Furthermore,

$$\frac{2}{U(x+1) - U(x)} = \frac{U(x+2) - U(x)}{U(x+1) - U(x)} = \frac{U(x+2) - U(x+1)}{U(x+1) - U(x)} + 1 \leq 13k^4 + 1 \leq 14k^4.$$

Therefore $U(x+1) - U(x) \geq 1/7k^4$. Suppose then that $t \geq 1$, and let $n \geq 1$ be the integer such that $n \leq t < n+1$. Then $t/n \geq 1$, and using the above result n times, we obtain the inequalities

$$\frac{U(x+t) - U(x)}{t} \geq \frac{n/(7k^4)}{n+1} \geq \frac{1}{14k^4}.$$

Also, since $t/n < (n+1)/n \leq 2$, we deduce from the recursion relation for $U(x)$ that

$$\frac{U(x) - U(x-t)}{t} \leq \frac{2n}{n} = 2.$$

Therefore $Q(x, t) \geq 1/28k^4$. A similar argument shows that $Q(x, t) \leq 28k^4$, which completes the proof.

Before constructing a continuously differentiable quasisymmetric continuation, we need a preliminary result. Suppose $u(x)$ is k-quasisymmetric on the entire line. For fixed a, define

$$V(x) = \int_0^1 u[(x-a)s + a]\,ds.$$

LEMMA 1. *The function* $V(x)$ *is* k-*quasisymmetric, has a continuous derivative for* $x \neq a$, *and satisfies the inequalities*

$$\frac{1}{1+k}[u(x) - u(a)] \leq V(x) - u(a) \leq \frac{k}{1+k}[u(x) - u(a)]. \tag{6}$$

Proof. Since, by a change of variable,

$$V(x) = \frac{1}{x-a}\int_a^x u(t)\,dt$$

for $x \neq a$, the differentiability is clear. The k-quasisymmetry of $V(x)$ follows if we integrate inequality (1) after multiplying by the denominator of the middle member.

Now fix x $(x \neq a)$ and consider the function

$$\alpha(s) = \frac{u[(x-a)s + a] - u(a)}{u(x) - u(a)}.$$

By the "conformal" invariance of quasisymmetric functions, $\alpha(s)$ is k-quasisymmetric. Furthermore, $\alpha(0) = 0$ and $\alpha(1) = 1$. Hence, by a result of Beurling and Ahlfors [4, page 137],

$$\frac{1}{1+k} \leq \int_0^1 \alpha(s)\,ds \leq \frac{k}{1+k}, \tag{7}$$

which yields (6).

THEOREM 6. *Let* u(x) *be as in Theorem* 5. *Then there exists a quasisymmetric continuation* V(x) *that is continuously differentiable outside* [0, 1].

Proof. Let U(x) be the quasisymmetric continuation in Theorem 5. Define

$$V(x) = \begin{cases} u(x) & (0 \leq x \leq 1), \\ \dfrac{1}{x-1} \displaystyle\int_1^x U(s)\,ds & (x > 1), \\ \dfrac{1}{x} \displaystyle\int_0^x U(s)\,ds & (x < 0). \end{cases}$$

By Lemma 1, V(x) is quasisymmetric and continuously differentiable on $(-\infty, 0)$ and $(1. \infty)$. We shall prove that there exist constants k_1 and k_2 $(k_1 \geq 1,\ k_2 \geq 1)$ such that

$$\text{(a)} \qquad \frac{1}{k_1} \leq Q_V(1, t) \leq k_1 \qquad (0 < t < 1),$$

and

$$\text{(b)} \qquad \frac{1}{k_2} \leq Q_V(0, t) \leq k_2 \qquad (0 < t < \infty),$$

where $Q_V(x, t)$ is defined by equation (2). Then Theorem 4 and (a) imply that V(x) is quasisymmetric on $(0, \infty)$, and the result follows from Theorem 3 and (b).

To prove (b), we first assume $0 < t \leq 1$. Then

$$Q_V(0, t) = \frac{V(0+t) - V(0)}{V(0) - V(0-t)} = \frac{u(t)}{\dfrac{1}{t}\displaystyle\int_0^{-t} U(s)\,ds} = \frac{u(t)}{\dfrac{1}{t}\displaystyle\int_0^{t} u(s)\,ds}.$$

Hence Lemma 1 implies that

$$1 + \frac{1}{k} \leq Q_V(0, t) \leq k + 1.$$

Next assume $t > 1$. Then, again by Lemma 1,

$$Q_V(0, t) = \frac{\dfrac{1}{t-1}\displaystyle\int_1^t U(s)\,ds}{\dfrac{1}{t}\displaystyle\int_0^t U(s)\,ds} \leq \frac{U(1) + \dfrac{k}{1+k}[U(t) - U(1)]}{U(t)/(1+k)} = \frac{1}{U(t)} + k \leq 1 + k,$$

and

$$Q_V(0, t) \geq \frac{U(1) + \dfrac{1}{1+k}[U(t) - U(1)]}{\dfrac{k}{1+k}\,U(t)} = \frac{1}{U(t)} + \frac{1}{k} \geq \frac{1}{k}.$$

To prove (a), we recall that $U(x) = 2 - u(2 - x)$ for $1 \leq x \leq 2$, and therefore

$$Q_V(1, t) = \frac{1 - \frac{1}{t}\int_{1-t}^{1} u(s)\,ds}{1 - u(1 - t)}.$$

If we replace s by $1 - s$ and apply Lemma 1 to $1 - u(1 - s)$, which is also k-quasi-symmetric on $(0, 1)$, we see that

$$\frac{1}{1+k} \leq Q_V(1 - t) \leq \frac{k}{1+k},$$

which completes the proof.

6. DENSE FAMILIES OF QUASISYMMETRIC FUNCTIONS

The continuously differentiable K-quasiconformal mappings of a domain D are known to be dense in the class of K-quasiconformal mappings of D, in the sense of uniform convergence on compact sets.

THEOREM 7. *The class of infinitely differentiable* k-*quasisymmetric functions on an interval* (a, b) *is dense in the class of* k-*quasisymmetric functions on* (a, b).

Proof. Suppose first that (a, b) is the entire line. Let

$$f_n(x) = \begin{cases} k_n \exp \dfrac{-1}{1 - n^2x^2} & (|x| \leq 1/n), \\ 0 & (|x| \geq 1/n), \end{cases}$$

where k_n is chosen so that $\int_{-\infty}^{\infty} f_n = 1$. Let $u(x)$ be k-quasisymmetric on $(-\infty, \infty)$, and define

$$g_n(x) = \int_{-\infty}^{\infty} u(t) f_n(x - t)\,dt = \int_{-\infty}^{\infty} u(x - s) f_n(s)\,ds.$$

It is easy to see that $\{g_n(x)\}$ converges to $u(x)$ uniformly on compact sets, and it remains only to show that each $g_n(x)$ is k-quasisymmetric. But for each s,

$$0 \leq [u(x + t - s) - u(x - s)] f_n(s) \leq k[u(x - s) - u(x - t - s)] f_n(s).$$

Integrating this, we see that

$$g_n(x + t) - g_n(x) \leq k[g_n(x) - g_n(x - t)].$$

The reverse inequality follows similarly.

Suppose next that (a, b) is a finite interval, which we may take as the unit interval. We define $g_n(x)$ as above, except that we use $f_n(x - t + \alpha_n(x))$ for the kernel, where

$$\alpha_n(x) = \frac{1}{n}(1 - 2x) \qquad (0 \leq x \leq 1).$$

Then $g_n(x)$ is defined and k-quasisymmetric, and $\{g_n(x)\}$ converges uniformly to $u(x)$ on $(0, 1)$. The details are similar to those above, and we omit them.

A similar technique can be used for the case where (a, b) is semi-infinite.

7. COMPACTNESS CHARACTERIZATIONS, COMPOSITION, AND INVERSES

Beurling and Ahlfors [4] characterize k-quasisymmetric functions on the entire line by means of a certain compactness condition. We derive analogous characterizations for functions k-quasisymmetric on rays and bounded intervals, respectively, and we use these characterizations to show that quasisymmetry is preserved under the operations of composition and inverse mapping.

Denote by $F(a, b)$ any family of strictly increasing, continuous functions that map (a, b) onto itself.

Definition 2. A family $F(-\infty, \infty)$ of functions is said to be *closed under linear transformations* if for each $u(x)$ in $F(-\infty, \infty)$ and each pair of linear functions T and S, the composed function SuT is again in $F(-\infty, \infty)$.

Definition 3. A family $F(0, \infty)$ of functions is said to be *closed under linear transformations* if for each $u(x)$ in $F(0, \infty)$, each interval $J = (a, \infty)$ $(a \geq 0)$, each linear function T from $(0, \infty)$ onto J, and each linear function S from $J' = u(J)$ onto $(0, \infty)$, the composed function SuT is again in $F(0, \infty)$.

Definition 4. A family $F(0, 1)$ of functions is said to be *closed under linear transformations* if for each $u(x)$ in $F(0, 1)$, each subinterval J of $(0, 1)$, each linear function T from $(0, 1)$ onto J, and each linear function S from $J' = u(J)$ onto $(0, 1)$, the composed function SuT is again in $F(0, 1)$.

We shall use the following compactness conditions.

(A) Every infinite set of functions u in a family $F(-\infty, \infty)$ with $u(0) = 0$ and $u(1) = 1$, or in a family $F(0, \infty)$ with $u(1) = 1$, or in a family $F(0, 1)$, contains a sequence that converges to a strictly increasing limit function.

THEOREM 8. *The functions* u *in a family* $F(-\infty, \infty)$, $F(0, \infty)$, *or* $F(0, 1)$ *that is closed under linear transformations satisfy condition* (A) *if and only if each function is* k-*quasisymmetric for some fixed* k.

Proof. The proof for a family $F(-\infty, \infty)$ is given in [4, Theorem 2]. We omit the proof for a family $F(0, \infty)$, since it is similar to the proof for a family $F(0, 1)$.

For a family $F(0, 1)$, the necessity follows easily if we continue each function in $F(0, 1)$ to a quasisymmetric function of the entire line. This process yields a family $F(-\infty, \infty)$ for which Theorem 8 is valid.

For the sufficiency, suppose $F = F(0, 1)$ satisfies condition (A). Let $0 < x - t < x < x + t < 1$, and let $u(x)$ be a function in F. Denote by u_1, u_2, u_3 the images of $x - t$, x, and $x + t$, respectively, under $u(x)$. Map the intervals $J = (x - t, x + t)$ and $J' = (u_1, u_3)$ linearly onto $(0, 1)$. Then x and u_2 are carried onto the points $1/2$ and $(u_2 - u_1)/(u_3 - u_1)$, respectively, and the resulting mapping from $(0, 1)$ onto itself is in F. This procedure gives rise to a family $F' \subset F$ of functions that map the point $1/2$ onto $(u_2 - u_1)/(u_3 - u_1)$. Let

$$\alpha = \inf\{u(1/2)\} \quad \text{and} \quad \beta = \sup\{u(1/2)\},$$

where u ranges over all functions in F'. Then there exist two sequences $\{u_n\}$ and $\{v_n\}$ of functions in F' such that $u_n(1/2) \to \alpha$ and $v_n(1/2) \to \beta$. Since there exist subsequences converging to a homeomorphism, $\alpha > 0$ and $\beta < 1$. We have shown that for every u(x) in F,

$$\alpha \leq \frac{u_2 - u_1}{u_3 - u_1} \leq \beta,$$

or

$$0 < \frac{1-\beta}{\beta} \leq \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leq \frac{1-\alpha}{\alpha} < \infty.$$

Therefore each u(x) in F is k-quasisymmetric, where k is the maximum of $(1-\alpha)/\alpha$ and $\beta/(1-\beta)$.

It is easy to prove the following result by means of quasiconformal mapping theory. However, it is partly our purpose to derive the properties of quasisymmetric functions from the basic definition.

THEOREM 9. *Inverses and compositions of quasisymmetric functions are quasisymmetric.*

Proof. Suppose f and g are quasisymmetric functions. By Theorem 5, we may assume that they are defined on the entire line. If h = gf, the family of functions $F = \{ThR\}$, where T and R range over all possible linear functions, is certainly closed under linear transformations. Let $\{U_n\} = \{T_n hR_n\}$ be a sequence of functions from F such that $U_n(0) = 0$ and $U_n(1) = 1$ (such functions of the entire line are said to be *normalized*). It follows from Theorem 8 that h is quasisymmetric if we can find a subsequence $\{U_{n_k}\}$ that converges to a strictly increasing function.

Let S_n be the linear function that carries the points $fR_n(0)$ and $fR_n(1)$ onto 0 and 1, respectively. Then

$$T_n hR_n = (T_n gS_n^{-1})(S_n fR_n) = P_n Q_n,$$

and each P_n and each Q_n is normalized. Since each P_n is k(g)-quasisymmetric and each Q_n is k(f)-quasisymmetric, there exist subsequences $\{P_{n_k}\}$ and $\{Q_{n_k}\}$ that converge to normalized functions. The Q_{n_k} are equicontinuous on compact sets [4, page 127], and hence the composed functions U_{n_k} converge to a normalized function.

The proof that the inverse of a quasisymmetric function is quasisymmetric is similar to the above, and we omit it.

An unfortunate feature of quasisymmetric functions is that the maximal dilatation of a function is not necessarily the same as that of its inverse, nor is the dilatation of a composed function bounded by the product of the dilatations (both conditions hold for quasiconformal mappings). For example, if $f(x) = x^\alpha$ $(x > 0)$, then $k(f) = 2^\alpha - 1$ if $\alpha > 1$, and $k(f) = 1/(2^\alpha - 1)$ if $0 < \alpha < 1$ [4]. Hence, if $\alpha = 2$, then $k(f) = 3$ and $k(f^{-1}) = \sqrt{2} + 1$. Similarly, $k(f(f)) = 15$ and $k(f)k(f) = 9$. The exact bounds for $k(f^{-1})$ and $k(f(g))$ are unknown. However, using the results of [4], one can easily show that

$$k(f^{-1}) \leq P^{-1}(k(f)^2) \quad \text{and} \quad k(f(g)) \leq P^{-1}(k(f)^2 k(g)^2),$$

where P is the ratio of certain hypergeometric functions.

8. DISTORTION

In this section we establish some Hölder inequalities for *normalized* k-quasisymmetric functions, that is, for functions k-quasisymmetric on the entire line with fixed points 0 and 1. The results resemble rather closely analogous results for quasiconformal mappings of the entire plane under which the origin is a fixed point and the unit disc is invariant (see [8] and [9]).

THEOREM 10. *Suppose* $u(x)$ *is normalized and* k-*quasisymmetric on the entire line. Then*

$$2^{-\alpha} x^{\alpha} \leq u(x) \leq 2x^{\beta} \tag{8}$$

for $0 \leq x \leq 1$,

$$8^{-\alpha}(x_2 - x_1)^{\alpha} \leq u(x_2) - u(x_1) \leq 8^{\alpha}(x_2 - x_1)^{\beta} \tag{9}$$

for $0 \leq x_1 \leq x_2 \leq 1$, *and*

$$x^{\beta}/2 \leq u(x) \leq (2x)^{\alpha} \tag{10}$$

for $x \geq 1$, *where*

$$\alpha = \log_2(1+k), \qquad \beta = \log_2\left(1 + \frac{1}{k}\right). \tag{11}$$

Furthermore, the exponents α *and* β *are best possible.*

Proof. If we substitute $x = t = 2^{-n-1}$ $(n = 0, 1, \cdots)$ into inequality (1), it is easy to show that

$$\frac{1}{(1+k)^n} \leq u(2^{-n}) \leq \frac{1}{\left(1+\frac{1}{k}\right)^n}, \tag{12}$$

or equivalently,

$$(2^{-n})^{\alpha} \leq u(2^{-n}) \leq (2^{-n})^{\beta}.$$

If $0 \leq x \leq 1$, there exists $n \geq 0$ such that $-n-1 \leq \log_2 x \leq -n$. Hence

$$u(x) \leq u(2^{-n}) \leq (2^{-n})^{\beta} \leq (2^{\log_2 x + 1})^{\beta} = 2^{\beta} x^{\beta} \leq 2x^{\beta},$$

and similarly

$$u(x) \geq (2^{\log_2 x - 1})^{\alpha} = 2^{-\alpha} x^{\alpha}.$$

Suppose next that $0 \leq x_1 \leq x_2 \leq 1$. Since

$$f(x) = \frac{u(x + x_1) - u(x_1)}{u(x_1 + 1) - u(x_1)}$$

is a normalized k-quasisymmetric function, we obtain for $x = x_2 - x_1$ the inequalities

$$u(x_2) - u(x_1) \leq 2u(2)(x_2 - x_1)^\beta \leq 2k(x_2 - x_1)^\beta \leq 2^{\alpha+1}(x_2 - x_1)^\beta .$$

Since $\alpha > 1$, the right-hand inequality of (9) follows. An elementary computation shows that $u(x_1 + 1) - u(x_1) \geq \frac{1}{k(1+k)}$. Hence

$$u(x_2) - u(x_1) \geq \frac{(x_2 - x_1)^\alpha}{2^\alpha k(1+k)} \geq 8^{-\alpha}(x_2 - x_1)^\alpha .$$

Inequality (10) is derived in much the same way as (8).

Since the functions x^α and x^β are k-quasisymmetric for $x > 0$ [4], it follows that the exponents α and β are best possible.

9. INTEGRABILITY

Boyarskiĭ has pointed out [3], [5] that there exists $p > 1$ such that the Jacobian of each plane K-quasiconformal mapping is locally p-integrable. The supremum of admissible values of p is not known. The result is false for quasisymmetric functions. Hence the proof in the plane must utilize the added restrictions on the mapping that quasiconformality implies.

THEOREM 11. *For each* $k > 1$, *each* $p > 1$, *and each compact set* E *of positive measure, there exists a function* $u(x)$, *k-quasisymmetric on the entire line, such that*

$$\int_E u'(x)^p \, dx = \infty .$$

Proof. Let $\{u_n(x)\}$ be a sequence of normalized k-quasisymmetric functions that converge to a completely singular function (see [4] for a construction of such a sequence). Since the class of all k-quasisymmetric functions on a given interval is evidently closed under addition and under pointwise passage to limits, the limit function is k-quasisymmetric. Then

$$\limsup_{n \to \infty} \int_E u_n'(x)^p dx = \infty ,$$

since otherwise the $u_n(x)$ would be uniformly absolutely continuous on E, and therefore the limit function would be absolutely continuous. We may assume that

$$\int_E u_n'(x)^p dx \geq n^{p^2+1} .$$

Since the $u_n(x)$ are normalized, they are uniformly bounded on compact sets (Theorem 10). Hence the function

$$u(x) = \sum_{n=1}^{\infty} n^{-p} u_n(x)$$

is defined and k-quasisymmetric on the entire line. Therefore, by a theorem of Fubini,

$$u'(x) = \sum_{n=1}^{\infty} n^{-p} u_n'(x) \geq n^{-p} u_n'(x)$$

almost everywhere, for all n. Thus

$$\int_E u'(x)^p\, dx \geq n^{-p^2} \int_E u_n'(x)^p\, dx \geq n,$$

and the result follows if we let n approach infinity.

10. LOCAL QUASISYMMETRY

A strictly increasing continuous function is said to be *locally* k*-quasisymmetric at a point* if it is k-quasisymmetric in an interval containing the point. A function is *locally* k*-quasisymmetric* if it is locally k-quasisymmetric at each point of its domain. Locally K-quasiconformal plane mappings are K-quasiconformal in the large. However, the function $u(x) = e^x$ is locally k-quasisymmetric on the entire line for all $k > 1$, but since it does not preserve the domain type, it is not k-quasisymmetric for any $k > 1$.

Throughout this section we consider only functions that map the entire line onto itself. It is shown in [4] that if such a function is k-quasisymmetric, it can be extended to a k^2-quasiconformal mapping of the upper half-plane onto itself. We shall prove that if such a function is locally k-quasisymmetric, it can be extended to a k^2-quasiconformal mapping between two horizontal strips (by a *horizontal strip* we mean a simply connected domain in the upper half-plane whose boundary contains the entire real axis).

THEOREM 12. *Suppose* f(x) *is a locally* k*-quasisymmetric function that maps the entire line onto itself. Then* f(x) *can be extended to a* k^2*-quasiconformal mapping between two horizontal strips.*

Proof. We shall produce a quasiconformal extension of undetermined dilatation. The computations needed to obtain a k^2-quasiconformal extension are found in [4]. Define

$$(13)\quad u(x, y) = \frac{1}{2}\int_0^1 [f(x + ty) + f(x - ty)]\, dt, \qquad v(x, y) = \frac{1}{2}\int_0^1 [f(x + ty) - f(x - ty)]\, dt.$$

Then $w(z) = u + iv$ is a homeomorphism of the closed upper half z-plane onto the closed upper half w-plane with f(x) as the boundary correspondence [4]. Furthermore, $w(\infty) = \infty$, u and v are continuously differentiable, and if $\Im z > 0$, then

$$(14)\qquad H(z) + \frac{1}{H(z)} = \frac{1}{(\xi + \eta)}\left[\frac{\alpha}{\beta}(1 + \xi^2) + \frac{\beta}{\alpha}(1 + \eta^2)\right],$$

where $H(z) \geq 1$ is the dilatation of the mapping, $\xi = \alpha'/\alpha$, $\eta = \beta'/\beta$, and

$$u_x(z) = (\alpha + \beta)/2, \quad u_y(z) = (\alpha' - \beta')/2,$$

$$v_x(z) = (\alpha - \beta)/2, \quad v_y(z) = (\alpha' + \beta')/2.$$

The theorem is proved if we can show that ξ, η, and α/β are bounded away from zero and from infinity in some horizontal strip, for then $H + 1/H$ and therefore H is also bounded there.

Fix $z_0 = x_0 + iy_0$ $(y_0 > 0)$. Assume that $f(x)$ is k-quasisymmetric on the interval $(x_0 - y_0, x_0 + y_0)$. A simple calculation shows that

$$\frac{\alpha}{\beta} = \frac{f(x_0 + y_0) - f(x_0)}{f(x_0) - f(x_0 - y_0)},$$

and that α'/α and β'/β are the integrals over the unit interval of appropriate normalized k-quasisymmetric functions. From (1) and (7) we obtain

$$\frac{1}{k} \leq \frac{\alpha}{\beta} \leq k \quad \text{and} \quad \frac{1}{1+k} \leq \frac{\alpha'}{\alpha}, \frac{\beta'}{\beta} \leq \frac{k}{1+k}. \tag{15}$$

Let D be the set of points (x, y) $(y > 0)$ for which (15) holds, and such that if (x, y) is in D and $y' < y$, then (x, y') is in D. The set D contains a horizontal strip if for each finite interval $[a, b]$ there exists an $\varepsilon > 0$ such that (x, y) is in D whenever $a \leq x \leq b$ and $y < \varepsilon$. Let ε be the Lebesgue number of the covering of $[a, b]$ induced by the local quasisymmetry. For each $a \leq x \leq b$, $f(x)$ is k-quasisymmetric on $(x - \varepsilon, x + \varepsilon)$, and thus if $y < \varepsilon$, (15) is satisfied at the point (x, y), which is therefore in D. This completes the proof.

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