GENERATION OF FULL VARIETIES

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Philip Hall has shown that a class of algebras closed under formation of homomorphic images and subdirect products must be a variety; see Cohn [1, p. 171], though there is a gap in that proof (which will be filled here). P. M. Cohn asked [1] whether a category of algebras closed under these constructions must be full, that is, must contain all homomorphisms between its algebras. The main purpose of this paper is to show that it must be, except that some homomorphisms with empty domain may be missing. The artificial convention that every algebra must be nonempty is sometimes adopted; according to it, varieties need not be complete categories, but the categorical form of Hall's theorem holds without exception.

Cohn's argument that Hall's classes of algebras include free algebras of every rank fails for rank 0. The remainder of Cohn's proof shows that every nonempty algebra in the variety is present. If an empty algebra occurs, it is a subdirect product of the empty family of algebras; therefore Hall's theorem holds.

We call a category of algebras and homomorphisms $\mathcal{K}$ a *Hall category* if (i) every onto homorphism with domain in $\mathcal{K}$ belongs to $\mathcal{K}$, (ii) f: $A \to B$ in $\mathcal{K}$ implies f: $A \to f(A)$ and $f(A) \subset B$ are in $\mathcal{K}$, and (iii) for every subdirect product $B \subset \prod C_\lambda$ with all C_λ in $\mathcal{K}$, $\mathcal{K}$ contains the coordinate projections $B \to C_\lambda$ and every homomorphism $A \to B$ all of whose coordinates belong to $\mathcal{K}$.

THEOREM. *If $\mathcal{K}$ is a Hall category of algebras,* f: $A \to B$ *is a homomorphism,* A *and* B *are in $\mathcal{K}$, and* A *is nonempty, then* f *is in $\mathcal{K}$.*

Proof. Since f: $A \to f(A)$ must be in $\mathcal{K}$, it remains to reach $f(A) \to B$. Since A is nonempty, so is f(A). Let Y be a generating set of B containing a generating set X of f(A); then there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ f(A) & \rightarrow & B, \end{array}$$

which gives rise to another:

$$\begin{array}{ccc} F_X & \longrightarrow & F_Y \\ \downarrow & & \downarrow \\ f(A) & \rightarrow & B\ , \end{array}$$

where F_X, F_Y are free on X, Y, respectively, and $F_X \to F_Y$ is the natural injection. Here F_X, F_Y may clearly be replaced (if necessary) by algebras of infinite rank. Because any two words w, w' in F_Y depend only on finitely many generators, there always exists a retraction $r(w, w') = r$ of F_Y upon F_X distinguishing w and w'. These morphisms r are the coordinates of a subdirect product representation of F_Y such that every coordinate of the injection $F_X \to F_Y$ is onto, and therefore in

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$\mathcal{K}$. Therefore $F_X \to F_Y$ is in $\mathcal{K}$. Finally, $F_Y \to B$ is in $\mathcal{K}$ because it is onto, and the conclusion follows by factorization of $F_X \to B$.

COROLLARY. *Every Hall category of algebras that is closed under formation of direct products is a full variety.*

Of course, "formation of direct products" is meant to include (as with subdirect products) the coordinate projections. To prove the corollary, note that every homomorphism with empty domain is a coordinate projection of a direct product upon a factor.

Not every Hall category is full. For a counterexample, treat the rings with unit as defined by binary operations +, -, ·, and unary ε, with the usual laws (including $x - x = y - y$) and $x\varepsilon \cdot y = y \cdot x\varepsilon = y$. The variety has two objects not usually considered rings with unit: the necessary one-element algebra T and also an empty algebra S. Take the category of those homomorphisms $A \to B$ in this variety such that if $A = S$ then B is S or T.

For any Hall category $\mathcal{K}$ on the objects of a variety $\mathcal{V}$ having an empty algebra S, the objects T such that $S \to T$ is in $\mathcal{K}$ form a subvariety $\mathcal{V}'$. (They are closed under subdirect product and homomorphic image.) It is easy to characterize the pairs $(\mathcal{V}, \mathcal{V}')$ that correspond to Hall categories, in terms of congruence relations on the $\mathcal{V}$-free algebra on two generators. The identities valid in $\mathcal{V}'$ are all deducible in $\mathcal{V}$ from identities in one variable valid in $\mathcal{V}'$.

REFERENCE

1. P. M. Cohn, *Universal Algebra*, Harper and Row, New York, 1965.

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