AN ELEMENTARY PROPERTY OF CLOSED COVERINGS OF MANIFOLDS

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So far as I have been able to determine, the following property of the n-sphere S^n $(n > 1)$ has not been noticed:

PROPOSITION A. *Let* $\mathfrak{U}$ *be a finite covering of* S^n $(n > 1)$ *by at least three* $(n - 1)$-*acyclic closed sets* $A, B_1, \cdots, B_k$, *each of which is essential to the covering. Then at least three elements of* $\mathfrak{U}$ *have a common point. Indeed, every element of* $\mathfrak{U}$ *has a boundary point lying in two other elements.*

[A set on S^n is $(n - 1)$-*acyclic* if and only if it does not separate S^n. The algebraic topology used in this paper is Čech homology theory, modulo 2.]

Of course, if it were known that all elements of $\mathfrak{U}$ are sufficiently small, there would be nothing novel in Proposition A, since dimension-theoretic results showed some time ago that in this case at least $n + 1$ elements of $\mathfrak{U}$ would have a common point. But even if *all but one* of the elements of $\mathfrak{U}$ are small, this does not hold. For example, on the 2-sphere let A be an equatorial band ("annulus") split into two "rectangular" disks B_1 and B_2 by arcs along two different longitudes, and let B_3 and B_4 be the two polar caps bounded by the respective circles bounding A. At most three of the sets B_1, B_2, B_3, B_4 have a common point, and clearly B_1, B_2, and B_3 may be deformed into arbitrarily small sets of the same topological character while B_4 increases in size. And for $n > 2$, the example may be duplicated. Incidentally, this example shows that three is the maximum number for which Proposition A holds.

Before proving Proposition A, let us note the necessity of the conditions imposed. In the example given above, the sets A, B_3, and B_4 constitute a covering in which at most two elements have a common point, but A is not 1-acyclic. If we let $A' = A \cup B_3$, then A', B_3, and B_4 constitute a covering for which Proposition A fails, but here B_3 is not essential to the covering.

Proof of Proposition A. Since $\bigcup_{i=1}^k B_i \not\supset A$, there exists $x \in A - \bigcup_{i=1}^k B_i$. Let A_1 denote the component of A that contains x. Then A_1 is nondegenerate, since some neighborhood of x must lie in A; also, A_1 is an $(n - 1)$-acyclic continuum, since $n > 1$. And since S^n has the Brouwer property [3, pp. 47, 60], the boundary F of A_1 is a continuum.

If any point of F is also a point of two of the sets B_i, then the desired conclusion follows. Since $S^n - A \subset \bigcup_{i=1}^k B_i$ and since the sets B_i are closed, every point of F must lie in some B_i. Now F must be nondegenerate, and no continuum is the union of a finite collection of disjoint, closed sets (at least two in number) each of which is a proper subset of it. Consequently, if we assume that there exists no triple of elements of $\mathfrak{U}$ satisfying the conclusion of Proposition A, then every point of F must lie in some single B_i, say B_1.

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In the exact sequence

$$H_n(A_1) \to H_n(A_1, A_1 \cap B_1) \to H_{n-1}(A_1 \cap B_1) \to H_{n-1}(A_1),$$

the end-terms are zero, so that the two middle-terms are isomorphic. Also, the groups $H_n(A_1, A_1 \cap B_1)$ and $H_n(S^n, (S^n - A_1) \cup A_1 \cap B_1)$ are isomorphic, and the latter group is not 0, since x is interior to the set $S^n - (S^n - A_1) \cup A_1 \cap B_1$. Consequently $H_{n-1}(A_1 \cap B_1) \neq 0$.

In the Mayer-Vietoris sequence

$$H_{n-1}(A_1) + H_{n-1}(B_1) \leftarrow H_{n-1}(A_1 \cap B_1) \leftarrow H_n(A_1 \cup B_1) \leftarrow H_n(A_1) + H_n(B_1),$$

the sums on the left and right are zero, and consequently the groups $H_{n-1}(A_1 \cap B_1)$ and $H_n(A_1 \cup B_1)$ are isomorphic. But since the former group is nontrivial, $H_n(A_1 \cup B_1)$ is also nontrivial, which can be the case only if $A_1 \cup B_1 = S^n$. But this is excluded by hypothesis. We infer, then, that there exist i and j $(i \neq j)$ such that some point of F —and hence of A_1 —lies in both B_i and B_j. It follows that $A \cap B_i \cap B_j \neq \emptyset$.

It is natural to ask whether these propositions hold in case S^n is replaced by an n-manifold. Examination of the above proof reveals that this would be the case if the manifold were orientable and a certain analogue of the Brouwer property were available. The latter can take the following form.

GENERALIZED BROUWER PROPERTY. *Let* A *be a subcontinuum of an orientable* n-gcm M *such that* $p_{n-1}(A) = 0$. *Then* M - A *is connected, and the boundary of* A *is a subcontinuum of* A.

[By an n-gcm we mean an n-dimensional generalized closed manifold in the sense of [3].]

Proof. From the duality $q_0(M - A) = q_{n-1}(A)$ it follows that M - A is connected. [By $q_r(A)$ we denote the dimension of the vector space of r-cycles of A that bound in M, reduced modulo the subspace of r-cycles that bound in A. Similar numbers based on compact cycles and compact homologies of M - A may be defined. For orientable M, the duality $q_r(A) = q_{n-r-1}(M - A)$ holds (see [4]). Clearly $p_r(A) \geq q_r(A)$.]

Let F denote the common boundary of A and M - A. If A = F, the desired conclusion follows. Suppose $A \neq F$. The following relations hold.

(1) $q_0(A) = q_{n-1}(M - A) = 0$,

(2) $q_0(\overline{M - A}) = q_{n-1}(A - F) = 0$,

(3) $q_{n-1}[(M - A) \cup (A - F)] = q_{n-1}(M - F)$,

(4) $q_{n-1}[(M - A) \cup (A - F)] = q_{n-1}(M - A) + q_{n-1}(A - F)$.

Relations (1) and (2) follow from the connectedness of A and M - A, respectively. Property (3) is obvious.

The relation (4) may actually fail if $p_{n-1}(A) \neq 0$, as we can see if we take for M a torus and for A an annulus bounded by two meridional circles on M. To prove (4) under our hypothesis, we may argue as follows: In the first place, (4) holds with the "=" replaced by "$\geq$". For if an (n - 1)-cycle of M - A, say, bounds on M but does not bound in M - A, then it certainly does not bound in (M - A) ∪ (A - F), since the sets M - A and A - F are separated. In the second place, (4) holds with "="

replaced by "$\leq$". To see this, only two cases need to be considered:

Case 1. Let Z be an $(n-1)$-cycle of $M - A$ that bounds on M but not in $(M - A) \cup (A - F)$. Then Z does not bound in $M - A$, *a fortiori*.

Case 2. Let Z_1 and Z_2 be cycles of $M - A$ and $A - F$, respectively, such that $Z_1 + Z_2 \sim 0$ on M but not in $(M - A) \cup (A - F)$. By hypothesis, Z_2 bounds on A (since $p_{n-1}(A) = 0$) and hence on M. But then Z_1 must bound on M. Therefore either $Z_1 \not\sim 0$ in $M - A$ or $Z_2 \not\sim 0$ in $A - F$.

From (1), (2), (3), and (4) it follows that $q_{n-1}(M - F) = 0$, and since $q_0(F) = q_{n-1}(M - F)$, we conclude that F is connected.

It is well known that every locally orientable n-gcm modulo 2 is orientable modulo 2 [local orientability is defined in [3, p. 281]; that an n-gcm locally orientable mod 2 is orientable mod 2 was shown independently by H. B. Griffiths (unpublished) and F. A. Raymond [1]]. Hence we can make the following assertion.

PROPOSITION B. *Proposition* A *continues to hold if, instead of* S^n, *the space is either a topological closed* n-*manifold or a locally orientable* n-gcm.

As an application of Proposition B we can give an extremely simple proof of the following.

PROPOSITION C. *If* M^n $(n > 1)$ *is a topological closed* n-*manifold* (*or a locally orientable* n-gcm) *that is the union of two finite collections* $\{A_i\}$ *and* $\{B_j\}$ *of disjoint, closed,* $(n - 1)$-*acyclic sets, then* $M^n = A_i \cup B_j$ *for some pair of indices* i *and* j.

[This result is also stated in [5], but the proof given there is not intrinsic, being dependent on imbedding in a euclidean space.]

Proof. We may suppose the collections $\{A_i\}$ and $\{B_j\}$ reduced in number, if necessary, so that each is essential to the covering of M^n. If either collection then contains at least two sets, then by Proposition B, three of them would have a common point. But this is impossible, since the A_i are disjoint and the B_j are disjoint.

It is natural to ask whether the conclusion of Proposition A continues to hold if in the hypothesis the word "finite" is replaced by "countable". In this direction, we first prove the following result.

PROPOSITION D. *Let* $\mathfrak{U}$ *be a countable covering of* S^n $(n > 1)$ *by at least three* $(n - 1)$-*acyclic closed sets* $A_1, A_2, \cdots, A_n, \cdots$, *each of which is essential to the covering. Then, if no three elements of* $\mathfrak{U}$ *have a common point, the set* $S^n - \bigcup_{i,j} A_i \cap A_j$ *is a connected, nonempty set; and if the elements of* $\mathfrak{U}$ *are continua, then the set* $\bigcup_{i,j} A_i \cap A_j$ *is itself a connected, nonempty set.*

Proof. Since no three of the elements of $\mathfrak{U}$ have a common point, the sets $A_i \cap A_j$ form a countable collection of disjoint, closed sets. Moreover, every such set $A_i \cap A_j$ is $(n - 1)$-acyclic. For consider the portion

$$H_{n-1}(A_i) + H_{n-1}(A_j) \leftarrow H_{n-1}(A_i \cap A_j) \leftarrow H_n(A_i \cup A_j)$$

of the Mayer-Vietoris sequence of the triad $(A_i \cup A_j, A_i, A_j)$. Since $A_i \cup A_j \neq S^n$, and since both A_i and A_j are $(n - 1)$-acyclic, the extremes of the above sequence are zero, so that $H_{n-1}(A_i \cap A_j) = 0$.

Now $S^n - \bigcup_{i,j} A_i \cap A_j \neq \emptyset$, since by a theorem of Sierpinski [2], no continuum is the union of a countable collection of (at least two) disjoint, closed sets. If $S^n - \bigcup_{i,j} A_i \cap A_j$ were not connected, then it would be the union of two separated sets S_1 and S_2. Consequently [3, pp. 50, 60], there would exist a continuum K that is a subset of $\bigcup_{i,j} A_i \cap A_j$ and separates a point of S_1 from a point of S_2. But, again by the theorem of Sierpinski cited above, K could not meet more than one of the sets $A_i \cap A_j$ (since $K = \bigcup_{i,j} K \cap A_i \cap A_j$ and the sets $K \cap A_i \cap A_j$ are disjoint). But then the particular set $A_i \cap A_j$ that contains K would not be $(n - 1)$-acyclic. We conclude, then, that $S^n - \bigcup_{i,j} A_i \cap A_j$ is a nonempty, connected set.

Turning to the set $\bigcup_{i,j} A_i \cap A_j$, we observe first that it is nonempty, since by Sierpinski's theorem S^n cannot be the union of a countable set of at least two closed disjoint sets A_i. Consequently, we have only to show that if the sets A_i are continua, then the set $\bigcup_{i,j} A_i \cap A_j$ is connected. As in the proof above, if the set were not connected, there would exist in $S^n - \bigcup_{i,j} A_i \cap A_j$ a continuum K separating two points x and y of $\bigcup_{i,j} A_i \cap A_j$. But in this case, K would be a subset of a single A_i; for $(K \cap A_i) \cap (K \cap A_j) = K \cap (A_i \cap A_j) = \emptyset$, and by the Sierpinski theorem cited above, K could not be the union of two or more sets $K \cap A_i$. Suppose $K \subset A_1$.

By the Alexander-Pontrjagin duality, a cycle Z_{n-1} of K is linked with the nontrivial cycle Z_0 carried by x and y. Since A_1 is $(n - 1)$-acyclic, $Z_{n-1} \sim 0$ on A_1, so that at least one of the points x and y, say x, lies in A_1.

For some $i \neq 1$, the point x lies in $A_1 \cap A_i$, and A_i is a continuum not meeting K, since $K \cap A_i = (K \cap A_1) \cap A_i = K \cap (A_1 \cap A_i) = \emptyset$. But since every element of $\mathfrak{U}$ is essential to the covering, some point z of A_i lies in $S^n - A_1$. Similarly, the point y must lie in an A_j $(j \neq 1)$ containing a point w of $S^n - A_1$. The set $S^n - A_1$ is an open connected subset of S^n, since A_1 is $(n - 1)$-acyclic, and consequently the set $A_i \cup A_1 \cup A_j$ is a connected subset of $S^n - K$ containing both x and y, contrary to the fact that K separates x and y in S^n. We conclude that $\bigcup_{i,j} A_i \cap A_j$ is connected.

That the conclusion of Proposition D is realizable is shown by an example. [For this example I am indebted to Messrs. Andrew C. Connor and William Transue of the University of Georgia. My original example was more complicated, and its extension to higher dimensions led to covering elements not $(n - 2)$-acyclic, while the obvious extension of the present example yields sets that are acyclic in all dimensions.] We cover S^2 by an infinite collection of closed sets, as we shall describe with the help of Figure 1. Both the front half (at the left) and the back are shown. The sets $A_1, A_2, A_3, \cdots$ are topological disks converging to the arc K; likewise, $B_1, B_2, B_3, \cdots$ are disks converging to the arc L, and $K \cup L$ forms the 1-sphere separating the front and back halves of S^2. The elements of the covering $\mathfrak{U}$ are the A_i, the B_j, and the individual points p and q. Analogous decompositions of S^n for $n > 2$ are obvious.

We observe that the above examples establish the following conclusion, in contrast to Proposition C.

PROPOSITION E. *Every* S^n $(n > 1)$ *is the union of two countably infinite collections* $\{A_i\}$ *and* $\{B_j\}$ *of disjoint, closed,* r*-acyclic sets* $(r = 0, 1, 2, \cdots)$ *such that every* A_i *and every* B_j *is essential to the covering of* S^n *formed by the combined collection* $\{A_i\} \cup \{B_j\}$.

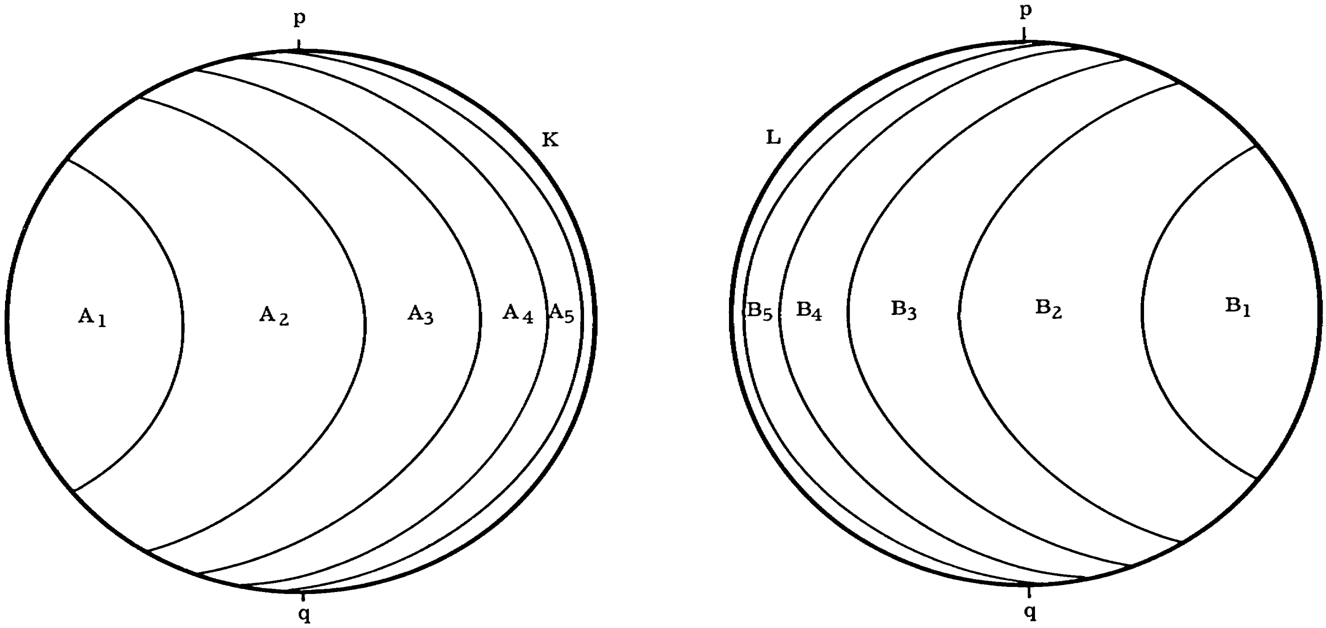


Figure 1.

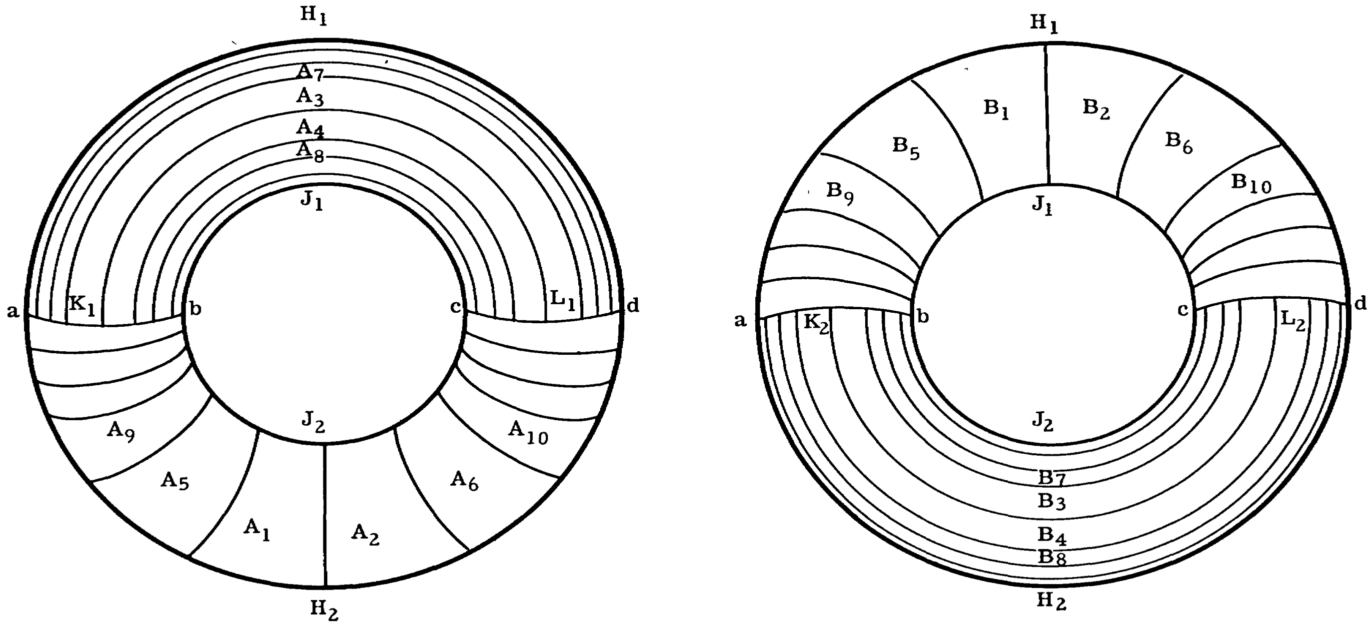
[For example, in Figure 1, we may let the sets A_i of Proposition E be the sets A_i and B_j of Figure 1 having even subscripts, together with the arc K, while the sets B_j of Proposition E are the sets A_i and B_j of Figure 1 having odd subscripts, together with the arc L.]

Remark. Proposition E forms an interesting contrast to the theorem of Sierpinski cited in the proof of Proposition D. For whereas no continuum whatsoever can be the union of a countable collection of (at least two) disjoint closed sets, the euclidean n-sphere S^n $(n > 1)$ is the union of two collections of disjoint, closed sets each of which is acyclic in all dimensions.

In conclusion, we may ask whether such decompositions are possible for other manifolds than S^n. That such is the case is exemplified by Figure 2. The part labelled 2a represents the front half of a torus T, and 2b the back half of T. The sets $K_1 \cup K_2$ and $L_1 \cup L_2$ form meridional circles of T, and $H_1 \cup H_2$, $J_1 \cup J_2$ form equatorial circles. The points a and b are endpoints of both arcs K_1 and K_2, while c and d are endpoints of arcs L_1 and L_2; the endpoints of H_1 and H_2 are a and d, while those of J_1 and J_2 are b and c. In 2a, the sets A_i are topological disks; those with subscripts of the form $4n+1$ $(n \geq 0)$ converge to K_1, those of the form $4n+2$ converge to L_1, those of the form $4n+3$ to H_1, and those of the form $4n$ to J_1. In 2b, an analogous decomposition by disks B_i is indicated; however, those with subscripts $4n+1$ converge to J_2, those with subscripts $4n+2$ to H_2, and so forth. The covering of T whose elements are the disks A_i and B_i together with the sets $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ has the property that no three elements have a common point.

The last example raises the question whether the analogue of Proposition D holds for manifolds more general than the n-sphere. This is in fact the case; moreover, the proposition holds even for generalized manifolds.

PROPOSITION F. *Proposition* D *continues to hold if* "S^n" *is replaced by* "*an orientable* n-gcm M".



2a 2b

Figure 2.

Inspection of the proof of Proposition D reveals that if one attempts to apply to the present case the argument given there, it breaks down at the point (in the second paragraph) where it is assumed that points of separated sets S_1 and S_2 are separated by a subcontinuum of the complement of $S_1 \cup S_2$. Since not all manifolds have this property of the n-sphere, we need the following lemma.

LEMMA. *If* M *is an orientable* n-gcm, X *and* Y *are separated subsets of* M, *and* $x \in X$ *and* $y \in Y$, *then a finite number* [*not greater than* $p_1(M) + 1$] *of subcontinua of* $M - (X \cup Y)$ *separate* x *and* y.

Since this lemma is a corollary of a more general theorem in a forthcoming paper, we omit its proof.

Applying the lemma, we can replace the K of the second paragraph of the proof of Proposition D with a finite collection of continua $K_1, K_2, \cdots, K_m$, each of which lies in some $A_i \cap A_j$. By the duality theorem of [4], an (n - 1)-cycle of $\bigcup_{j=1}^m K_j$ is linked with the nontrivial 0-cycle carried by x and y; this contradicts the (n - 1)-acyclicity of the $A_i \cap A_j$.

The remainder of the proof—that $\bigcup_{i,j} A_i \cap A_j$ is connected if the A_i are connected--is not quite so obvious. The K of the third paragraph in the proof of Proposition D is again replaced by a finite number of continua $K_1, K_2, \cdots, K_m$, by virtue of the above lemma, and again each K_j lies in a single A_i. Suppose $K_1 \subset A_1$. We may assume that $\bigcup_{j=1}^m K_j = K$ is an irreducible cut of M between x and y, and by the duality of [4], an (n - 1)-cycle Z_{n-1} of K is linked with the nontrivial 0-cycle carried by x and y.

The cycle Z_{n-1} is the sum of cycles Z_{n-1}^j, where Z_{n-1}^j is on K_j. Suppose that $Z_{n-1}^1 \sim 0$ in $M - x - y$. Then $Z_{n-1} - Z_{n-1}^1$ is in the same homology class as Z_{n-1} in $M - x - y$, and accordingly it is linked with Z_0. But this would imply that $K - K_1$

is a cut of M between x and y, contrary to the fact that K is an irreducible cut of M between x and y. [See [3, p. 375, Lemma 3.12]; while the lemma cited assumes that the manifold is 1-acyclic, a reading of its proof reveals that all that is needed is that the cycle in question bound--which is the case above, since A_1 is (n - 1)-acyclic.] We conclude, then, that $Z^1_{n-1} \not\sim 0$ in $M - x - y$. It follows that K_1 separates x and y in M and that $K = K_1$. The proof may now be concluded as in the corresponding part of the proof for Proposition D.

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