UNILATERAL VARIATIONAL PROBLEMS WITH SEVERAL INEQUALITIES

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1. INTRODUCTION

In a previous paper [6] we have considered nonparametric problems of the calculus of variations in which the "controls" are chosen from a compact Hausdorff space and the admissible curves satisfy given boundary conditions and are restricted to lie in a closed set $A = \{x \in E_n \mid a(x) \leq 0\}$. Here E_n denotes euclidean n-space, and $a(x)$ is a prescribed continuous function on an open subset of E_n with continuous first- and second-order partial derivatives. We shall now extend the results of [6] to the more general problem in which the set A is defined by the simultaneous inequalities $a^k(x) \leq 0$ $(k = 1, \cdots, m)$ and the functions $a^k(x)$ $(k = 1, \cdots, m)$, defined over an open subset of E_n, are twice continuously differentiable.

We shall describe the problem in greater detail and state our assumptions in Section 2. Our basic results are contained in Theorem 3.1, which generalizes Theorem 3.1 of [6] and states that, in a large class of unilateral control problems, there exists a "relaxed" (or generalized) minimizing curve, that this curve can be approximated by solutions of the differential equations of the original problem, and that this relaxed curve satisfies "constructive" necessary conditions for a minimum, including two that are analogous to the Weierstrass E-condition and transversality conditions.

We carry out the proof of Theorem 3.1 in the remaining sections of the paper. Large parts of the proof, especially those contained in Sections 4 and 7, differ only in small details from the arguments of [6].

We refer the reader to [6, Section 1] for a brief discussion of prior work in this general area by Young [8], McShane [3], Filippov [1], Warga [4], and Gamkrelidze [2].

2. STATEMENT OF THE PROBLEM AND ASSUMPTIONS

Let R be a compact Hausdorff space, E_n the euclidean n-space, T the closed interval $[t_0, t_1]$ of the real axis, V an open set in E_n, and B_0 and B_1 closed sets in V. We are also given a function

$$g(x, t, \rho) = (g^1(x, t, \rho), \cdots, g^n(x, t, \rho))$$

from $V \times T \times R$ to E_n and a function $a(x) = (a^1(x), \cdots, a^m(x))$ from V to E_m.

Let $G(x, t) = \{g(x, t, \rho) \mid \rho \in R\}$ $(x \in V,\ t \in T)$, and let $F(x, t)$ be the convex closure of $G(x, t)$.

We define an *original admissible curve with respect to* $a(x)$ as any absolutely continuous function $x(t)$ from T to V such that, for some function $\rho(t)$ from T to R,

$$\text{(2.1.1)} \qquad dx(t)/dt = \dot{x}(t) = g(x(t), t, \rho(t)) \quad \text{a.e. in } T,$$

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or equivalently,

(2.1.1 Orig.) $$\dot{x}(t) \in G(x(t), t) \quad \text{a.e. in } T$$

and

(2.1.2) $$x(t_0) \in B_0, \quad x(t_1) \in B_1,$$

(2.1.3) $$a^k(x(t)) \leq 0 \quad (k = 1, 2, \cdots, m;\ t \in T).$$

We similarly define a *relaxed admissible curve with respect to* $a(x)$ except that relation (2.1.1), respectively (2.1.1 Orig.), is replaced by

(2.1.1 Relaxed) $$\dot{x}(t) \in F(x(t), t) \quad \text{a.e. in } T.$$

An *original* (respectively, *relaxed*) *minimizing curve with respect to* $a(x)$ is a curve that minimizes the value $x^1(t_1)$, among all original (respectively, relaxed) admissible curves with respect to $a(x)$.

We now state our basic assumptions.

Assumption 2.2. There exist a finite or denumerable collection of disjoint (Lebesgue) measurable subsets T_r $(r = 1, 2, \cdots)$ of T whose union T' has measure $|T| = t_1 - t_0$, positive constants c_1 and ε_1, a function $\varepsilon(h)$ $(h > 0)$ converging to 0 as $h \to +0$, and a compact set $D \subset V$ such that the following five conditions are satisfied.

(2.2.1) The functions $g^i(x, t, \rho)$ and $\partial g^i(x, t, \rho)/\partial x^j$ $(i, j = 1, \cdots, n)$ exist over $V \times T' \times R$, and over that set they are continuous functions of (x, t), uniformly in ρ, and continuous functions of ρ for each (x, t); furthermore,

$$\|g(x, t, \rho) - g(x, t', \rho)\| \leq \varepsilon(|t - t'|)$$

provided t and t' belong to the same set T_r (where $\|g\| = \|(g^1, \cdots, g^n)\| = \sum_{i=1}^{n} |g^i|$).

(2.2.2) $\|g(x, t, \rho)\| \leq c_1$ and $\|g_x(x, t, \rho)\| \leq c_1$ on $V \times T' \times R$; here g_x is the matrix $(\partial g^i/\partial x^j)$ $(i, j = 1, \cdots, n)$, and $\|g_x\| = \sum_{i,j=1}^{n} |\partial g^i/\partial x^j|$.

(2.2.3) The functions

$$a^k(x), \quad \frac{\partial a^k(x)}{\partial x^i}, \quad \frac{\partial^2 a^k(x)}{\partial x^i \partial x^j} \quad (k = 1, \cdots, m;\ i, j = 1, \cdots, n)$$

exist and are continuous on V; furthermore,

$$|a^k| \leq c_1, \quad \|a_x^k\| \leq c_1, \quad |a_x^k g| \leq c_1 \quad (k = 1, \cdots, m);$$

here a_x^k is the gradient of a^k, $\|a_x^k\| = \sum_{j=1}^{n} \left|\frac{\partial a^k}{\partial x^j}\right|$, and $a_x^k g = \sum_{j=1}^{n} g^j \frac{\partial a^k}{\partial x^j}$.

(2.2.4) There exists at least one relaxed admissible curve with respect to $a(x)$.

(2.2.5) All relaxed admissible curves with respect to $(a^1(x) - \varepsilon_1, \cdots, a^m(x) - \varepsilon_1)$ are contained in D.

As in [5] and [6], we shall replace relation (2.1.1 Relaxed) by an equivalent system of differential equations. We shall also replace relation (2.1.2) by a relation involving only convex sets. To do so, we restate the definitions of *proper representations* of $F(x, t)$ and of B_0 and B_1, introduced in [6].

Definition 2.3. A function $f(x, t, \sigma)$ from $V \times T \times S$ to E_n is a *proper representation of* $F(x, t)$ if

(2.3.1) $$F(x, t) = \{f(x, t, \sigma) \mid \sigma \in S\} \qquad (x \in V,\ t \in T);$$

(2.3.2) for every absolutely continuous curve $x(t)$ satisfying (2.1.1 Relaxed), there exists a function $\sigma(t)$ from T to S such that

$$\dot{x}(t) = f(x(t), t, \sigma(t)) \qquad \text{a.e. in } T,$$

and for all $x \in V$ and almost all $t \in T$, $f(x, t, \sigma(\tau))$ is a (Lebesgue) measurable function of τ on T;

(2.3.3) $f^i(x, t, \sigma)$ and $\partial f^i(x, t, \sigma)/\partial x^j$ $(i, j = 1, \cdots, n)$ exist and are continuous functions of (x, t) on $V \times T'$ for every σ in S;

(2.3.4) $$\|f(x, t, \sigma)\| \leq c_1 \text{ and } \|f_x(x, t, \sigma)\| \leq c_1 \text{ on } V \times T' \times S;$$

(2.3.5) the set

$$H(x, t, \alpha) = \{(f(x, t, \sigma), f_x^T(x, t, \sigma)\alpha) \mid \sigma \in S\}$$

in $E_n \times E_n$ is compact and convex for every $(x, t, \alpha) \in V \times T' \times E_n$ (here f_x^T is the transpose of the matrix f_x).

Definition 2.4. Let $B \subset E_n$. We shall say that $(C, c(\xi))$ is a *proper representation of* B *at* x if

(2.4.1) C is a compact and convex set in some euclidean space;

(2.4.2) $c(\xi)$ is a continuous and continuously differentiable function from C to B;

(2.4.3) $x = c(\xi)$ for some $\xi \in C$.

All of the conditions stated in the Definitions 2.3 and 2.4 are directly verifiable, except for condition (2.3.2). We therefore indicate two methods of constructing proper representations of $F(x, t)$.

2.5. *The Filippov representation.* Let S be a compact set in some euclidean space, and let $f(x, t, \sigma)$ be continuous on $V \times T' \times S$ and satisfy conditions (2.3.1), (2.3.3), (2.3.4), and (2.3.5). Then condition (2.3.2) follows from a lemma of Filippov [1, p. 78].

2.6. *The Young representation.* Let S be the class of probability measures defined on the Borel subsets of R, and let $f(x, t, \sigma) = \int_R g(x, t, \rho)\, d\sigma$. Then conditions (2.3.1) to (2.3.4) follow from Assumption 2.2 and from [4, Theorem 4.1, p. 124]. Condition (2.3.5) is easily verified, since S is a convex set and $f(x, t, \sigma)$ is linear in σ.

3. EXISTENCE OF A MINIMIZING CURVE. NECESSARY CONDITIONS FOR A MINIMUM.

THEOREM 3.1. *Let Assumption* 2.2 *be satisfied. Then there exists a curve* $x(t)$ *that is a relaxed minimizing curve with respect to* $a(x)$, *and this curve can be uniformly approximated by solutions of the differential equations* (2.1.1).

Let $f(x, t, \sigma)$ *be a proper representation of* $F(x, t)$, *let* $(C_i, c_i(\xi_i))$ *be a proper representation of* B_i *at* $x(t_i)$ $(i = 0, 1)$, *and let*

$$Z^k = \{t \in T \mid a^k(x(t)) = 0\} \quad (k = 1, \cdots, m), \qquad Z = \bigcup_{\ell=1}^{m} Z^\ell,$$

$$K(t) = \{k \mid a^k(x(t)) = 0\} \qquad (t \in T).$$

Finally, let $\delta_i = (\delta_i^1, \delta_i^2, \cdots, \delta_i^n)$ $(i = 1, \cdots, n)$, *where* $\delta_i^j = 0$ $(i \neq j)$ *and* $\delta_i^i = 1$. *Then either*

(3.1.1) *there exist a point* ξ_1^* *in* C_1 *and numbers* γ^a, γ^k $(k \in K(t_1))$ *such that*

$$c_1(\xi_1^*) = x(t_1), \qquad \gamma^a \geq 0, \qquad \gamma^k \geq 0 \quad (k \in K(t_1)), \qquad \gamma^a + \sum_{\ell \in K(t_1)} \gamma^\ell \neq 0,$$

and

$$\left(\gamma^a \delta_1 + \sum_{\ell \in K(t_1)} \gamma^\ell a_x^\ell(x(t_1))\right) \cdot c_{1,\xi}(\xi_1^*)\,\xi_1^*$$

$$= \operatorname*{Min}_{\xi_1 \in C_1} \left(\gamma^a \delta_1 + \sum_{\ell \in K(t_1)} \gamma^\ell a_x^\ell(x(t_1))\right) \cdot c_{1,\xi}(\xi_1^*)\,\xi_1$$

(*where* $c_{1,\xi}^i$ *is the gradient of* c_1^i *and* $c_{1,\xi} = (c_{1,\xi}^1, \cdots, c_{1,\xi}^n)$), *or*

(3.1.2) *there exist a function* $\sigma(t)$ *from* T *to* S, *a function*

$$\mu(t) = (\mu^1(t), \cdots, \mu^m(t))$$

from T *to* E_m, *a function* $z(t)$ *from* T *to* E_n, *a closed subset* M *of* Z, *points* $\xi_0^* \in C_0$ *and* $\xi_1^* \in C_1$, *and a nonnegative number* γ^1 *such that*

(3.1.2.1) $\mu^k(t) \geq 0$ $(k = 1, \cdots, m)$ *and* $\|z(t)\| + \|\mu(t)\| > 0$ $(t \in T)$, *where*

$$\|\mu(t)\| = \sum_{\ell=1}^{m} |\mu^\ell(t)|;$$

(3.1.2.2) $z(t)$ *is absolutely continuous on every closed subinterval of* $T - M$;

(3.1.2.3) *for every* k $(k = 1, \cdots, m)$, $\mu^k(t)$ *is nonincreasing on every subinterval of* $T - M$, $\mu^k(t)$ *is constant on every subinterval of* $T - M - Z^k$, *and* $\mu^k(t_1)\,a^k(x(t_1)) = 0$;

(3.1.2.4) $$\dot{x}(t) = f(x(t), t, \sigma(t)) \qquad \text{a.e. } in\ T, \quad and$$

$$\dot{z}(t) = -f_x^T(x(t), t, \sigma(t))\,z(t) - \sum_{\ell=1}^{m} \mu^\ell(t)\,b_x^\ell(x(t), t, \sigma(t)) \qquad \text{a.e. } in\ T - M,$$

where

$$b^k(x, t, \sigma) = a_x^k(x) \cdot f(x, t, \sigma), \qquad b_x^k = \left(\frac{\partial b^k}{\partial x^1}, \cdots, \frac{\partial b^k}{\partial x^n}\right),$$

f_x^T *denotes the transpose of the matrix* f_x,

$$z(t) = O = (0, \cdots, 0), \quad \mu^k(t) = 0 \quad (k \notin K(t)) \quad \textit{for } t \in M,$$

$$z(t-0) = \lim_{\substack{\tau \to t \\ \tau < t}} z(\tau) = O \quad \textit{and} \quad \mu^k(t-0) = 0 \quad (k \notin K(t))$$

if $t \in M$ *and* t *is the right endpoint of some open subinterval of* $T - M$;

(3.1.2.5) *(the Weierstrass* E*-condition)*

$$v(t) \cdot f(x(t), t, \sigma(t)) = \operatorname*{Min}_{\sigma \in S} v(t) \cdot f(x(t), t, \sigma) \quad \text{a.e. } \textit{in } T,$$

where $v(t) = z(t) + \sum_{\ell=1}^{m} \mu^\ell(t) a_x^\ell(x(t))$;

(3.1.2.6) *(support (transversality) conditions)*

$$c_1(\xi_1^*) = x(t_1), \qquad c_0(\xi_0^*) = x(t_0);$$

(3.1.2.6.1) $$v(t_0) \cdot c_{0,\xi}(\xi_0^*)\xi_0^* = \operatorname*{Min}_{\xi_0 \in C_0} v(t_0) \cdot c_{0,\xi}(\xi_0^*)\xi_0;$$

(3.1.2.6.2) $$(\gamma^1 \delta_1 - z(t_1)) \cdot c_{1,\xi}(\xi_1^*)\xi_1^* = \operatorname*{Min}_{\xi_1 \in C_1} (\gamma^1 \delta_1 - z(t_1)) \cdot c_{1,\xi}(\xi_1^*)\xi_1;$$

(3.1.2.7) *there exists a point* t_0^* *in* T $(t_0^* < t_1)$ *such that* $\|v(t)\| \neq 0$ $(t_0^* < t \leq t_1)$ *and*

either $t_0^* = t_0$,

or $t_0^* \in Z$ *and* $z(t_0^*) = -\sum_{\ell=1}^{m} \bar\gamma^\ell a_x^\ell(x(t_0^*))$ *for some numbers* $\bar\gamma^k \geq \mu^k(t_0^*)$ $(k \in K(t_0^*))$ *and* $\bar\gamma^k = \mu^k(t_0^*)$ $(k \notin K(t_0^*))$,

or $t_0^* \in Z$ *and* $\left\|\sum_{\ell \in K(t_0^*)} \bar\gamma^\ell a_x^\ell(x(t_0^*))\right\| = 0$ *for some numbers* $\bar\gamma^k$ $(k \in K(t_0^*))$ *such that* $\bar\gamma^k \geq 0$ *and* $\sum_{\ell \in K(t_0^*)} \bar\gamma^\ell = 1$;

(3.1.2.8) *if there exists a negative number* β *such that, for every subset* K *of* $\{1, 2, \cdots, m\}$, *the relations*

$$x \in V, \quad t \in T, \quad a^k(x) < 0 \ (k \notin K), \quad a^k(x) = 0 \ (k \in K), \quad \gamma^k \geq 0 \ (k \in K), \quad \sum_{\ell \in K} \gamma^\ell = 1$$

imply $\operatorname*{Min}_{\sigma \in S} \sum_{\ell \in K} \gamma^\ell a_x^\ell(x) \cdot f(x, t, \sigma) < \beta$, *then the set* M *is empty or contains the single point* t_0.

3.2. We shall complete this section by discussing briefly the meaning and interpretation of some of the statements in Theorem 3.1.

The alternative (3.1.1) applies in all cases where the only true limitations on the minimization of $x^1(t_1)$ are imposed by the conditions

$$x(t_1) \in B_1 \quad \text{and} \quad x(t_1) \in A = \{x \in E_n \mid a^k(x) \leq 0 \ (k = 1, \cdots, m)\}.$$

If the point $\bar{x}$ minimizes $\bar{x}^1$ on $B_1 \cap A$ and if this point can be "reached" at time t_1 by each curve of a large family of curves $y(t)$ such that

$$y(t_0) \in B_0, \quad y(t) \in A \quad (t \in T), \quad \dot{y}(t) \in F(y(t), t) \quad \text{a.e. in } T,$$

then clearly no more precise information about the nature of such curves can in general be expected.

Consider now the case where the alternative (3.1.2) holds. If an interval I is contained in T - Z, then $\mu^k(t)$ $(k = 1, \cdots, m)$ is constant on I, by (3.1.2.3). It follows then easily from (3.1.2.4) that

$$\dot{v}(t) = -f_x^T(x(t), t, \sigma(t))v(t) \quad \text{a.e. in I}, \tag{3.2.1}$$

where $v(t) = z(t) + \sum_{\ell=1}^{m} \mu^\ell(t) a_x^\ell(x(t))$.

Let us now designate as an *arc of an extremal* any absolutely continuous curve $(y(t), w(t))$ in E_{2n}, defined over some subinterval J of T, such that, a.e. in J,

$$w(t) \neq 0, \quad \dot{y}(t) = f(y(t), t, \sigma(t)), \quad \dot{w}(t) = -f_x^T(y(t), t, \sigma(t))\, w(t),$$

$$w(t) \cdot f(y(t), t, \sigma(t)) = \operatorname*{Min}_{\sigma \in S} w(t) \cdot f(y(t), t, \sigma).$$

It was shown in [5, Theorem 6.1, pp. 142-143] that if $x(t)$, the relaxed minimizing curve with respect to $a(x)$, is contained in the interior of A, then $(x(t), v(t))$ is an arc of an extremal. We now observe, as a consequence of (3.1.2.4), (3.1.2.5), and (3.2.1), that the curve $(x(t), v(t))$ is an arc of an extremal over each subinterval of T - Z, that is, over each subinterval I of T for which $x(t)$ is contained in the interior of A.

Next we make a few remarks about the set M. Assume, for the sake of simplicity, that the set T' of Assumption 2.2 is the entire set T, so that $f(x, t, \sigma)$ is continuous in t on $V \times T \times S$. It will be shown in Lemma 6.5 that

$$\sum_{\ell \in K(t)} \mu^\ell(t) a_x^\ell(x(t)) f(x(t), t, \sigma) \geq 0$$

for all $\sigma \in S$ and all $t \in M \cap T'$, $t > t_0$. It follows then that if $t > t_0$ and $t \in M$, then there exists no value of σ for which $a_x^k(x(t)) f(x(t), t, \sigma) \leq 0$ $(k \in K(t))$ and $a_x^\ell(x(t)) f(x(t), t, \sigma) < 0$ for some $\ell \in K(t)$. Thus no choice of a control $\tilde{\sigma}(t)$ can be made in the vicinity of a point t ($t \in M$, $t > t_0$) that would permit the trajectory to re-enter the interior of A other than tangentially to the boundary of A. In other words, at any point t of M, $t > t_0$, the conditions (2.1.1 Relaxed) and (2.1.3) are barely consistent. It follows, however, from statements (3.1.2.8) that such "near conflicts" between conditions (2.1.1 Relaxed) and (2.1.3) are impossible if the assumption of (3.1.2.8) is satisfied.

Finally, we shall make a brief comment about statement (3.1.2.7). If, at any point $\bar{t}$, the vector

$$v(\bar{t}) = z(\bar{t}) + \sum_{\ell=1}^{m} \mu^{\ell}(\bar{t}) a_x^{\ell}(x(\bar{t}))$$

vanishes, then the functions $\bar{z}(t) = z(\bar{t})$ and $\bar{\mu}^k(t) = \mu^k(\bar{t})$ ($t_0 \leq t \leq \bar{t}$, $k = 1, \cdots, m$) satisfy conditions (3.1.2.4), (3.1.2.5), and (3.1.2.6.1) on $[t_0, \bar{t}]$. Moreover, the generalized Weierstrass E-condition (3.1.2.5) and the support condition (3.1.2.6.1) are then trivially satisfied. This reflects the fact that, in many unilateral problems, a minimizing unilateral curve may be quite arbitrary within a certain subinterval of T, but may be uniquely determined over some interval $[t_0^*, t_1]$. In fact, as we observed in discussing the alternative (3.1.1), the entire unilateral minimizing curve may, in some cases, be chosen rather arbitrarily.

4. Q-MINIMIZING CURVES

4.1. We now proceed to derive the proof of Theorem 3.1. As in [6], our approach will be to consider first the problem of minimizing $x^1(t_1)$ subject to condition (2.1.1 Relaxed), to a slightly modified form of condition (2.1.2), and with condition (2.1.3) ($a^k(x(t)) \leq 0$ for $k = 1, \cdots, m$; $t \in T$) replaced by the weaker condition

$$a^k(x(t)) \leq 0 \quad (k = 1, \cdots, m;\ t \in Q),$$

where Q is a finite subset of T. We next consider a sequence of such problems for ever denser nested sets Q_s ($s = 1, 2, \cdots$) that converge to a denumerable dense subset of T. The basic statements of Theorem 3.1 are then derived by a passage to the limit as $s \to \infty$ over appropriate sequences of the positive integers.

It follows from Assumption 2.2 and from [4, Theorem 3.3, p. 123] that there exists a curve $y(t)$ that is a relaxed minimizing curve with respect to $a(x)$. Let $f(x, t, \sigma)$ be a proper representation of $F(x, t)$, and let $(C_i, c_i(\xi_i))$ be a proper representation of B_i at $y(t_i)$ ($i = 0, 1$).

Consider a finite set $Q = \{\tau_1, \tau_2, \cdots, \tau_q\}$, where

$$t_0 = \tau_0 < \tau_1 < \tau_2 \cdots < \tau_q < \tau_{q+1} = t_1, \qquad \tau_{i+1} - \tau_i \leq \varepsilon_1/c_1 \quad (i = 0, \cdots, q),$$

and where ε_1 and c_1 are defined as in Assumption 2.2. Let $u_i(t)$ ($i = 1, \cdots, q$) be the characteristic function of the interval $[t_0, \tau_i]$ in T; that is, let $u_i(t) = 1$ for $t_0 \leq t \leq \tau_i$ and $u_i(t) = 0$ for $\tau_i < t \leq t_1$ ($i = 1, 2, \cdots, q$). Let $\tilde{t}_0 \in T$, let $(\tilde{C}_0, \tilde{c}_0(\tilde{\xi}_0))$ be a proper representation of some subset $\tilde{B}$ of V at some point $\tilde{y} \in \tilde{B}$, and let $\mathscr{I}(\tilde{t}_0)$ be the set of indices i such that $1 \leq i \leq q$ and $\tau_i \in Q \cap (\tilde{t}_0, t_1)$. We shall call an absolutely continuous curve $x(t)$ ($t \in [\tilde{t}_0, t_1]$) a *Q-admissible curve* $(\tilde{t}_0, \tilde{C}_0, \tilde{c}_0(\tilde{\xi}_0))$ if there exist points $\tilde{\xi}_{0,Q} \in \tilde{C}_0$ and $\xi_{1,Q} \in C_1$, a function $\sigma(t)$

from $[\tilde{t}_0, t_1]$ to S, and absolutely continuous functions $\eta_i(t) = (\eta_i^1(t), \cdots, \eta_i^m(t))$ $(i \in \mathscr{I}(\tilde{t}_0))$ from $[\tilde{t}_0, t_1]$ to E_m such that

$$(4.1.1)\quad \begin{cases} \dot{x}(t) = f(x(t), t, \sigma(t)) \quad \text{a.e. in } [\tilde{t}_0, t_1], \\ \dot{\eta}_i^k(t) = a_x^k(x(t))\cdot f(x(t), t, \sigma(t))u_i(t) = b^k(x(t), t, \sigma(t))u_i(t) \\ \qquad\qquad \text{a.e. in } [\tilde{t}_0, t_1] \quad (i \in \mathscr{I}(\tilde{t}_0);\ k = 1, \cdots, m), \\ x(\tilde{t}_0) = \tilde{c}_0(\tilde{\xi}_{0,Q}), \eta_i^k(\tilde{t}_0) = a^k(\tilde{c}_0(\tilde{\xi}_{0,Q})) \quad (i \in \mathscr{I}(\tilde{t}_0);\ k = 1, \cdots, m), \\ x(t_1) = c_1(\xi_{1,Q}), \quad \eta_i^k(t_1) \leq 0 \quad (i \in \mathscr{I}(\tilde{t}_0);\ k = 1, \cdots, m), \end{cases}$$

where $b^k(x, t, \sigma) = a_x^k(x)\cdot f(x, t, \sigma)$ and a_x^k is the gradient of $a^k(x)$.

We observe that $\eta_i^k(t_1) = a^k(x(\tau_i))$ $(i \in \mathscr{I}(\tilde{t}_0);\ k = 1, \cdots, m)$, hence $a^k(x(t)) \leq 0$ $(k = 1, \cdots, m)$ for $t \in Q \cap (\tilde{t}_0, t_1)$.

A *Q-minimizing curve* $(\tilde{t}_0, \tilde{C}_0, \tilde{c}_0(\tilde{\xi}_0))$ is a Q-admissible curve $(\tilde{t}_0, \tilde{C}_0, \tilde{c}_0(\tilde{\xi}_0))$ that minimizes, among all such curves, the value $x^1(t_1)$.

We can easily verify (see (2.2.3), (2.3.1), (2.3.4), (2.4.2), and (2.4.3)) that any Q-admissible curve $(t_0, C_0, c_0(\xi_0))$ is a relaxed admissible curve with respect to $(a^1(x) - \varepsilon_1, \cdots, a^m(x) - \varepsilon_1)$ and that it is therefore contained in the compact set D (see (2.2.5)). Furthermore, there exists at least one Q-admissible curve $(t_0, C_0, c_0(\xi_0))$, namely $y(t)$, the previously mentioned relaxed minimizing curve with respect to $a(x)$. It follows then, by [4, Theorem 3.3, p. 123], that there exists a Q-minimizing curve $(t_0, C_0, c_0(\xi_0))$.

We may now consider a sequence of successively finer sets $Q_1 \subset Q_2 \subset \cdots$ that become everywhere dense on T. Specifically, for $s = 1, 2, \cdots$, let

$$Q_s = \{t_0 + k\,2^{-s-s_1}(t_1 - t_0) \mid k = 1, 2, \cdots, 2^{s+s_1} - 1\},$$

where s_1 is sufficiently large so that $2^{-s_1}(t_1 - t_0) \leq \varepsilon_1/c_1$. As was just shown, for each s $(s = 1, 2, \cdots)$ there exists a curve $x_s(t)$ $(t \in T)$ that is a Q_s-minimizing curve $(t_0, C_0, c_0(\xi_0))$, and each of these curves is contained in the compact set D. Then, by [4, Theorems 3.1 and 3.2, pages 119 and 122], Assumption 2.2, and Definitions 2.3 and 2.4, there exist an infinite sequence P of integers, an absolutely continuous curve $x^*(t)$ $(t \in T)$, and a function $\sigma(t)$ from T to S such that the curves $x_s(t)$ (s in P) converge uniformly to $x^*(t)$ over T and, furthermore,

$$(4.1.2)\quad \begin{aligned} &\dot{x}^*(t) = f(x^*(t), t, \sigma(t)) \quad \text{a.e. in } T, \\ &x^*(t) \in D \quad (t \in T), \\ &x^*(t_0) = c_0(\xi_0^*), \quad x^*(t_1) = c_1(\xi_1^*) \quad \text{for some } \xi_0^* \in C_0 \text{ and some } \xi_1^* \in C_1. \end{aligned}$$

LEMMA 4.2. *The curves* $x_s(t)$ (*s in* P) *satisfy a uniform Lipschitz condition on* T, *and the curve* $x^*(t)$ *is a relaxed minimizing curve with respect to* $a(x)$.

Proof. The first part of the lemma follows directly from (4.1.1) and (2.3.4). Since the sets Q_s become everywhere dense on T, it easily follows that $a^k(x^*(t)) \leq 0$ $(t \in T;\ k = 1, \cdots, m)$. Thus, by (4.1.2), $x^*(t)$ is a relaxed admissible curve with respect to $a(x)$. Furthermore, as previously observed, the curve $y(t)$ (which is a relaxed minimizing curve with respect to $a(x)$) is also a Q_s-admissible curve $(t_0, C_0, c_0(\xi_0))$ for every s, hence $x_s^1(t_1) \leq y^1(t_1)$ $(s = 1, 2, \cdots)$, implying $\dot{x}^{*1}(t_1) \leq y^1(t_1)$. This proves that $x^*(t)$ is a relaxed minimizing curve with respect to $a(x)$.

We now investigate certain properties of Q-minimizing curves.

LEMMA 4.3. *Let* $x_s(t)$ *be a* Q_s*-minimizing curve* $(t_0, C_0, c_0(\xi))$ *for a fixed* s, *let*

$$Z_s^k = \{t \in Q_s \mid a^k(x_s(t)) = 0\} \qquad (k = 1, \cdots, m),$$

and let $Z_s = \bigcup_{\ell=1}^{m} Z_s^\ell$. *Then either there exists a point* $\xi_{1,s}$ *in* C_1 *such that*

$$c_1(\xi_{1,s}) = x_s(t_1) \quad \text{and} \quad c_{1,\xi}^1(\xi_{1,s})\xi_{1,s} = \min_{\xi_1 \in C_1} c_{1,\xi}^1(\xi_{1,s})\xi_1$$

(where $c_{1,\xi} = (c_{1,\xi}^1, \cdots, c_{1,\xi}^n)$ *and* $c_{1,\xi}^i$ *is the gradient of* $c_1^i(\xi_1)$ $(i = 1, \cdots, n)$*), or there exist a nonnegative number* γ_s^1*, points* $\xi_{0,s}$ *in* C_0 *and* $\xi_{1,s}$ *in* C_1*, a function* $\sigma_s(t)$ *from* T *to* S*, a set* $L_s \subset Z_s$*, and functions* $z_s(t)$ *and* $\mu_s(t)$ *from* T *to* E_n *and* E_m*, respectively, such that*

(4.3.1) $x_s(t_0) = c_0(\xi_{0,s})$ *and* $x_s(t_1) = c_1(\xi_{1,s})$;

(4.3.2) $\mu_s^k(t) \geq 0$ *and* $\|z_s(t)\| + \|\mu_s(t)\| > 0$ *for* $t \in T$ *and* $k = 1, \cdots, m$ *(where* $\|z\| = \sum_{i=1}^{n} |z^i|$ *and* $\|\mu\| = \sum_{\ell=1}^{m} |\mu^\ell|$*)*;

(4.3.3) $z_s(t)$ *is absolutely continuous on any closed subinterval of* $T - L_s$, *and* $z_s(t)$ *is continuous from the left on* T;

(4.3.4) *for each* k $(k = 1, \cdots, m)$, $\mu_s^k(t_1) = 0$, *and on every subinterval of* $T - L_s$, $\mu_s^k(t)$ *is a nonincreasing step function, continuous from the left, with no discontinuities except possibly at points of* Z_s^k;

(4.3.5) $z_s(t) = O = (0, \cdots, 0)$ $(t \in L_s)$ *and* $\mu_s^k(t) \cdot a^k(x_s(t)) = 0$ $(k = 1, \cdots, m;$ $t \in L_s)$;

(4.3.6) $\dot{x}_s(t) = f(x_s(t), t, \sigma_s(t))$ a.e. *in* T,

$$\dot{z}_s(t) = -f_x^T(x_s(t), t, \sigma_s(t))\, z_s(t) - \sum_{\ell=1}^{m} \mu_s^\ell(t)\, b_x^\ell(x_s(t), t, \sigma_s(t)) \quad \text{a.e. } in\ T,$$

where f_x^T *is the transpose of the matrix* f_x *and* $b^k(x, t, \sigma) = a_x^k(x) \cdot f(x, t, \sigma)$ $(k = 1, \cdots, m)$;

(4.3.7) $v_s(t) \cdot f(x_s(t), t, \sigma_s(t)) = \min_{\sigma \in S} v_s(t) \cdot f(x_s(t), t, \sigma)$ a.e. *in* T, *where*

$$v_s(t) = z_s(t) + \sum_{\ell=1}^{m} \mu_s^\ell(t)\, a_x^\ell(x_s(t)) \quad (t \in T);$$

(4.3.8) $v_s(t_0)\cdot c_{0,\xi}(\xi_{0,s})\xi_{0,s} = \underset{\xi_0\in C_0}{\text{Min}}\ v_s(t_0)\cdot c_{0,\xi}(\xi_{0,s})\xi_0$;

(4.3.9) $(\gamma_s^1\,\delta_1 - z_s(t_1))\cdot c_{1,\xi}(\xi_{1,s})\xi_{1,s} = \underset{\xi_1\in C_1}{\text{Min}}\ (\gamma_s^1\,\delta_1 - z_s(t_1))\cdot c_{1,\xi}(\xi_{1,s})\xi_1$,

where $\delta_1 = (\delta_1^1, \cdots, \delta_1^n)$, $\delta_1^1 = 1$, *and* $\delta_1^j = 0$ $(j = 2, \cdots, n)$.

Proof. Let us write $x(t)$, $\sigma(t)$, $z(t)$, $\mu(t)$, $v(t)$ instead of $x_s(t), \cdots, v_s(t)$. In Section 4.1 we showed that $x(t)$, a Q_s-minimizing curve $(t_0, C_0, c_0(\xi_0))$, exists and is contained in the compact set D. It follows then easily from relations (4.1.1) that

$$\eta_i^k(t) \qquad (i = 1, 2, \cdots, q = 2^{s_1+s} - 1;\ t \in T;\ k = 1, \cdots, m)$$

also exist, and that $\eta_i(t) = a(x(t))$ on $[t_0, \tau_i]$ and $\eta_i(t) = a(x(\tau_i))$ on $[\tau_i, t_1]$. Thus for the problem defined by relations (4.1.1), the curve $(x(t), \eta_1(t), \cdots, \eta_q(t))$ is a relaxed minimizing curve in the sense of [5, Theorem 6.1, p. 142] (where V is replaced by $V \times V_\eta$, $A = D \times D_\eta$, $V_\eta = I_a \times I_a \times \cdots \times I_a$ (q times),

$$D_\eta = R_a \times R_a \times \cdots \times R_a \quad (q \text{ times}),$$

R_a is the range of $a(x)$ for $x \in D$, and I_a is a bounded open m-dimensional interval containing R_a). After some manipulation, it follows then from [5, Theorem 6.1, p. 142] that either there exists a point $\xi_{1,s}$ in C_1 such that

(4.3.10) $c_1(\xi_{1,s}) = x(t_1)$ and $c^1_{1,\xi}(\xi_{1,s})\cdot\xi_{1,s} = \underset{\xi_1\in C_1}{\text{Min}}\ c^1_{1,\xi}(\xi_{1,s})\cdot\xi_1$,

or there exist a nonnegative number γ_s, points $\xi_{0,s}$ in C_0 and $\xi_{1,s}$ in C_1, and absolutely continuous functions $z(t)$, $\nu_1(t), \cdots, \nu_q(t)$ from T to $E_n, E_m, \cdots, E_m$, respectively, such that

(4.3.11) $$\|z(t)\| + \sum_{i=1}^{q} \|\nu_i(t)\| \neq 0 \qquad (t \in T);$$

(4.3.12) $$\begin{cases} \dot{x}(t) = f(x(t), t, \sigma(t)) & \text{a.e. in } T, \\ \dot{z}(t) = -f_x^T(x(t), t, \sigma(t))z(t) - \sum_{\ell=1}^{m} b_x^\ell(x(t), t, \sigma(t)) \sum_{i=1}^{q} \nu_i^\ell(t)u_i(t) & \text{a.e. in } T, \\ \dot{\nu}_i(t) = 0 \quad (i = 1, 2, \cdots, q) & \text{a.e. in } T; \end{cases}$$

(4.3.13) $v(t)\cdot f(x(t), t, \sigma(t)) = \underset{\sigma\in S}{\text{Min}}\ v(t)\cdot f(x(t), t, \sigma)$ a.e. in T,

where $v(t) = z(t) + \sum_{\ell=1}^{m} a_x^\ell(x(t)) \sum_{i=1}^{q} \nu_i^\ell(t)u_i(t)$;

(4.3.14) $c_0(\xi_{0,s}) = x(t_0)$ and $v(t_0)\cdot c_{0,\xi}(\xi_{0,s})\xi_{0,s} = \underset{\xi_0\in C_0}{\text{Min}}\ v(t_0)\cdot c_{0,\xi}(\xi_{0,s})\xi_0$;

(4.3.15) $$\begin{cases} c_1(\xi_1, s) = x(t_1), \\ (\gamma_s\,\delta_1 - z(t_1))\cdot c_{1,\xi}(\xi_{1,s})\xi_{1,s} = \underset{\xi_1\in C_1}{\text{Min}}\ (\gamma_s\,\delta_1 - z(t_1))\cdot c_{1,\xi}(\xi_{1,s})\xi_1, \\ \nu_i^k(t_1) \geq 0 \quad (i = 1, 2, \cdots, q;\ k = 1, \cdots, m), \\ \nu_i^k(t_1) = 0 \text{ if } \eta_i^k(t_1) = a^k(x(\tau_1)) < 0 \quad (i = 1, \cdots, q;\ k = 1, \cdots, m). \end{cases}$$

Remark. The last (third) line of (6.1.3) in [5, p. 143] should read "for some $\zeta^1 \geq 0$" instead of "for some $\xi^1 \geq 0$". The derivation of statements 4.3.10 through 4.3.15 from [5, Theorem 6.1, p. 142] is quite similar to the analogous derivation in the proof of [7, Lemma 4.1, statements 4.1.10 through 4.1.16].

Assume that (4.3.10) does not hold. Let

$$\mu^k(t) = \sum_{i=1}^{q} \nu_i^k(t) u_i(t) \qquad (k = 1, \cdots, m),$$

and let $\tau_0 = t_0$, $\tau_{q+1} = t_1$. We observe that, by (4.3.12) and (4.3.15), $\nu_i(t) = \nu_i$ is constant and $\nu_i^k \geq 0$ $(i = 1, 2, \cdots, q;\ k = 1, \cdots, m)$, and thus $\mu^k(t)$ $(k = 1, \cdots, m)$ is a nonnegative nonincreasing step function, continuous from the left, with its discontinuities (if any) restricted to points of Z_s^k. Furthermore, $\mu^k(t_1) = 0$ $(k = 1, \cdots, m)$.

We shall now show that there exists an integer j $(1 \leq j \leq q + 1)$ such that $\|z(t)\| + \|\mu(t)\| > 0$ on $[t_0, \tau_j]$. Indeed, if

$$\mu^k(\tau_j) > 0 \qquad \text{for some k and j } (j = 1, \cdots, q+1;\ k = 1, \cdots, m),$$

then $\mu^k(t) \geq \mu^k(\tau_j) > 0$ for all t in $[t_0, \tau_j]$. If $\|\mu(\tau_j)\| = 0$ for all j $(j = 1, \cdots, q + 1)$, then

$$\nu_j^k = 0 \qquad (j = 1, \cdots, q;\ k = 1, \cdots, m),$$

hence, by (4.3.11), $\|z(t)\| \neq 0$ and $\|z(t)\| + \|\mu(t)\| > 0$ over $[t_0, t_1]$.

Let now $\theta_1^{(0)}$ be the largest of the numbers τ_j $(j = 1, \cdots, q + 1)$ such that $\|z(t)\| + \|\mu(t)\| > 0$ on $[t_0, \tau_j]$. If $\theta_1^{(0)} = t_1$, we define L_s to be the empty set, and the lemma follows easily from relations (4.3.11) to (4.3.15). If $\theta_1^{(0)} = \tau_j < t_1$, then $\|z(\tau_{j+1})\| + \|\mu(\tau_{j+1})\| = 0$, and this implies that $\|\mu(t)\| = 0$ on $[\tau_{j+1}, t_1]$. It follows then from (4.3.12) and (4.3.15) that $\|z(t)\| = 0$ on $[\theta_1^{(0)}, t_1]$ and that

$$\mu^k(\theta_1^{(0)}) a^k(x(\theta_1^{(0)})) = 0 \qquad (k = 1, \cdots, m).$$

Furthermore $\|\nu_j\| \neq 0$, hence, by (4.3.15), $\theta_1^{(0)} \in Z_s$.

We observe that, trivially, x(t) is a Q'-minimizing curve

$$(\theta_1^{(0)}, \{x(\theta_1^{(0)})\}, \text{identity}),$$

where $Q' = Q_s \cap (\theta_1^{(0)}, t_1)$, where $\{x(\theta_1^{(0)})\}$ is the set with the single element $x(\theta_1^{(0)})$, and where the mapping is the identity mapping of $\{x(\theta_1^{(0)})\}$ into itself. It follows now, by our previous argument, that there exist a point

$$\theta_1^{(1)} \in \{t_1\} \cup Z_s \cap (\theta_1^{(0)}, t_1],$$

functions $z^{(1)}(t)$ and $\mu^{(1)}(t)$ on $[\theta_1^{(0)}, \theta_1^{(1)}]$, and a nonnegative number $\gamma_s^{(1)}$ with the following properties: $\mu^{(1)k}(t)$ $(k = 1, \cdots, m)$ is a nonnegative nonincreasing step function on $[\theta_1^{(0)}, \theta_1^{(1)}]$, continuous from the left, and with its only possible discontinuities on $(\theta_1^{(0)}, \theta_1^{(1)}] \cap Z_s$; $z^{(1)}(t)$ is absolutely continuous on $[\theta_1^{(0)}, \theta_1^{(1)}]$;

$$\|z^{(1)}(t)\| + \|\mu^{(1)}(t)\| > 0 \quad \text{on } (\theta_1^{(0)}, \theta_1^{(1)}];$$

$$\mu^{(1)k}(\theta_1^{(1)})a^k(x(\theta_1^{(1)})) = 0 \quad (k = 1, \cdots, m) \quad \text{if } \theta_1^{(1)} < t_1;$$

and relations (4.3.6), (4.3.7), and (4.3.9) hold with $[\theta_1^{(0)}, \theta_1^{(1)}]$ replacing T and $\gamma_s^{(1)}$ replacing γ_s.

If $\theta_1^{(1)} < t_1$, we continue in this manner. After ℓ steps (where $\ell \leq q + 1$), we determine a point $\theta_1^{(\ell)} = t_1$, since $\theta_1^{(\alpha)} \in Q_s$ $(\alpha = 0, 1, 2, \cdots)$ and $\theta_1^{(\alpha)} < \theta_1^{(\alpha+1)}$ $(\alpha = 0, 1, 2, \cdots)$. We now redefine the functions $z(t)$ and $\mu(t)$ to equal $z^{(\alpha)}(t)$ and $\mu^{(\alpha)}(t)$, respectively, on $(\theta_1^{(\alpha)}, \theta_1^{(\alpha+1)}]$ $(\alpha = 1, 2, \cdots, \ell - 1)$, we let $\gamma_s^1 = \gamma_s^{(\ell)}$, and we let

$$L_s = \{\theta^{(\alpha)} \mid \theta^{(\alpha)} < t_1, \ 0 \leq \alpha < \ell\}.$$

The lemma now follows directly.

(4.3.16). *Remark.* We shall continue to use the notation introduced in Sections 4.1 to 4.3. If for infinitely many values of s in P there exists a point $\xi_{1,s}$ in C_1 such that

$$c_1(\xi_{1,s}) = x_s(t_1) \quad \text{and} \quad c^1_{1,\xi}(\xi_{1,s})\xi_{1,s} = \min_{\xi_1 \in C_1} c^1_{1,\xi}(\xi_{1,s})\xi_1,$$

then, since C_1 is compact, we may extract an infinite subsequence P' of P such that $\xi_{1,s}$ converges to some ξ_1^* over P', and we have the relations

$$c_1(\xi_1^*) = x^*(t_1) \quad \text{and} \quad c^1_{1,\xi}(\xi_1^*)\xi_1^* = \min_{\xi_1 \in C_1} c^1_{1,\xi}(\xi_1^*)\xi_1.$$

The alternative (3.1.1) of Theorem 3.1 is then satisfied with $\gamma^a = 1$ and $\gamma^k = 0$ $(k = 1, \cdots, m)$.

Henceforth we shall therefore assume, unless it is otherwise specified, that the second alternative of Lemma 4.3 holds for all sufficiently large values of s in P that constitute a sequence P_1 of integers.

LEMMA 4.4. *Let* s *be in* P_1, *and let* t *and* t' *belong to a subinterval of* $T - L_s$. *Then there exists a positive constant* c_2, *depending only on* $a(x)$, $t_1 - t_0$, c_1, *and* D (of Assumption 2.2), *such that*

$$\|z_s(t)\| \leq c_2 \|z_s(t')\| + c_2 \left| \int_t^{t'} \|\mu_s(\tau)\| \, d\tau \right|.$$

Proof. As was observed in Section 4.1, the curve $x_s(t)$ is contained in the compact set D (of Assumption 2.2). Since $a(x)$ has continuous first- and second-order partial derivatives on V, and since by (2.3.4) $\|f(x, t, \sigma)\| \leq c_1$ and $\|f_x(x, t, \sigma)\| \leq c_1$ on $V \times T' \times S$, it follows that there exists a constant c_2' such that

$$|b^k(x_s(t), t, \sigma_s(t))| \leq c_2' \quad \text{and} \quad \|b_x^k(x_s(t), t, \sigma_s(t))\| \leq c_2' \quad \text{a.e. in } T \ (k = 1, \cdots, m).$$

It can easily be verified that the second equation of (4.3.6) yields

$$z_s(t) = U_s^T(t, t')z_s(t') + \int_t^{t'} U_s^T(t, \tau) \sum_{\ell=1}^{n} b_x^\ell(x_s(\tau), \tau, \sigma_s(\tau))\mu_s^\ell(\tau)\,d\tau ,$$

where the matrix $U_s(t, \tau)$ is the solution of the system

$$\frac{d}{d\tau} U_s(t, \tau) = f_x(x_s(\tau), \tau, \sigma_s(\tau))\, U_s(t, \tau)$$

that reduces to the unit matrix for $\tau = t$. It follows easily that there exists a constant $c_2 = c_2(c_1, c_2')$ such that

$$\|U_s(t, t')\| \leq c_2 \quad \text{and} \quad \|U_s^T(t, \tau)b_x^k(x_s(\tau), \tau, \sigma_s(\tau))\| \leq c_2 \quad (k = 1, \cdots, m).$$

The lemma now follows directly.

5. PASSING TO THE LIMIT

5.1. We shall continue to use the notation introduced in Section 4. Let

$$\|z, \mu\| = \|z\| + \|\mu\| \qquad \text{for } z \in E_n \text{ and } \mu \in E_m,$$

and let $Q_\infty = \bigcup_{s=1}^{\infty} Q_s$. Since Q_∞ is denumerable and C_0 and C_1 are compact, there exists an infinite subsequence P_2 of P_1 such that the sequences $\{\xi_{0,s}\}$ and $\{\xi_{1,s}\}$ converge over P_2 to points $\xi_0^* \in C_0$ and $\xi_1^* \in C_1$, the sequences

$$\{\|z_s(t)\|/\|z_s(t), \mu_s(t)\|\}, \qquad \{\mu_s^k(t)/\|z_s(t), \mu_s(t)\|\} \quad (k = 1, \cdots, m),$$

$$\{\|z_s(t), \mu_s(t)\|/\|z_s(t'), \mu_s(t')\|\}$$

converge over P_2 to finite limits or to ∞, for all t and t' in Q_∞, and the sets $L_s \cap [t, t']$ are either all empty or all nonempty for all sufficiently large s in P_2, provided $t \in Q_\infty$, $t' \in Q_\infty$, $t \leq t'$.

Let now

$$L = \bigcap_{i=1}^{\infty} \mathrm{Cl}\left(\bigcup_{\substack{s \text{ in } P_2 \\ s \geq i}} L_s\right)$$

(where Cl denotes closure) and let *open* henceforth mean "open relative to T" when it refers to a subset of T. Let N be the set of points t in $T - L$ with the property that t belongs to an open interval $I(t)$ such that

$$\limsup_{P_2} \|z_s(t'), \mu_s(t')\|/\|z_s(t''), \mu_s(t'')\| < \infty$$

for all $t' \in I(t)$ and $t'' \in I(t)$. Clearly, L is closed and N is open. Finally, let $M = T - N$.

Since the set N is open, it is a denumerable union of disjoint open intervals, which we shall call *maximal subintervals* of N or, briefly, *maximal*. Let J be maximal, and let $\tau_0(J)$, $\tau_1(J)$, and $\tau^*(J)$ be the left and right endpoints and the midpoint of J, respectively. Then $\tau_0(J) \in J$ if and only if $\tau_0(J) = t_0$ and $t_0 \in N$, and $\tau_1(J) \in J$ if and only if $\tau_1(J) = t_1$ and $t_1 \in N$.

For all maximal J and for all s in P_2, let

$$z_s^*(t) = z_s(t)/\|z_s(\tau^*(J)), \mu_s(\tau^*(J))\| \qquad (t \in J),$$

$$\mu_s^{*k}(t) = \mu_s^k(t)/\|z_s(\tau^*(J)), \mu_s(\tau^*(J))\| \qquad (k = 1, \cdots, m;\ t \in J),$$

$$v_s^*(t) = z_s^*(t) + \sum_{\ell=1}^{m} \mu_s^{*\ell}(t)\, a_x^k(x_s(t)) \qquad (t \in J).$$

This defines $z_s^*(t)$, $\mu_s^{*k}(t)$ $(k = 1, \cdots, m)$, and $v_s^*(t)$ on the set N. The definitions are permissible since the denominator is positive, by (4.3.2). We also observe that, by (4.3.6),

$$(5.1.2)\quad \dot{z}_s^*(t) = -f_x^T(x_s(t), t, \sigma_s(t))\, z_s^*(t) - \sum_{\ell=1}^{m} \mu_s^{*\ell}(t)\, b_x^\ell(x_s(t), t, \sigma_s(t)) \quad \text{a.e. in } N.$$

LEMMA 5.2. *Let* J *be maximal, and let* $[\theta_0, \theta_1] \subset J$. *Then, for all sufficiently large* s *in* P_2, *the sets* $L_s \cap [\theta_0, \theta_1]$ *are empty, the functions* $z_s^*(t)$ *are absolutely continuous on* $[\theta_0, \theta_1]$, *and the functions* $\mu_s^{*k}(t)$ $(k = 1, \cdots, m)$ *are nonincreasing on* $[\theta_0, \theta_1]$.

Proof. Since $J \subset N \subset T - L$, the lemma follows from (4.3.3) and (4.3.4).

LEMMA 5.3. *For every fixed* t *in* N,

$$0 < \liminf_{P_2} \|z_s^*(t), \mu_s^*(t)\| \le \limsup_{P_2} \|z_s^*(t), \mu_s^*(t)\| < \infty.$$

Proof. Let

$$u_s(\tau, \tau') = \|z_s(\tau), \mu_s(\tau)\| / \|z_s(\tau'), \mu_s(\tau')\| \qquad (s \text{ in } P_2,\ \tau \in T,\ \tau' \in T).$$

We shall prove that if the closed interval $I(\tau, \tau')$ joining τ and τ' is contained in N, then $\limsup_{P_2} u_s(\tau, \tau') < \infty$, which implies the statement of the lemma.

By the definition of N, every point t in $I(\tau, \tau')$ belongs to an open interval I(t) such that

$$\limsup_{P_2} u_s(t', t'') < \infty \qquad \text{for all } t' \in I(t) \text{ and } t'' \in I(t).$$

Since $I(\tau, \tau')$ is compact, it can be covered by a finite number of such open intervals, say by $I_1, \cdots, I_\ell$. Therefore there exist points $\tau_0 = \tau, \tau_1, \cdots, \tau_{\ell-1}, \tau_\ell = \tau'$ such that $I(\tau_i, \tau_{i+1}) \subset I_{i+1}$ $(i = 0, 1, \cdots, \ell - 1)$. Since

$$u_s(\tau, \tau') = \prod_{i=0}^{\ell-1} u_s(\tau_i, \tau_{i+1}) \qquad (s \text{ in } P_2),$$

it follows that $\limsup_{P_2} u_s(\tau, \tau') < \infty$.

LEMMA 5.4. *There exist an infinite subsequence* P_3 *of* P_2 *and functions* $\mu^{*k}(t)$ $(k = 1, \cdots, m)$, $z^*(t)$, *and* $v^*(t)$ *such that* $\mu_s^{*k}(t)$ $(k = 1, \cdots, m)$, $z_s^*(t)$, *and* $v_s^*(t)$ *converge on the set* N, *as* $s \to \infty$ *over* P_3, *to* $\mu^{*k}(t)$ $(k = 1, \cdots, m)$, $z^*(t)$, *and*

$$v^*(t) = z^*(t) + \sum_{\ell=1}^{m} \mu^{*\ell}(t) a_x^{\ell}(x^*(t)),$$

respectively. The functions $\mu_s^{*k}(t)$ $(k = 1, \cdots, m)$, $z_s^*(t)$, *and* $v_s^*(t)$ *are uniformly bounded on every closed subinterval of* N *for* s *in* P_3. *On every closed subinterval of* N *the functions* $\mu^{*k}(t)$ $(k = 1, \cdots, m)$ *are nonnegative and nonincreasing, and the function* $z^*(t)$ *is Lipschitz-continuous. Finally,* $\|z^*(t), \mu^*(t)\| > 0$.

Proof. Let $0 < \eta < 1/2$, let P' be an infinite subsequence of P_2, and let J be maximal. Consider the closed interval $[\theta_0, \theta_1]$, where

$$\theta_0 = \tau_0(J) + \eta(\tau_1(J) - \tau_0(J)) \quad \text{and} \quad \theta_1 = \tau_1(J) - \eta(\tau_1(J) - \tau_0(J)).$$

By Lemma 5.2 and by (4.3.4), the $\mu_s^{*k}(t)$ $(k = 1, \cdots, m)$ are nonnegative and nonincreasing on $[\theta_0, \theta_1]$ for sufficiently large s and, by Lemma 5.3,

$$\psi(t) = \limsup_{P_2} \|\mu_s^*(t)\| < \infty \quad \text{on } [\theta_0, \theta_1].$$

Thus $\psi(t)$ exists and is nonnegative and nonincreasing on $[\theta_0, \theta_1]$, hence the $\|\mu_s^*(t)\|$ (s in P_2) are nonincreasing and uniformly bounded on $[\theta_0, \theta_1]$. By Lemma 4.4,

$$\|z_s^*(t)\| \le c_2 + c_2 |t - \tau^*| \, \|\mu_s^*(\mathrm{Min}(t, \tau^*))\| \qquad (t \in [\theta_0, \theta_1]),$$

where $\tau^* = \tau^*(J)$; hence, the $z_s^*(t)$ are uniformly bounded on $[\theta_0, \theta_1]$. Furthermore, by Lemma 5.2, the $z_s^*(t)$ are absolutely continuous on $[\theta_0, \theta_1]$ for sufficiently large s. It was observed in Section 4.1 that the curves $x_s(t)$ $(t \in T)$ are contained in the compact set D for all s, hence, by Assumption 2.2 and by (2.3.4),

$$\|f_x(x_s(t), t, \sigma_s(t))\| \text{ and } \|b_x^k(x_s(t), t, \sigma_s(t))\| \quad (t \in T';\ k = 1, \cdots, m;\ s \text{ in } P_2)$$

are uniformly bounded. By (5.1.2) it follows that the $z_s^*(t)$ satisfy a uniform Lipschitz condition on $[\theta_0, \theta_1]$, for all sufficiently large s.

Thus $\mu_s^*(t)$ and $z_s^*(t)$ are uniformly bounded and of uniformly bounded variation on $[\theta_0, \theta_1]$, for all s in P'. Hence, by Helly's selection theorem, there exists an infinite subsequence P" = P"(P', J, η) of P' such that $\{\mu_s^*(t)\}$ and $\{z_s^*(t)\}$ converge on $[\theta_0, \theta_1]$ to limit functions $\mu^*(t)$ and $z^*(t)$, respectively, as $s \to \infty$ over P". The function $\mu^*(t)$ is nonnegative and nonincreasing and $z^*(t)$ is Lipschitz-continuous on $[\theta_0, \theta_1]$.

We now consider a sequence $\{\eta_j\}_{j=1}^{\infty}$ converging to $+0$. The open set N is a denumerable union of maximal subintervals J_i $(i = 1, 2, \cdots)$. We now let $P_1^1 = P_2$ and, recursively,

$$P^i_{j+1} = P''(P^i_j, J_i, \eta_j) \qquad (i, j = 1, 2, \cdots).$$

By P_1^{i+1} we denote the diagonal subsequence of $P_1^i, P_2^i, P_3^i, \cdots$ $(i = 1, 2, \cdots)$. Letting P_3 be the diagonal subsequence of $P_1^1, P_1^2, P_1^3, \cdots$, we conclude that $\{\mu_s^*(t)\}$ and $\{z_s^*(t)\}$ converge on N, as $s \to \infty$ over P_3, to limit functions $\mu^*(t)$ and $z^*(t)$ that satisfy the statements of the lemma.

Now, $v_s^*(t) = z_s^*(t) + \sum_{\ell=1}^m \mu_s^{*\ell}(t) a_x^\ell(x_s(t))$ for all s in P_2, $\lim_P x_s(t) = x^*(t)$, P_3 is a subsequence of P and of P_2, $x_s(t) \in D$ $(t \in T$, s in P), and the $a_x^k(x)$ $(k = 1, \cdots, m)$ are uniformly continuous on the compact set D. It follows that

$$\lim_{P_3} v_s^*(t) = v^*(t) = z^*(t) + \sum_{\ell=1}^m \mu^{*\ell}(t) a_x^\ell(x^*(t)) \qquad \text{on } N.$$

Finally, by Lemma 5.3, $\liminf_{P_2} \|z_s^*(t), \mu_s^*(t)\| > 0$ on N, and this implies that $\|z^*(t), \mu^*(t)\| > 0$ on N.

6. CONDITIONS ON THE SET M

Let the sets L, N, and $M = T - N$ and the sequences P_2 and P_3 be defined as in Section 5, and let

$$u_s(t', t'') = \|z_s(t'), \mu_s(t')\| / \|z_s(t''), \mu_s(t'')\| \qquad (t' \in T,\ t'' \in T,\ s \text{ in } P_2).$$

We verify, as an immediate consequence of the definition, that $\theta \in M$ implies that either $\theta \in L$ or that every neighborhood of θ contains points t' and t'' such that $\limsup_{P_2} u_s(t', t'') = \infty$.

Henceforth, we let

$$K(t) = \{k \mid a^k(x^*(t)) = 0\}, \qquad Z^k = \{t \mid a^k(x^*(t)) = 0\}, \qquad Z = \bigcup_{\ell=1}^m Z^\ell.$$

LEMMA 6.1. *There exists a positive constant* c_3 *such that* $u_s(t', t'') \geq c_3$ *whenever* $t' \leq t''$, $[t', t''] \subset T - L_s$, *and* s *is in* P_2.

Proof. By Lemma 4.4, $\|z_s(t'')\| \leq c_2 \|z_s(t')\| + c_2 (t_1 - t_0) \|\mu_s(t')\|$, and by (4.3.4), $\|\mu_s(t'')\| \leq \|\mu_s(t')\|$; hence

$$u_s(t', t'') \geq \frac{\|z_s(t')\| + \|\mu_s(t')\|}{c_2 \|z_s(t')\| + (c_2(t_1 - t_0) + 1)\|\mu_s(t')\|} \geq \frac{1}{c_2 + c_2(t_1 - t_0) + 1} = c_3.$$

LEMMA 6.2. *If* $\theta \in M - L$, *then there exists a positive* $\varepsilon = \varepsilon(\theta)$ *such that* $\lim_{P_3} u_s(t', t'') = \infty$ *for* $\theta - \varepsilon \leq t' < t'' \leq \theta + \varepsilon$. *If* $\theta = t_0$, *then* $\lim_{P_3} u_s(t_0, t'') = \infty$ *for* $t_0 < t'' \leq t_0 + \varepsilon$, *and if* $\theta = t_1$ *then* $\lim_{P_3} u_s(t', t_1) = \infty$ *for* $t_1 - \varepsilon \leq t' < t_1$.

Proof. Let $\theta \in M - L$ and $t_0 < \theta < t_1$, and let $\varepsilon = \varepsilon(\theta)$ be such that $[\theta - \varepsilon, \theta + \varepsilon] \subset (T - L) \cap (T - L_s)$ for sufficiently large s in P_2, say for s in P'.

Let $\theta - \varepsilon \leq t' < \theta < t'' \leq \theta + \varepsilon$. Since $\theta \in M - L$, there exist θ' and θ'' in $(t', t'') \subset (\theta - \varepsilon, \theta + \varepsilon)$ such that $\limsup_{P_2} u_s(\theta', \theta'') = \infty$. Lemma 6.1 implies that $\theta' < \theta''$. Let now

$$\tau' \in Q_\infty, \qquad \tau'' \in Q_\infty, \qquad t' < \tau' < \theta' < \theta'' < \tau'' < t''.$$

By Lemma 6.1,

$$u_s(\tau', \tau'') = u_s(\tau', \theta')u_s(\theta', \theta'')u_s(\theta'', \tau'') \geq c_3^2 u_s(\theta', \theta'') \qquad (s \text{ in } P').$$

It follows that $\limsup_{P_2} u_s(\tau, \tau'') = \infty$, and since $\tau' \in Q_\infty$ and $\tau'' \in Q_\infty$,

$$\lim_{P_2} u_s(\tau', \tau'') = \lim_{P_3} u_s(\tau', \tau'') = \infty .$$

Furthermore, $u_s(t', t'') = u_s(t', \tau')u_s(\tau', \tau'')u_s(\tau'', t'') \geq c_3^2 u_s(\tau', \tau'')$, hence $\lim_{P_3} u_s(t', t'') = \infty$.

The argument is analogous when $\theta = t_0$ or $\theta = t_1$.

LEMMA 6.3. *The set* M *is contained in* Z. *Furthermore, for every* θ *in* M *there exist a number* $\varepsilon = \varepsilon(\theta)$ $(0 < \varepsilon < 1)$ *and an infinite subsequence* $P' = P'(\theta)$ *of* P_3 *such that*

$$\limsup_{P'} \|z_s(t)\| / \sum_{\ell \in K(\theta)} \mu_s^\ell(t) \leq 2c_2\eta \quad \textit{and} \quad \lim_{P'} \mu_s^k(t) / \sum_{\ell \in K(\theta)} \mu_s^\ell(t) = 0 \quad (k \notin K(\theta))$$

uniformly on $[\tau_1, \tau_2]$, *where* $\tau_1 = \tau_1(\theta, \eta) = \mathrm{Max}(\theta - \eta, t_0)$,

$$\tau_2 = \tau_2(\theta, \eta) = \mathrm{Max}\,(\theta - \eta^2, t_0),$$

and $0 < \eta \leq \varepsilon(\theta)$.

Proof. Since the $a^k(x^*(t))$ $(k = 1, 2, \cdots, m)$ are continuous, θ is in the interior (relative to T) of an interval $[\theta_1, \theta_2]$ in T such that $K(t) \subset K(\theta)$ for $t \in [\theta_1, \theta_2]$.

If $\theta \in L$, let $\varepsilon = 1/2$ if $\theta = t_0$, and let $\varepsilon(\theta) = \mathrm{Min}(\theta - \theta_1, 1/2)$ otherwise. Since P_3 is a subsequence of P_2, it follows easily from the definition of L and of P_2 that there exist an infinite subsequence $P' = P'(\theta)$ of P_3 and points $\ell_s' \in L_s$ (s in P') such that $\theta = \lim_{P'} \ell_s'$. Let $0 < \eta \leq \varepsilon$ and $\tau \in [\tau_1, \tau_2]$. Then $\tau_2 \leq \ell_s' \leq \theta_2$ for all sufficiently large s in P', say for s in $P'' = P''(\theta, \eta)$.

Let now ℓ_s (s in P") be the point in $L_s \cap [\tau, \ell_s']$ nearest to τ. By (4.3.4) and (4.3.5),

$$\|z_s(\ell_s)\| = 0 \quad \text{and} \quad \mu_s^k(t) = 0 \qquad (k \notin K(\theta),\ t \in [\tau, \ell_s],\ s \text{ in } P''),$$

since $K(t) \subset K(\theta)$. Furthermore, by (4.3.2), $\|z_s(\tau)\| + \|\mu_s(\tau)\| \neq 0$ (s in P_3). Thus, by Lemma 4.4 and by (4.3.4),

$$\|z_s(\tau)\| \leq c_2 \|z_s(\ell_s)\| + c_2 \int_\tau^{\ell_s} \|\mu_s(t)\|\,dt \leq c_2(\ell_s - \tau) \sum_{\ell \in K(\theta)} \mu_s^\ell(\tau) \quad (s \text{ in } P''),$$

hence

$$\sum_{\ell \in K(\theta)} \mu_s^\ell(\tau) \neq 0, \qquad \limsup_{P'} \|z_s(\tau)\| / \sum_{\ell \in K(\theta)} \mu_s^\ell(\tau) \leq c_2(\theta - \tau) \leq c_2\eta,$$

$$\lim_{P'} \mu_s^k(\tau) / \sum_{\ell \in K(\theta)} \mu_s^\ell(\tau) = 0 \qquad (k \notin K(\theta)).$$

Since the sum of $\mu_s^\ell(\tau)$ over $\ell \in K(\theta)$ is positive, $K(\theta)$ is nonempty, hence $\theta \in Z$.

We now consider the case where $\theta \in M - L$. Let ε' be defined as $\varepsilon(\theta)$ in Lemma 6.2, and let $\varepsilon = \varepsilon(\theta)$ be such that $0 < \varepsilon < 1$,

$$[\theta - \varepsilon, \theta + \varepsilon] \cap T \subset [\theta_1, \theta_2] \cap (T - L),$$

and $\varepsilon \leq \varepsilon'$. Let $0 < \eta \leq \varepsilon$, $\tau \in [\tau_1, \tau_2]$, and $\tau' = \mathrm{Min}\,(\theta + \eta, t_1)$. It follows from Lemmas 6.1 and 6.2 that $\lim_{P_3} u_s(\tau, \tau') = \infty$ uniformly for $\tau \in [\tau_1, \tau_2]$, hence $\lim_{P_3} u_s(\tau', \tau) = 0$ uniformly for $\tau \in [\tau_1, \tau_2]$. For sufficiently large s in P_3, (4.3.2), (4.3.4), and Lemma 4.4 imply that

$$\|z_s(\tau')\| + \|\mu_s(\tau')\| \neq 0, \qquad \|z_s(\tau)\| \leq c_2 \|z_s(\tau')\| + 2c_2\eta \|\mu_s(\tau)\|,$$

and $\|\mu_s(\tau')\| \leq \|\mu_s(\tau)\|$, hence

$$0 = \lim_{P_3} u_s(\tau', \tau) \geq \lim_{P_3} \|\mu_s(\tau')\| / \{c_2 \|z_s(\tau')\| + (2c_2\eta + 1)\|\mu_s(\tau)\|\}$$

and

$$0 = \lim_{P_3} u_s(\tau', \tau) \geq \lim_{P_3} \|z_s(\tau')\| / \{c_2 \|z_s(\tau')\| + (2c_2\eta + 1)\|\mu_s(\tau)\|\}.$$

This implies that $\|\mu_s(\tau)\| \neq 0$ for sufficiently large s in P_3, and

$$\text{(6.3.1)} \qquad \lim_{P_3} \|z_s(\tau')\| / \|\mu_s(\tau)\| = \lim_{P_3} \|\mu_s(\tau')\| / \|\mu_s(\tau)\| = 0$$

uniformly in $[\tau_1, \tau_2]$.

Since $\|z_s(\tau)\| \leq c_2 \|z_s(\tau')\| + 2c_2\eta \|\mu_s(\tau)\|$ for sufficiently large s in P_3, we see that

$$\limsup_{P_3} \|z_s(\tau)\| / \|\mu_s(\tau)\| \leq 2c_2\eta \qquad \text{uniformly on } [\tau_1, \tau_2].$$

Now, by (4.3.4), $\sum_{\ell \notin K(\theta)} \mu_s^\ell(t)$ is constant on $[\tau, \tau']$; hence, as an easy consequence of (6.3.1), $\sum_{\ell \in K(\theta)} \mu_s^\ell(\tau) \neq 0$ for sufficiently large s in P_3, $\theta \in Z$,

$$\limsup_{P_3} \|z_s(\tau)\| / \sum_{\ell \in K(\theta)} \mu_s^\ell(\tau) \leq 2c_2\eta,$$

and

$$\lim_{P_3} \mu_s^k(\tau)/\sum_{\ell\in K(\theta)} \mu_s^\ell(\tau) = 0 \qquad (k \notin K(\theta)) \quad \text{uniformly on } [\tau_1, \tau_2].$$

LEMMA 6.4. *Let* $\theta \in Z$ *and* $t_0 \leq \tau_1 \leq \tau_2 < \theta$, *and assume that* $K(t) \subset K(\theta)$ *and* $\sum_{\ell\in K(\theta)} \mu_s^\ell(t) \neq 0$ *on* $[\tau_1, \tau_2]$ *for sufficiently large* s *in* P_3, *say for* s *in* P_3'. *Let*

$$\alpha_s^k(t) = \mu_s^k(t)/\sum_{\ell\in K(\theta)} \mu_s^\ell(t) \qquad (k \in K(\theta);\ t \in [\tau_1, \tau_2];\ s \text{ in } P_3').$$

Then

$$\liminf_{P_3} \beta_s^k = \liminf_{P_3} \int_{\tau_1}^{\tau_2} \alpha_s^k(t)\, a_x^k(x_s(t))\, \dot{x}_s(t)\, dt \geq -c_1(\theta - \tau_2) \qquad (k \in K(\theta)),$$

where c_1 *is defined as in Assumption* 2.2.

Proof. Let $k \in K(\theta)$, and let s be in P_3'. Since $a^k(x_s(t))$ is continuous on T and $\alpha_s^k(t)$ is a step function, we have the relation

$$(6.4.1)\quad \beta_s^k = \int_{\tau_1}^{\tau_2} \alpha_s^k(t)\, da^k(x_s(t))$$

$$= \alpha_s^k(\tau_2 - 0)\, a^k(x_s(\tau_2)) - \alpha_s^k(\tau_1 + 0)\, a^k(x_s(\tau_1)) - \int_{\tau_1+0}^{\tau_2-0} a^k(x_s(t))\, d\alpha_s^k(t).$$

Now, by (4.3.4), $\alpha_s^k(t)$ is a step function with no discontinuities except at points of

$$\bigcup_{\ell\in K(\theta)} Z_s^\ell.$$

Let τ in (τ_1, τ_2) be a point of discontinuity of $\alpha_s^k(t)$. If $\tau \notin Z_s^k$ and $\tau \notin L_s$, then by (4.3.4) $\mu_s^k(t)$ is constant in some neighborhood of τ, and $\sum_{\ell\in K(\theta)} \mu_s^\ell(t)$ is nonincreasing in that neighborhood of τ; hence $d\alpha_s^k(\tau) \geq 0$. If $\tau \notin Z_s^k$ and $\tau \in L_s$, then $\mu_s^k(t) = 0$ immediately to the left of τ, by (4.3.5). Hence, by (4.3.2),

$$d\alpha_s^k(\tau) = \alpha_s^k(\tau + 0) \geq 0.$$

Since

$$\tau \in \bigcup_{\ell\in K(\theta)} Z_s^\ell \subset Q_s,$$

we see that $a^k(x_s(\tau)) \leq 0$. Thus $a^k(x_s(t))\, d\alpha_s^k(t) \leq 0$ on (τ_1, τ_2), and by (6.4.1),

$$\beta_s^k \geq \alpha_s^k(\tau_2 - 0)\, a^k(x_s(\tau_2)) - \alpha_s^k(\tau_1 + 0)\, a^k(x_s(\tau_1)).$$

By (4.3.2), $\mu_s^k(t) \geq 0$ $(k \in K(\theta))$, and it follows that $0 \leq \alpha_s^k(t) \leq 1$ $(k \in K(\theta))$. By (4.3.6) and condition (2.2.3),

$$|a_s^k(x_s(t))\dot{x}_s(t)| = \left|\frac{d}{dt} a^k(x_s(t))\right| \leq c_1 \quad \text{a.e. in } T.$$

Now every point of T is within a distance at most $2^{-s}(t_1 - t_0)$ of some point of Q_s, and $a^k(x_s(t)) \leq 0$ $(k = 1, \cdots, m)$ on Q_s. Furthermore, $\lim_{P_3} a^k(x_s(\theta)) = 0$ $(k \in K(\theta))$. It follows that

$$\lim_{P_3} a^k(x_s(\tau_2)) \geq -c_1(\theta - \tau_2) \quad (k \in K(\theta)) \quad \text{and} \quad -\lim_{P_3} a^k(x_s(\tau_1)) \geq 0.$$

LEMMA 6.5. *Let* T' *be defined as in Assumption* 2.2. *There exist scalar functions* $\mu^{*k}(t)$ $(k = 1, \cdots, m)$, *defined for each* $t \in M \cap T'$, $t > t_0$, *such that*

$$\mu^{*k}(t) \geq 0 \quad (k = 1, \cdots, m), \qquad \mu^{*k}(t) = 0 \quad (k \notin K(t)), \qquad \sum_{\ell \in K(t)} \mu^{*\ell}(t) = 1,$$

and $\sum_{\ell \in K(t)} \mu^{*\ell}(t) a_x^\ell(x^*(t)) \cdot f(x^*(t), t, \sigma) \geq 0$ *for all* $\sigma \in S$.

Proof. By (4.3.6) and (4.3.7), we have for every s in P_3 the relation

(6.5.1) $$v_s(t) \cdot \dot{x}_s(t) = \operatorname*{Min}_{\sigma \in S} v_s(t) \cdot f(x_s(t), t, \sigma) \text{ a.e. in } T,$$

where

$$v_s(t) = z_s(t) + \sum_{\ell=1}^{m} \mu_s^\ell(t) a_x^\ell(x_s(t)).$$

Let now $\theta \in M \cap T'$ and $\theta > t_0$, and let $\varepsilon = \varepsilon(\theta)$ and $P' = P'(\theta)$ be defined as in Lemma 6.3.

Let σ be an arbitrary point of S, let $0 < \eta < \operatorname{Min}(\varepsilon, \theta - t_0)$, let $\tau_1 = \theta - \eta$, $\tau_2 = \theta - \eta^2$, and let

$$\zeta_s(t) = z_s(t) / \sum_{\ell \in K(\theta)} \mu_s^\ell(t), \qquad \alpha_s^k(t) = \mu_s^k(t) / \sum_{\ell \in K(\theta)} \mu_s^\ell(t) \quad (k \in K(\theta),\ t \in [\tau_1, \tau_2]).$$

By (4.3.6) and Assumption 2.2, $|a_x^k(x_s(t))\dot{x}_s(t)| \leq c_1$ $(k = 1, \cdots, m)$ and $\|\dot{x}_s(t)\| \leq c_1$ a.e. in T for all s in P_3, and it follows from Lemmas 6.3 and 6.4 and from Assumption 2.2 that

$$\limsup_{P'} \left| \int_{\tau_1}^{\tau_2} (\zeta_s(t) + \sum_{\ell \notin K(\theta)} \alpha_s^\ell(t) a_x^\ell(x_s(t))) \cdot \dot{x}_s(t)\, dt \right| \leq 2c_1 c_2 \eta^2$$

and

$$\liminf_{P'} \sum_{\ell \in K(\theta)} \int_{\tau_1}^{\tau_2} \alpha_s^\ell(t) a_x^\ell(x_s(t)) \dot{x}_s(t)\, dt \geq -mc_1 \eta^2.$$

Similarly,

$$\limsup_{P'} \left| \int_{\tau_1}^{\tau_2} \left(\zeta_s(t) + \sum_{\ell \notin K(\theta)} \alpha_s^\ell(t) a_x^\ell(x_s(t) \right) f(x_s(t), t, \sigma) dt \right| \leq 2c_1 c_2 \eta^2 .$$

Dividing both sides of (6.5.1) by $\sum_{\ell \in K(\theta)} \mu_s^\ell(t)$, integrating from τ_1 to τ_2, and taking lim inf over P', we deduce that

$$\text{(6.5.2)} \quad \liminf_{P'} \sum_{\ell \in K(\theta)} \int_{\tau_1}^{\tau_2} \alpha_s^\ell(t) a_x^\ell(x_s(t)) f(x_s(t), t, \sigma) dt \geq -(4c_1 c_2 + mc_1)\eta^2 .$$

Let now $P'' = P''(\theta, \eta)$ be an infinite subsequence of P' over which the

$$\frac{1}{\eta} \int_{\tau_1}^{\tau_2} \alpha_s^k(t) dt \qquad (k \in K(\theta))$$

converge to a limit $\alpha^k(\theta, \eta)$. Since $\alpha_s^k(t) \geq 0$ $(k \in K(\theta))$ and

$$\sum_{\ell \in K(\theta)} \alpha_s^\ell(t) = 1 \qquad (t \in [\tau_1, \tau_2],\ s \text{ sufficiently large and in } P'),$$

such a sequence P'' exists, $\alpha^k(\theta, \eta) \geq 0$ $(k \in K(\theta))$, and

$$\sum_{\ell \in K(\theta)} \alpha^\ell(\theta, \eta) = 1 - \eta.$$

Since $\theta \in T'$, Assumption 2.2 and Lemma 4.2 imply that if $\delta > 0$, then

$$\sum_{\ell \in K(\theta)} \left| a_s^\ell(x_s(t)) f(x_s(t), t, \sigma) - a_x^\ell(x^*(\theta)) f(x^*(\theta), \theta, \sigma) \right| < \delta$$

on $[\theta - \eta^2, \theta - \eta] = [\tau_1, \tau_2]$ for all sufficiently large s in P_3 and all sufficiently small η. We now conclude from (6.5.2) that

$$\text{(6.5.3)} \quad \sum_{\ell \in K(\theta)} \alpha^\ell(\theta, \eta) a_x^\ell(x^*(\theta)) f(x^*(\theta), \theta, \sigma) \geq -(4c_1 c_2 + mc_1)\eta - \delta$$

for all δ and all sufficiently small η.

We now choose a sequence $\{\eta_j\}_{j=1}^\infty$ converging to $+0$ as $j \to \infty$, and such that $\mu^{*k}(\theta) = \lim_{j \to \infty} \alpha^k(\theta, \eta_j)$ exists for every $k \in K(\theta)$. Since the choice of $\mu^{*k}(\theta)$ $(k \in K(\theta))$ is made independently of σ, the statement of the lemma follows directly from (6.5.3).

7. THE GENERALIZED WEIERSTRASS E-CONDITION

7.1. Let $\mu^*(t)$, $z^*(t)$, and $v^*(t)$ $(t \in N)$ be functions having the properties described in Lemma 5.4, and let $\mu^*(t)$ $(t \in M \cap T' \cap (t_0, t_1])$ be a function having the properties described in Lemma 6.5. Let us also set $z^*(t) = O$ $(t \in M)$ and

$$\mu^{*k} = 1/k(t) \qquad (k \in K(t),\ t \in (M - T') \cap (t_0, t_1]),$$

where $k(t)$ denotes the number of elements in $K(t)$. Finally, let

$$v^*(t) = \sum_{\ell \in K(t)} \mu^{*\ell}(t) a_x^\ell(x^*(t)) \qquad \text{on } (M - T') \cap (t_0, t_1].$$

The function $z^*(t)$ is then defined everywhere on T, and the functions $\mu^*(t)$ and $v^*(t)$ are defined everywhere on $T - M \cap \{t_0\}$.

LEMMA 7.2. *There exists a function* $\sigma^*(t)$ *from* T *to* S *such that*

$$\dot{x}^*(t) = f(x^*(t), t, \sigma^*(t)) \qquad \text{a.e. } in\ T,$$

$$\dot{z}^*(t) = -f_x^T(x^*(t), t, \sigma^*(t)) z^*(t) - \sum_{\ell=1}^{m} \mu^{*\ell}(t) b_x^\ell(x^*(t), t, \sigma^*(t)) \qquad \text{a.e. } in\ N.$$

Furthermore, for every maximal J, $z^*(\tau_1(J) - 0) = O$ *and* $\mu^{*k}(\tau_1(J) - 0) = 0$ $(k \in K(\tau_1(J)))$ *if* $\tau_1(J) \in M$.

Proof. Let E be a euclidean space, $\mathfrak{T}$ a closed interval of the real axis, $\mathfrak{T}' \subset \mathfrak{T}$, $|\mathfrak{T}'| = |\mathfrak{T}|$, $\mathfrak{B}$ an open set in E, $\mathfrak{A}$ a compact subset of $\mathfrak{B}$, and $\mathfrak{F}(\mathfrak{x}, t)$ $(\mathfrak{x} \in \mathfrak{B},\ t \in \mathfrak{T})$ a compact and convex set in E. Assume that $\mathfrak{F}(\mathfrak{x}, t)$ is uniformly bounded for $(\mathfrak{x}, t) \in \mathfrak{A} \times \mathfrak{T}'$ and that $\mathfrak{F}(\mathfrak{x}, t)$ is *quasi-continuous* [4, p. 119] (in the language of Filippov [1, p. 76], *upper-semicontinuous with respect to inclusion*) at $(\mathfrak{x}, t)$ for all $(\mathfrak{x}, t) \in \mathfrak{A} \times \mathfrak{T}'$; that is, corresponding to each positive δ, let there exist a positive $\eta(\delta, \mathfrak{x}, t)$ such that $\mathfrak{F}(\mathfrak{x}', t')$ is contained in a δ-neighborhood of $\mathfrak{F}(\mathfrak{x}, t)$ in $\mathfrak{B}$ provided

$$|t - t'| + \|\mathfrak{x} - \mathfrak{x}'\| \leq \eta(\delta, \mathfrak{x}, t) \qquad \text{and} \qquad (\mathfrak{x}, t) \in \mathfrak{A} \times \mathfrak{T}'.$$

Let $\{\mathfrak{x}_j(t)\}$ $(t \in \mathfrak{T},\ j = 1, 2, \cdots)$ be a sequence of curves, all contained in $\mathfrak{A}$, such that

$$(7.2.1) \qquad \dot{\mathfrak{x}}_j(t) \in \mathfrak{F}(\mathfrak{x}_j(t), t) \text{ a.e. in } \mathfrak{T} \quad (j = 1, 2, \cdots).$$

Finally, let $\mathfrak{x}(t)$ be a uniform limit of the curves $\mathfrak{x}_j(t)$ $(t \in \mathfrak{T})$.

It was shown in [4, Theorem 3.1, p. 119] and, effectively, in [1, Theorem 1, p. 77] that then

$$(7.2.2) \qquad \dot{\mathfrak{x}}(t) \in \mathfrak{F}(\mathfrak{x}(t), t) \qquad \text{a.e. in } \mathfrak{T}.$$

Now let $\mathfrak{F}_j(\mathfrak{x}, t)$ $((\mathfrak{x}, t) \in \mathfrak{A} \times \mathfrak{T},\ j = 1, 2, \cdots)$ be convex and compact sets converging uniformly, as $j \to \infty$, to $\mathfrak{F}(\mathfrak{x}, t)$ on $\mathfrak{A} \times \mathfrak{T}'$, in the sense that to each positive δ there corresponds a $j(\delta)$ such that each of the two sets $\mathfrak{F}_j(\mathfrak{x}, t)$ and $\mathfrak{F}(\mathfrak{x}, t)$ is in a δ-neighborhood of the other provided $j \geq j(\delta)$. Furthermore, let the assumption (7.2.1) be replaced by

(7.2.3) $\dot{\mathfrak{x}}_j(t) \in \mathfrak{F}_j(\mathfrak{x}_j(t), t)$ a.e. in $\mathfrak{T}$ $(j = 1, 2, \cdots)$.

Then we can easily show that the conclusion (7.2.2) still holds. Indeed, if $\varepsilon > 0$ and $\mathfrak{F}_\varepsilon(\mathfrak{x}, t)$ is the ε-neighborhood of $\mathfrak{F}(\mathfrak{x}, t)$, then there exists an integer $j(\varepsilon)$ such that $\mathfrak{F}_j(\mathfrak{x}, t) \subset \mathfrak{F}_\varepsilon(\mathfrak{x}, t)$ for $(\mathfrak{x}, t) \in \mathfrak{A} \times \mathfrak{T}'$ and $j \geq j(\varepsilon)$. Thus $\dot{\mathfrak{x}}_j(t) \in \mathfrak{F}_\varepsilon(\mathfrak{x}_j(t), t)$ a.e. in $\mathfrak{T}$ for $j \geq j(\varepsilon)$, whence it follows, by the quoted arguments, that $\dot{\mathfrak{x}}(t) \in \mathfrak{F}_\varepsilon(\mathfrak{x}(t), t)$ a.e. in $\mathfrak{T}$. Since ε can be chosen arbitrarily small, relation (7.2.2) now follows.

We now apply this result as follows. Let the infinite sequence P_3 of Lemma 5.4 be $s_1, s_2, \cdots$; let $E = E_n \times E_n$; let $\mathfrak{T}$ be an arbitrary closed subinterval of N over which the functions $\mu^*_{s_j}(t)$ and $z^*_{s_j}(t)$ $(j = 1, 2, \cdots)$ converge uniformly to $\mu^*(t)$ and $z^*(t)$, respectively; let $\mathfrak{T}' = \mathfrak{T} \cap T'$, $\mathfrak{B} = V \times E_n$,

$$D_z = \mathrm{Cl}\,\{z \in E_n \mid z = z^*_s(t) \text{ or } z = z^*(t) \text{ for some } s \text{ in } P_3 \text{ and some } t \in \mathfrak{T}\},$$

$$\mathfrak{A} = D \times D_z, \qquad \mathfrak{x} = (x, z),$$

$$\mathfrak{F}(\mathfrak{x}, t) = \{(\xi, \eta) \mid \xi = f(x, t, \sigma),\ \eta = -f_x^T(x, t, \sigma)z - \sum_{\ell=1}^{m} \mu^{*\ell}(t) b_x^\ell(x, t, \sigma)$$
$$\text{for some } \sigma \text{ in } S\},$$
$$(x, z, t) \in D \times D_z \times \mathfrak{T},$$

$$\mathfrak{F}_j(\mathfrak{x}, t) = \{(\xi, \eta) \mid \xi = f(x, t, \sigma),\ \eta = -f_x^T(x, t, \sigma)z - \sum_{\ell=1}^{m} \mu^{*\ell}_{s_j}(t) b_x^\ell(x, t, \sigma)$$
$$\text{for some } \sigma \text{ in } S\}.$$

Furthermore, let $\mathfrak{x}_j(t) = (x_{s_j}(t), z^*_{s_j}(t))$ $(t \in \mathfrak{T},\ j = 1, 2, \cdots)$ and $\mathfrak{x}(t) = (x^*(t), z^*(t))$. Then it easily follows from Assumption 2.2, Definition 2.3, Section 4.1, formula (5.1.2), Lemma 5.4, and our previous argument, that

$$\dot{\mathfrak{x}}(t) \in \mathfrak{F}(\mathfrak{x}, t) \quad \text{a.e. in } \mathfrak{T},$$

or, equivalently, that there exists a function $\sigma^*(t)$ from T to S such that

$$\dot{x}^*(t) = f(x^*(t), t, \sigma^*(t)) \quad \text{a.e. in } \mathfrak{T},$$

$$\dot{z}^*(t) = -f_x^T(x^*(t), t, \sigma^*(t))z^*(t) - \sum_{\ell=1}^{m} \mu^{*\ell}(t) b_x^\ell(x^*(t), t, \sigma^*(t)) \quad \text{a.e. in } \mathfrak{T}.$$

Since, by an easy consequence of Egoroff's theorem, N can be covered (except for a set of measure 0) by a finite or denumerable collection of closed intervals over each of which $\mu^*_{s_j}(t)$ and $z^*_{s_j}(t)$ $(j = 1, 2, \cdots)$ converge uniformly to $\mu^*(t)$ and $z^*(t)$, respectively, it follows that these differential equations hold a.e. in N.

We now complete the definition of $\sigma^*(t)$ by setting $\sigma^*(t) = \sigma(t)$ $(t \in M)$, where $\sigma(t)$ is defined as in (4.1.2).

Finally, since $z_s^*(t)$ and $\mu_s^*(t)$ converge to $z^*(t)$ and $\mu^*(t)$ on N, as $s \to \infty$ over P_3, it follows from Lemma 6.3 that

$$\|z^*(\tau_1(J) - 0)\| = \|\mu^{*k}(\tau_1(J) - 0)\| = 0 \quad (k \notin K(\tau_1(J))) \qquad \text{if } \tau_1(J) \in M.$$

This completes the proof of the lemma.

LEMMA 7.3. *(the generalized Weierstrass* E*-condition).*

$$v^*(t) \cdot f(x^*(t), t, \sigma^*(t)) = \operatorname*{Min}_{\sigma \in S} v^*(t) \cdot f(x^*(t), t, \sigma) \qquad \text{a.e. } in \text{ T}$$

(here $v^*(t) = z^*(t) + \sum_{\ell=1}^{m} \mu^{*\ell}(t) a_x^\ell(x^*(t))$).

Proof. We shall first prove that the relation holds a.e. in N. By (4.3.6) and (4.3.7),

(7.3.1) $$v_s^*(t) \cdot f(x_s(t), t, \sigma_s(t)) = v_s^*(t) \cdot \dot{x}_s(t) = \operatorname*{Min}_{\sigma \in S} v_s^*(t) \cdot f(x_s(t), t, \sigma)$$

a.e. in T for all s in P_3.

Let T_1 be the subset of $T' \cap N$ over which the above relation is satisfied, over which $v^*(t)$ is continuous, and over which $\dot{x}^*(t)$ exists and satisfies the first equation of Lemma 7.2. Clearly, $|T_1| = |N|$. Let $\theta < t_1$, let $\theta \in T_1$, and let h be positive and sufficiently small so that $[\theta, \theta + h] \subset N$. Then

$$x^*(\theta + h) - x^*(\theta) = \lim_{P_3} (x_s(\theta + h) - x_s(\theta)) = \lim_{P_3} \int_\theta^{\theta+h} \dot{x}_s(\tau) d\tau,$$

hence

(7.3.2) $$\frac{1}{h} v^*(\theta) \cdot (x^*(\theta + h) - x^*(\theta))$$

$$= \lim_{P_3} \frac{1}{h} \int_\theta^{\theta+h} [v_s^*(\tau) \dot{x}_s(\tau) + (v^*(\tau) - v_s^*(\tau)) \dot{x}_s(\tau) + (v^*(\theta) - v^*(\tau)) \dot{x}_s(\tau)] d\tau.$$

Now $|\dot{x}_s(\tau)| \leq c_1$ a.e. in T, and by Lemma 5.4, the $v_s^*(\tau)$ are uniformly bounded on $[\theta, \theta + h]$ for s in P_3 and $\lim_{P_3} v_s^*(t) = v^*(t)$ on $[\theta, \theta + h]$. It follows thus from (7.3.1) and (7.3.2) that, for every σ in S,

$$\frac{1}{h} v^*(\theta) \cdot (x^*(\theta + h) - x^*(\theta))$$

$$\leq \frac{1}{h} \int_\theta^{\theta+h} v^*(\tau) \cdot f(x^*(\tau), \tau, \sigma) d\tau + \frac{c_1}{h} \int_\theta^{\theta+h} \|v^*(\theta) - v^*(\tau)\| d\tau.$$

By (2.3.3) and the definition of T_1, $v^*(t)$ and $f(x^*(t), t, \sigma)$ are continuous at θ, and $x^*(t)$ is differentiable at θ. Letting $h \to 0$, we conclude that

$$v^*(\theta) \cdot f(x(\theta), \theta, \sigma^*(\theta)) = v^*(\theta) \cdot \dot{x}^*(\theta) \leq v^*(\theta) \cdot f(x^*(\theta), \theta, \sigma)$$

for every σ in S and every θ in T_1. Thus the lemma holds on N.

We must now consider the set M. By Lemma 6.5 and Section 7.1,

$$(7.3.3)\qquad \sum_{\ell\in K(t)} \mu^{*\ell}(t)\, a^{\ell}_{x}(x^*(t))\cdot f(x^*(t), t, \sigma) = v^*(t)\cdot f(x^*(t), t, \sigma) \geq 0 \quad (\sigma \in S)$$

a. e. in M.

Since almost every point in a set is a limit point of the set, Lemma 7.2 implies that

$$\frac{d}{dt}\, a^{k}(x^*(t)) = a^{k}_{x}(x^*(t))\, f(x^*(t), t, \sigma^*(t)) = 0 \quad (k \in K(t)) \quad \text{a. e. in } T,$$

hence

$$\sum_{\ell\in K(t)} \mu^{*\ell}(t)\, a^{\ell}_{x}(x^*(t))\, f(x^*(t), t, \sigma^*(t)) = 0 \qquad \text{a. e. in } T.$$

Thus, by (7.3.3), $0 = v^*(t)\cdot f(x^*(t), t, \sigma^*(t)) = \operatorname*{Min}_{\sigma\in S} v^*(t)\cdot f(x^*(t), t, \sigma)$ a. e. in M.

8. SUPPORT (TRANSVERSALITY) CONDITIONS. COMPLETION OF THE PROOF OF THEOREM 3.1

LEMMA 8.1. *Either alternative* (3.1.1) *of Theorem* 3.1 *is satisfied, or there exists a nonnegative number* γ^1 *such that*

$$(8.1.1)\quad (\gamma^1 \delta_1 - z^*(t_1))\cdot c_{1,\xi}(\xi_1^*)\,\xi_1^* = \operatorname*{Min}_{\xi_1\in C_1} (\gamma^1 \delta_1 - z^*(t_1))\, c_{1,\xi}(\xi_1^*)\,\xi_1$$

and

$$(8.1.2)\quad v^*(t_0)\cdot c_{0,\xi}(\xi_0^*)\,\xi_0^* = \operatorname*{Min}_{\xi_0\in C_0} v^*(t_0)\cdot c_{0,\xi}(\xi_0^*)\,\xi_0\,,$$

where $v^*(t_0)$ *is defined as in Lemma* 5.4 *if* $t_0 \in N$, *while otherwise*

$$v^*(t_0) = \sum_{\ell\in K(t_0)} \mu^{*\ell}(t_0)\, a^{\ell}_{x}(x^*(t_0)), \qquad \mu^{*k}(t_0) \geq 0 \quad (k \in K(t_0)),$$

and $\displaystyle\sum_{\ell\in K(t_0)} \mu^{*\ell}(t_0) = 1.$

Proof. We observed, in Remark (4.3.16), that either the alternative (3.1.1) of Theorem 3.1 is satisfied or there exists an infinite sequence P_1 of integers s for which the second alternative of Lemma 4.3 holds. Consider, in the latter case, the statement (4.3.9) in Lemma 4.3. If, over some infinite subsequence P_3' of P_3,

$$\gamma_s^1 \neq 0 \qquad \text{and} \qquad \lim_{P_3'} \|z_s(t_1)\|/\gamma_s^1 = 0,$$

then it follows from (4.3.9) that

$$c^1_{1,\xi}(\xi_1^*)\,\xi_1^* = \operatorname*{Min}_{\xi_1\in C_1} c_{1,\xi}(\xi_1^*)\,\xi_1\,,$$

where $\xi_1^* = \lim_{P_2} \xi_{1,s} = \lim_{P_3'} \xi_{1,s}$.

In this case, therefore, the alternative (3.1.1) of Theorem 3.1 is also satisfied.

In view of the above argument it remains to consider the case where the second alternative of Lemma 4.3 holds over an infinite sequence P_1 and where, over some infinite subsequence P_3'' of P_3, $\|z_s(t_1)\| \neq 0$ and the $\gamma_s^1/\|z_s(t_1)\|$ are bounded.

If $t_1 \in M$, then $z^*(t_1) = O$ and relation (8.1.1) is trivially satisfied if we set $\gamma^1 = 0$. If $t_1 \in N$, let J be the maximal interval to which t_1 belongs, and let

$$\alpha_s = \|z_s(\tau^*(J))\| + \|\mu_s(\tau^*(J))\| \qquad (s \text{ in } P_3'').$$

Since, by (4.3.2), $\alpha_s \neq 0$, since $z_s(t_1)/\alpha_s = z_s^*(t_1) \to z^*(t_1)$ over P_3'', and since the $\gamma_s^1/\|z_s(t_1)\|$ are bounded, we can find an infinite subsequence P_3''' of P_3'' and a non-negative γ^1 such that $\lim_{P_3'''} \gamma_s^1/\alpha_s = \gamma^1$. Relation (8.1.1) now easily follows from (4.3.9).

If $t_0 \in N$, then relation (8.1.2) is derived from (4.3.8) by dividing both sides by $\|z_s(\tau^*(J))\| + \|\mu_s(\tau^*(J))\|$ (where J is the maximal interval containing t_0) and passing to the limit over P_3.

If $t_0 \in M$ then, by Lemma 6.3, there exists an infinite subsequence P' of P_3 such that

$$\lim_{P'} \|z_s(t_0)\| / \sum_{\ell \in K(t_0)} \mu_s^\ell(t_0) = \lim_{P'} \mu_s^k(t_0) / \sum_{\ell \in K(t_0)} \mu_s^\ell(t_0) = 0 \qquad (k \notin K(t_0)).$$

Let now P'' be an infinite subsequence of P' such that $\mu_s^k(t_0) / \sum_{\ell \in K(t_0)} \mu_s^\ell(t_0)$ converges to a limit $\mu^{*k}(t_0)$ over P'' for every k in $K(t_0)$. We now derive relation (8.1.2) from (4.3.8) by dividing both sides by $\sum_{\ell \in K(t_0)} \mu_s^\ell(t_0)$ and passing to the limit over P''.

8.2. *Completion of the Proof of Theorem* 3.1. Let $x(t) = x^*(t)$ $(t \in T)$. In Section 4.1 and Lemma 4.2, we showed that $x(t)$ exists and is a relaxed minimizing curve with respect to $a(x)$. By Assumption 2.2 and by [6, Theorem 2.2, p. 113], $x(t)$ can be uniformly approximated by solutions of the differential equations (2.1.1). We shall now show that if alternative (3.1.1) of Theorem 3.1 does not hold, then alternative (3.1.2) is satisfied if we set $\sigma(t) = \sigma^*(t)$, $\mu(t) = \mu^*(t)$, $z(t) = z^*(t)$,

$$v(t) = v^*(t) = z^*(t) + \sum_{\ell=1}^{m} \mu^{*\ell}(t)\, a_x^\ell(x^*(t)) \qquad (t \in T),$$

where, for $t_0 \in M$, $\mu^{*k}(t_0) = 0$ $(k \notin K(t_0))$ and $\mu^{*k}(t_0)$ $(k \in K(t_0))$ is defined as in Lemma 8.1.

By Lemma 6.3, $M \subset Z$. Statements (3.1.2.1) and (3.1.2.2) follow directly from Lemma 5.4 and from the definition of $\mu^*(t)$ and $z^*(t)$ on M (see Section 7.1). By statement (4.3.4) in Lemma 4.3, the $\mu_s^k(t)$ $(k = 1, \cdots, m)$ are, for all s in P_3, constant over every subinterval of $T - Z_s^k - L_s$, and $\mu_s^k(t_1) = 0$. We easily verify that every closed subinterval of $N - Z^k$ is contained in $T - Z_s^k - L_s$ for all sufficiently large s in P_3. Thus, $\mu^{*k}(t)$ $(k = 1, \cdots, m)$ is constant on every closed subinterval of $N - Z^k$. Statement (3.1.2.3) now follows from Lemma 5.4.

Statements (3.1.2.4), (3.1.2.5), and (3.1.2.6) follow from Lemmas 7.2, 7.3, and 8.1, respectively, and from (4.1.2).

8.2.1. *Proof of Statement* (3.1.2.7). We have just shown that either alternative (3.1.1) of Theorem 3.1 holds or alternative (3.1.2) is satisfied through (3.1.2.6). Assume now that alternative (3.1.1) does not hold.

For every $t \in Z$, let $H(t)$ be the convex hull of the points $a^k_x(x(t))$ $(k \in K(t))$ in E_n, and let

$$\alpha(t) = \operatorname*{Min}_{y \in H(t)} \|y\| = \operatorname{Min} \Big\| \sum_{\ell \in K(t)} \bar{\delta}^{\ell} a^{\ell}_x(x(t)) \Big\| ,$$

the minimum being taken over all $\bar{\delta}^k$ such that $\bar{\delta}^k \geq 0$ $(k \in K(t))$ and $\sum_{\ell \in K(t)} \bar{\delta}^{\ell} = 1$. Let $\alpha(t) = +\infty$ if $t \in T - Z$, that is, if $K(t)$ is empty.

Let U be the set of points θ in T with the property that $\alpha(\theta) \neq 0$ and

$$\Big\| z(\theta) + \sum_{\ell \notin K(\theta)} \mu^{\ell}(t) a^{\ell}_x(x(\theta)) + \sum_{\ell \in K(\theta)} \bar{\gamma}^{\ell} a^{\ell}_x(x(\theta)) \Big\| \neq 0$$

for all $\bar{\gamma}^k \geq \mu^k(\theta)$ $(k \in K(\theta))$.

We show that U contains t_1, and that for each θ $(\theta \in U,\ \theta > t_0)$ there exists a positive $\varepsilon(\theta)$ such that $[\theta - \varepsilon(\theta),\ \theta] \subset U$. Then, letting $t^*_0 = t_0$ if $U = T$ and $t^*_0 = \sup(T - U)$ if $U \neq T$, we can easily verify that statement (3.1.2.7) follows. To prove our assertion about U, we note first that

(8.2.1.1) $\liminf_{\tau \to t} \alpha(\tau) \geq \alpha(t)$.

Since $a^k(x(t))$ is continuous for every k $(k = 1, \cdots, m)$, we see that $K(\tau) \subset K(t)$ in some neighborhood of t. Thus, since $a^k_x(x(t))$ $(k = 1, \cdots, m)$ is continuous, for every $\varepsilon > 0$ there exists a δ-neighborhood of t such that $H(\tau)$ is contained in an ε-neighborhood of $H(t)$ if $|\tau - t| \leq \delta = \delta(\varepsilon)$. In other words, for every ε and for $|\tau - t| \leq \delta(\varepsilon)$, $\alpha(\tau) \geq \alpha(t) - \varepsilon$ if $H(\tau)$ is nonempty, and $\alpha(\tau) = +\infty$ if $H(\tau)$ is empty.

(8.2.1.2) $t_1 \in U$.

If $t_1 \in T - Z$, then, by (3.1.2.1) and (3.1.2.3),

$$\Big\| z(t_1) + \sum_{\ell=1}^{m} \mu^{\ell}(t_1) a^{\ell}_x(t_1)) \Big\| = \| z(t_1) \| \neq 0,$$

hence $t_1 \in U$. If $t_1 \in Z$, then $\alpha(t_1) \neq 0$, since otherwise (3.1.1) is satisfied with $\gamma^a = 0$. Furthermore,

$$\Big\| z(t_1) + \sum_{\ell=1}^{m} \bar{\gamma}^{\ell} a^{\ell}_x(x(t_1)) \Big\| = 0 \quad \text{for some } \bar{\gamma}^k \geq \mu^k(t_1) \qquad (k = 1, \cdots, m)$$

implies, by (3.1.2.1) and (3.1.2.6.2), that (3.1.1) is satisfied, contrary to our assumption. Thus $t_1 \in U$.

(8.2.1.3). Let $\theta \in N \cap U = (T - M) \cap U$, $\theta > t_0$. Then $[\theta - \varepsilon(\theta), \theta] \subset U$ for some $\varepsilon(\theta) > 0$.

Let J be the maximal interval to which θ belongs. Since $a(x(t))$ is continuous and $\alpha(\theta) \neq 0$, there exists, by (8.2.1.1), an $\varepsilon' = \varepsilon'(\theta) > 0$ such that $\alpha(t) \neq 0$ and $K(t) \subset K(\theta)$ on $[\theta - \varepsilon', \theta] \subset J$. Assume now, by way of contradiction, that there exists an increasing sequence $\{\theta_j\}_1^\infty$ in $[\theta - \varepsilon', \theta]$, converging to θ, and with $\theta_j \notin U$. Then

$$(8.2.1.3.1) \qquad \left\| z(\theta_j) + \sum_{\ell=1}^{m} \bar{\gamma}_j^\ell a_x^\ell(x(\theta_j)) \right\| = 0 \qquad (j = 1, 2, \cdots)$$

for some $\bar{\gamma}_j^k \geq \mu^k(\theta_j)$ $(k \in K(\theta);\ j = 1, 2, \cdots)$ and $\bar{\gamma}_j^k = \mu^k(\theta_j)$ $(k \notin K(\theta);$ $j = 1, 2, \cdots)$. By (3.1.2.2) and (3.1.2.3), $z(\theta_j) \to z(\theta)$, $\bar{\gamma}_j^k = \mu^k(\theta)$ $(k \notin K(\theta);$ $j = 1, 2, \cdots)$ and $\bar{\gamma}_j^k \geq \mu^k(\theta)$ $(k = 1, \cdots, m;\ j = 1, 2, \cdots)$. Let $j_1, j_2, \cdots$ be a sequence of positive integers such that the $\bar{\gamma}_{j_i}^k$ approach a finite or an infinite limit $\bar{\gamma}^k$ for each k $(k \in K(\theta))$. If $\bar{\gamma}^k = \infty$ for some $k \in K(\theta)$, then by (8.2.1.3.1), $\left\| \sum_{\ell \in K(\theta)} \gamma^{*\ell} a_x^\ell(x(\theta)) \right\| = 0$, where the γ^{*k} $(k \in K(\theta))$ are the limits, over some appropriate sequence of j's, of $\bar{\gamma}_{j_i}^k / \sum_{\ell \in K(\theta)} \bar{\gamma}_{j_i}^\ell$. This implies that $\alpha(\theta) = 0$, contrary to our assumption that $\theta \in U$. It follows that the $\bar{\gamma}^k$ $(k \in K(\theta))$ are finite, hence, by (8.2.1.3.1), that

$$\left\| z(\theta) + \sum_{\ell \notin K(\theta)} \mu^\ell(\theta) a_x^\ell(x(\theta)) + \sum_{\ell \in K(\theta)} \bar{\gamma}^\ell a_x^\ell(x(\theta)) \right\| = 0,$$

where $\bar{\gamma}^k \geq \mu^k(\theta)$ $(k \in K(\theta))$. This again contradicts the assumption that $\theta \in U$.

(8.2.1.4) Let $\theta \in M \cap U$ and $\theta > t_0$. Then $[\theta - \varepsilon(\theta), \theta] \subset U$ for some $\varepsilon(\theta) > 0$.

Since $\theta \in U$, it follows from (8.2.1.1) that there exists a positive $\varepsilon' = \varepsilon'(\theta)$ such that $\alpha(t) \neq 0$ and $K(t) \subset K(\theta)$ on $[\theta - \varepsilon', \theta]$. Furthermore, by the same argument as in (8.2.1.1), we can show that

$$\beta(t) > \frac{1}{2}\beta(\theta) = \frac{1}{2}\alpha(\theta) \qquad \text{on } [\theta - \varepsilon''(\theta), \theta],$$

where

$$\beta(t) = \operatorname{Min} \sum_{\ell \in K(\theta)} \bar{\gamma}^\ell a_x^\ell(x(t))$$

(the minimum being taken over all $\bar{\gamma}^k$ such that $\bar{\gamma}^k \geq 0$, $\sum_{\ell \in K(\theta)} \bar{\gamma}^\ell = 1$), and where $\varepsilon''(\theta)$ is sufficiently small. Let

$$\varepsilon = \varepsilon(\theta) = \operatorname{Min}\,(\varepsilon'(\theta), \varepsilon''(\theta), \alpha(\theta)/(4c_2), \theta - t_0),$$

where c_2 is defined as in Lemma 4.4, and let $t^{\#} \in [\theta - \varepsilon, \theta]$.

If $t^{\#} \in M$, then $\|z(t^{\#})\| = \mu^k(t^{\#}) = 0$ $(k \notin K(t^{\#}))$ and $\sum_{\ell \in K(t^{\#})} \mu^{\ell}(t^{\#}) > 0$, hence

$$\|z(t^{\#}) + \sum_{\ell \notin K(t^{\#})} \mu^{\ell}(t^{\#})\, a_x^{\ell}(x(t^{\#})) + \sum_{\ell \in K(t^{\#})} \bar{\gamma}^{\ell} a_x^{\ell}(x(t^{\#}))\| \geq \sum_{\ell \in K(t^{\#})} \bar{\gamma}^{\ell} \cdot \alpha(t^{\#}) > 0$$

for all $\bar{\gamma}^k \geq \mu^k(t^{\#})$ $(k \in K(t^{\#}))$. Thus $t^{\#} \in U$.

If $t^{\#} \in N$, let J be the maximal subinterval of N to which $t^{\#}$ belongs, and let $\tau_1(J)$ be its right endpoint. Then $\tau_1(J) \leq \theta$, hence $\tau_1(J) \in M$. It follows easily from (3.1.2.4) (or from Lemma 4.4) and from (3.1.2.2) that

$$\|z(t)\| \leq c_2 \|z(t')\| + c_2 \int_t^{t'} \|\mu(\tau)\|\, d\tau \leq c_2 \|z(t')\| + c_2 (t' - t) \|\mu(t)\|$$

for $t \in J$, $t' \in J$, $t < t'$. Furthermore, by (3.1.2.4), $\|z(\tau_1(J) - 0)\| = 0$. Thus

$$\|z(t^{\#})\| \leq c_2(\tau_1(J) - t^{\#}) \|\mu(t^{\#})\| \leq c_2 \varepsilon \|\mu(t^{\#})\| \leq \tfrac{1}{2} \beta(t^{\#}) \|\mu(t^{\#})\|,$$

hence, by (3.1.2.1), $\|\mu(t^{\#})\| > 0$. Since $K(t^{\#}) \subset K(\theta)$, we see that

$$\|\sum_{\ell \in K(\theta)} \bar{\gamma}^{\ell} a_x^{\ell}(x(t^{\#}))\| \geq \sum_{\ell \in K(\theta)} \bar{\gamma}^{\ell} \cdot \beta(t^{\#})$$

whenever $\bar{\gamma}^k \geq \mu^k(t^{\#})$ $(k \in K(\theta))$ and $\mu^k(t^{\#}) = 0$ $(k \notin K(\theta))$.

Let now Min represent the minimum over all values of γ^k $(k \in K(\theta))$ such that $\bar{\gamma}^k \geq \mu^k(t^{\#})$. Then

$$\text{Min} \|\sum_{\ell \in K(t^{\#})} \bar{\gamma}^{\ell} a_x^{\ell}(x(t^{\#})) + \sum_{\ell \notin K(t^{\#})} \mu^{\ell}(t^{\#})\, a_x^{\ell}(x(t^{\#}))\| \geq \text{Min} \|\sum_{\ell \in K(\theta)} \bar{\gamma}^{\ell} a_x^{\ell}(x(t^{\#}))\|$$

$$\geq \sum_{\ell \in K(\theta)} \bar{\gamma}^{\ell} \cdot \beta(t^{\#}) \geq \|\mu(t^{\#})\| \beta(t^{\#}) > 0.$$

Since $\|z(t^{\#})\| \leq \frac{1}{2} \|\mu(t^{\#})\| \beta(t^{\#})$ and $\alpha(t^{\#}) \neq 0$, we conclude that $t^{\#} \in U$.

8.2.2. *Proof of statement* (3.1.2.8). If the assumption of (3.1.2.8) is satisfied, then it follows directly from Lemma 6.5 that $M \subset (t_0, t_1] - T'$, hence that M is of measure 0 and $N = T - M$ is nonempty. Let J be some maximal subinterval of N, with the right endpoint $\tau_1(J)$, and assume, by way of contradiction, that $\theta = \tau_1(J) \in M$. Let

$$\zeta^*(t) = z^*(t)/\|\mu^*(t)\| \quad \text{and} \quad \nu^*(t) = \mu^*(t)/\|\mu^*(t)\| \quad (t \in J).$$

By Lemmas 5.4 and 6.3, $\theta \in Z$, $K(t) \subset K(\theta)$, and

$$\|\zeta^*(t)\| = \lim_{P_2} \|z_s(t)\|/\|\mu_s(t)\| = \lim_{P_2} \|z_s(t)\|/\sum_{\ell \in K(\theta)} \mu_s^{\ell}(t) \leq 2c_2(\theta - t)$$

for all $t < \theta$ and sufficiently close to θ, say for $t \in [\theta - \varepsilon', \theta]$. Similarly, $\nu^{*k}(t) = 0$ ($k \notin K(\theta)$, $t \in [\theta - \varepsilon'', \theta]$), where $\varepsilon'' > 0$.

Now, by Lemma 4.2 and Assumption 2.2,

$$f(x^*(t), \tau, \sigma) \quad \text{and} \quad a^k_x(x^*(t))f(x^*(t), \tau, \sigma) \quad (k \in K(\theta))$$

converge to

$$f(x^*(\theta), \tau, \sigma) \quad \text{and} \quad a^k_x(x^*(\theta))f(x^*(\theta), \tau, \sigma),$$

respectively, as $t \to \theta$ ($t < \theta$), uniformly in τ and σ, and $\|f(x, \tau, \sigma)\|$ and $\|a^k_x(x(t))\|$ are uniformly bounded. Therefore, let $0 < \varepsilon \leq \operatorname{Min}(\varepsilon', \varepsilon'')$, and let ε be sufficiently small so that $\tau^*(J) < \theta - \varepsilon$ (where $\tau^*(J)$ is the midpoint of J),

$$(8.2.2.1) \quad \left(\zeta^*(t) + \sum_{\ell=1}^{m} \nu^{*\ell}(t) a^\ell_x(x^*(t))\right) \cdot f(x^*(t), t, \sigma)$$

$$\leq \sum_{\ell \in K(\theta)} \nu^{*\ell}(t) a^\ell_x(x^*(\theta)) \cdot f(x^*(\theta), t, \sigma) - \frac{1}{2}\beta,$$

and $|\zeta^*(t) \cdot f(x^*(t), t, \sigma)| \leq -\frac{1}{4}\beta$ for all $\sigma \in S$ and for all $t \in [\theta - \varepsilon, \theta]$.

Since $K(t) \subset K(\theta)$ on $[\theta - \varepsilon, \theta]$, we see that $\mu^{*k}(t) = \nu^{*k}(t) = 0$ ($k \notin K(\theta)$), hence

$$\sum_{\ell \in K(\theta)} \nu^{*\ell}(t) = 1 \quad \text{and} \quad \nu^{*k}(t) \geq 0 \quad (t \in [\theta - \varepsilon, \theta],\ k \in K(\theta)).$$

It therefore follows from the assumption of (3.1.2.8) that

$$\operatorname*{Min}_{\sigma \in S} \sum_{\ell \in K(\theta)} \nu^{*\ell}(t) a^\ell_x(x^*(\theta)) \cdot f(x^*(\theta), t, \sigma) \leq \beta \quad \text{on } [\theta - \varepsilon, \theta].$$

Thus, letting $w(t) = \zeta^*(t) + \sum_{\ell=1}^{m} \nu^{*\ell}(t) a^\ell_x(x(t))$, we deduce from (3.1.2.5) and (8.2.2.1) that

$$w(t) \cdot f(x^*(t), t, \sigma^*(t)) = \operatorname*{Min}_{\sigma \in S} w(t) \cdot f(x^*(t), t, \sigma) \leq \frac{1}{2}\beta \quad \text{a.e. in } [\theta - \varepsilon, \theta],$$

and from (3.1.2.4) that

$$(8.2.2.2) \quad \sum_{\ell \in K(\theta)} \nu^{*\ell}(t) a^\ell_x(x^*(t)) \dot{x}(t) = \sum_{\ell \in K(\theta)} \nu^{*\ell}(t) a^\ell_x(x^*(t)) f(x^*(t), t, \sigma^*(t))$$

$$= w(t) \cdot f(x^*(t), t, \sigma^*(t)) - \zeta^*(t) \cdot f(x^*(t), t, \sigma^*(t)) \leq \frac{1}{4}\beta$$

$$\text{a.e. in } [\theta - \varepsilon, \theta].$$

Let $0 < \eta \leq \varepsilon$, $0 < \eta < 1$, $\tau_1 = \theta - \eta$, $\tau_2 = \theta - \eta^2$. Then integration of (8.2.2.2) from τ_1 to τ_2 yields the inequality

$$(8.2.2.3) \quad \int_{\tau_1}^{\tau_2} \sum_{\ell \in K(\theta)} \nu^{*\ell}(t)\, a_x^{\ell}(x^*(t))\, \dot{x}^*(t)\, dt \leq \tfrac{1}{4}\beta\,(\eta - \eta^2).$$

We now observe that $\|\mu^*(\tau_2)\| = \sum_{\ell \in K(\theta)} \mu^{*\ell}(\tau_2) \geq 0$, since otherwise (3.1.2.2), (3.1.2.3), and (3.1.2.4) would imply that $\|z^*(t)\| + \|\mu^*(t)\| = 0$ on (τ_2, θ), contrary to (3.1.2.1). Furthermore, by (3.1.2.3), the $\mu^{*k}(t)$ $(k \in K(\theta))$ are nonnegative and nonincreasing on $[\tau_1, \tau_2]$. We can then easily deduce that the functions

$$\nu^{*k}(t) = \mu^{*k}(t) \Big/ \sum_{\ell \in K(\theta)} \mu^{*\ell}(t) \qquad (k \in K(\theta))$$

are of bounded variation on $[\tau_1, \tau_2]$. Thus

$$(8.2.2.4) \quad \int_{\tau_1}^{\tau_2} \nu^{*k}(t)\, a_x^k(x^*(t))\, \dot{x}^*(t)\, dt = \int_{\tau_1}^{\tau_2} \nu^{*k}(t)\, da^k(x^*(t))$$

$$= \nu^{*k}(\tau_2 - 0)\, a^k(x^*(\tau_2)) - \nu^{*k}(\tau_1 + 0)\, a^k(x^*(\tau_1))$$

$$- \int_{\tau_1+0}^{\tau_2-0} a^k(x^*(t))\, d\nu^{*k}(t) \quad (k \in K(\theta)).$$

Now, by (3.1.2.3), $\nu^{*k}(t)$ is nondecreasing on each subinterval of $T - Z^k$ and, by Lemma 4.2, $a^k(x^*(t)) \leq 0$ on T. Thus

$$- \int_{\tau_1+0}^{\tau_2-0} a^k(x^*(t))\, d\nu^{*k}(t) \geq 0.$$

Furthermore, $0 \leq \nu^{*k}(t) \leq 1$ for $t \in [\tau_1, \tau_2]$, and by (3.1.2.4) and Assumption 2.2, $a^k(x^*(\tau_2)) \geq -c_1\eta^2$ for $k \in K(\theta)$. It follows then from (8.2.2.4) that

$$\int_{\tau_1}^{\tau_2} \nu^{*k}(t)\, a_x^k(x^*(t))\, \dot{x}^*(t)\, dt \geq -c_1\eta^2 \qquad (k \in K(\theta)),$$

which contradicts (8.2.2.3) for all sufficiently small η.

Thus the assumption that $\theta = \tau_1(J) \in M$ leads to a contradiction, and this implies that $(t_0, t_1] \subset N$.

This completes the proof of statement (3.1.2.8) and of Theorem 3.1.

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