POLYNOMIALS ORTHOGONAL OVER A CURVE

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Let $\mu(x)$ be a non-decreasing function whose moments

$$\int_{-\infty}^{\infty} x^n d\mu(x) \qquad (n = 0, 1, 2, \cdots)$$

all exist, and let $P_0(x)$, $P_1(x)$, $P_2(x)$, $\cdots$ be the unique polynomials for which

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) d\mu(x) = \delta_{nm}$$

and

$$P_n(x) = k_n x^n + \cdots, \qquad k_n > 0. \tag{1}$$

It is well known [3, p. 157] that these orthogonal polynomials satisfy a recursion relation of the form

$$a_n P_{n-1}(x) + (b_n - x) P_n(x) + c_n P_{n+1}(x) = 0, \tag{2}$$

where $a_n > 0$ $(n = 1, 2, \cdots)$. Conversely, there is a theorem of Favard [2] (see [4, p. 349]) that any sequence of polynomials $P_n(x)$ with the structure (1), generated by a recursion relation (2) with $a_n > 0$, must be orthogonal with respect to some non-decreasing function $\mu(x)$.

It is interesting to ask whether there is any analogous theory for polynomials orthogonal over a rectifiable Jordan curve in the complex plane. There are some examples of polynomials that are orthogonal over ellipses and satisfy relations

$$\alpha_n p_{n-1}(z) + (\beta_n - z) p_n(z) + \gamma_n p_{n+1}(z) = 0 \qquad (n = 1, 2, \cdots). \tag{3}$$

The polynomials $p_n(z) = z^n$ are of course orthogonal over any circle with center at the origin. Less trivial examples are provided by the Chebyshev polynomials of the first and second kinds, which are orthogonal with respect to certain positive continuous weight functions over a family of confocal ellipses [8], [6], [1]. (These are the Jacobi polynomials with parameters $\alpha = \beta = -1/2$ and $\alpha = \beta = 1/2$, respectively.) Nevertheless, the simultaneous occurrence of orthogonality and a recursion relation (3) is a very special phenomenon, as the following theorem indicates.

THEOREM. *Let* C *be an analytic Jordan curve in the complex plane, and let* $\omega(z)$ *be a positive continuous function on* C. *Let* $p_0(z)$, $p_1(z)$, $\cdots$ *be the orthogonal polynomials normalized so that*

$$\int_C p_n(z)\overline{p_m(z)}\omega(z)\,|dz| = \delta_{nm},$$

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$$p_n(z) = k_n z^n + \cdots, \qquad k_n > 0.$$

Suppose further that the $p_n(z)$ *satisfy some recursion relation of the form* (3). *Then* C *is an ellipse, and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *converge to finite limits.*

Proof. Let us begin by mentioning a few basic facts about polynomials orthogonal over a curve. It is readily seen, even if C is merely rectifiable, that all the zeros of $p_n(z)$ must lie inside the convex hull of C. (If $p_n(\alpha) = 0$, then $p_n(z) = (z - \alpha)q(z)$, where $q(z)$ is a polynomial of degree $n - 1$. Thus $p_n(z)$ is orthogonal to $q(z)$, which gives the result.) If C is *analytic*, it is possible to say more: for sufficiently large n, all the zeros of $p_n(z)$ are exterior to any preassigned closed subdomain of the exterior of C. This follows from Szegö's asymptotic formula for $p_n(z)$ as $n \to \infty$, valid for z in the exterior of C [5], [6], [7, p. 368]. The asymptotic formula implies, further, that for each z in the exterior of C,

$$\lim_{n \to \infty} \frac{p_{n+1}(z)}{p_n(z)} = \psi(z), \tag{4}$$

where

$$w = \psi(z) = cz + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots, \qquad c > 0,$$

is the (normalized) conformal mapping of the exterior of C onto $|w| > 1$. The relation (4) is the basis of our proof.

Suppose first that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are bounded. Choose a subsequence $\{n_k\}$ such that

$$\alpha_{n_k} \to \alpha, \qquad \beta_{n_k} \to \beta, \qquad \gamma_{n_k} \to \gamma$$

as $k \to \infty$. According to (3),

$$z = \alpha_n p_{n-1}(z)/p_n(z) + \beta_n + \gamma_n p_{n+1}(z)/p_n(z). \tag{5}$$

Now let n tend to infinity through the sequence $\{n_k\}$. In view of (4), we find

$$z = \alpha/\psi(z) + \beta + \gamma\,\psi(z).$$

In other words,

$$\phi(w) = \alpha/w + \beta + \gamma w \qquad (|w| > 1),$$

where $\phi(w)$ is the inverse of $\psi(z)$. This shows that C is an ellipse. If there were some other subsequence through which α_n, β_n, γ_n tend to different limits $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, it would follow by the same argument that

$$\phi(w) = \tilde{\alpha}/w + \tilde{\beta} + \tilde{\gamma}w.$$

But the coefficients of $\phi(w)$ are unique. Thus

$$\alpha_n \to \alpha, \qquad \beta_n \to \beta, \qquad \gamma_n \to \gamma.$$

To finish the proof, we now show that unboundedness of any of the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ would lead to a contradiction. Suppose first that $\{\alpha_n\}$ is unbounded, and select a sequence $\{n_k\}$ such that $\alpha_{n_k} \neq 0$ and $\alpha_{n_k} \to \infty$. There are four possibilities:

(i) $\{\beta_{n_k}/\alpha_{n_k}\}$ and $\{\gamma_{n_k}/\alpha_{n_k}\}$ both bounded. Here, passing to a further subsequence, we find by (5) that

$$1/\psi(z) + B + C\,\psi(z) \equiv 0,$$

which is impossible.

(ii) $\{\beta_{n_k}/\alpha_{n_k}\}$ bounded, $\{\gamma_{n_k}/\alpha_{n_k}\}$ unbounded. Here there is a further subsequence through which $\alpha_n/\gamma_n \to 0$ and $\beta_n/\gamma_n \to 0$. Hence, by (5), $\psi(z) \equiv 0$.

(iii) $\{\beta_{n_k}/\alpha_{n_k}\}$ unbounded, $\{\gamma_{n_k}/\alpha_{n_k}\}$ bounded. Here, through some further subsequence, $\alpha_n/\beta_n \to 0$ and $\gamma_n/\beta_n \to 0$. Relation (5) thus leads to the conclusion $0 = 1$.

(iv) $\{\beta_{n_k}/\alpha_{n_k}\}$ and $\{\gamma_{n_k}/\alpha_{n_k}\}$ both unbounded. We may assume, after passing to a further subsequence, that

$$\beta_{n_k}/\alpha_{n_k} \to \infty, \qquad \gamma_{n_k}/\alpha_{n_k} \to \infty, \qquad \beta_{n_k}/\gamma_{n_k} \to B \quad (B \text{ finite or infinite}).$$

If B is finite, (5) shows $B + \psi(z) \equiv 0$. If $B = \infty$, one concludes that $0 = 1$.

We have proved that the sequence $\{\alpha_n\}$ must be bounded. A similar argument shows the sequence $\{\gamma_n\}$ is also bounded. Finally, if there were a subsequence $\{\beta_{n_k}\}$ tending to infinity, (5) would imply $0 = 1$. Hence $\{\beta_n\}$ is bounded, and the proof is complete.

It seems likely that the theorem remains true for an arbitrary rectifiable Jordan curve. However, the validity of formula (4) in this more general situation appears to be an open question. For present purposes it would suffice to prove (4) for large z.

Our result bears on the theory of the invariant subspaces of tridiagonal operators, as presented in [1]. It shows that the class of "regular" tridiagonal operators is not so general as it might appear. This has no effect, however, on the discussion of the invariant subspaces of the multiplication operator in $H_2(D)$ [1, p. 244].

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