ISOPERIMETRIC INEQUALITIES FOR RELATED CONFORMAL MAPS

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1. INTRODUCTION

Let

$$(1) \qquad w = f(z) = f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic for $|z| < 1$. We shall say that

$$(2) \qquad \zeta = g(z) = z\frac{df}{dz} = zf'(z)$$

is the *star relative* of (1).

The above choice of terminology is based on the well-known fact that if the map given by (1) is univalent, or *schlicht*, then this map is convex if and only if the map given by (2) is starshaped with respect to the origin [4, pp. 357, 359]. This geometric result follows directly from the fact that $\partial f/\partial\theta = ig$.

Certain deformation theorems, ordinarily expressed in terms of $f(z)$ and $f'(z)$, admit interesting geometric interpretations that compare the maps given by (1) and (2). Thus if $f(z)$ is normalized by $a_0 = 0$, $a_1 = 1$, and if the map (1) is univalent for $|z| < 1$, then [3, pp. 4, 5]

$$(3) \qquad \frac{1-r}{1+r} \le \left|\frac{g(re^{i\theta})}{f(re^{i\theta})}\right| \le \frac{1+r}{1-r} \qquad (0 < r < 1).$$

If, further, the map (1) is convex, then [3, p. 13]

$$(4) \qquad \frac{1}{1+r} \le \left|\frac{g(re^{i\theta})}{f(re^{i\theta})}\right| \le \frac{1}{1-r} \qquad (0 < r < 1).$$

It therefore seems natural to investigate for their own sake relationships between an arbitrary analytic $f(z)$ and its star relative $g(z)$. In this note, we shall consider isoperimetric properties comparing $f(z)$, $g(z)$, and other related functions.

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2. THE ISOPERIMETRIC INEQUALITY REVERSED

Between the length

$$L(r; f) = \int_0^{2\pi} |f'(re^{i\theta})| \, r \, d\theta$$

of the image of $|z| = r$ and the area

$$A(r; f) = \int_0^{2\pi} \int_0^{r} |f'(\rho e^{i\theta})|^2 \rho \, d\rho \, d\theta$$

of the image of $|z| \le r$ $(0 < r < 1)$ under the transformation (1), the isoperimetric-inequality relationship [2] is

$$A(r; f) \le \frac{1}{2\pi} [L(r; f)]^2. \tag{5}$$

Similarly, for the transformation (2), if $L(r; g)$ denotes the length of the image of $|z| = r$, and $A(r; g)$ denotes the area of the image of $|z| \le r$ $(0 < r < 1)$, then

$$A(r; g) \le \frac{1}{2\pi} [L(r; g)]^2. \tag{6}$$

We shall show that the same isoperimetric inequality holds between $L(r; f)$ and $A(r; g)$, but this time with the sense reversed:

THEOREM 1. *If the function* $f(z)$ *is analytic for* $|z| < 1$ *and* $g(z)$ *is the star relative of* $f(z)$, *and if* $L(r; f)$ *is the length of the image of* $|z| = r$ *under the transformation* $w = f(z)$ *and* $A(r; g)$ *is the area of the image of* $|z| \le r$ *under the transformation* $\zeta = g(z)$, *then*

$$\frac{1}{2\pi} [L(r; f)]^2 \le A(r; g) \qquad (0 < r < 1). \tag{7}$$

The sign of equality holds in (7) *if and only if* $f(z)$ *is of the form*

$$g(z) = a_0 + a_1 z. \tag{8}$$

Proof. By the Schwarz inequality,

$$\frac{1}{4\pi} [L(r; f)]^2 = \frac{1}{4\pi} \left[\int_0^{2\pi} |f'(re^{i\theta})| \, r \, d\theta \right]^2 \le \pi r^2 \left[\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \right]. \tag{9}$$

The sign of equality holds if and only if $|f'(z)|$ is a constant on $|z| = r$, that is, if and only if $f(z)$ is of the form $f(z) = a_0 + a_k z^k$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{10}$$

Then

$$g(z) = z f'(z) = \sum_{n=0}^{\infty} n a_n z^n . \tag{11}$$

By Parseval's identity,

$$\pi r^2 \left[\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \right] = \pi \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}, \tag{12}$$

and similarly

$$A(r; g) = \int_0^r \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho \, d\rho \, d\theta = \pi \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n} . \tag{13}$$

Since $n^2 < n^3$ for $n > 1$,

$$\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n} \leq \pi \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n}, \tag{14}$$

the sign of equality holding if and only if $f(z)$ is of the form (8).

Now (7) follows from (9), (12), (13), and (14), the sign of equality holding if and only if $f(z)$ is of the form (8).

3. HADAMARD COMPOSITION AND HADAMARD MEANS

The *Hadamard composition*, or *Hadamard product* [4, p. 82], of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \phi(z) = \sum_{n=0}^{\infty} b_n z^n$$

is the function

$$f * \phi(z) = \sum_{n=0}^{\infty} a_n b_n z^n .$$

In particular, the Hadamard composition of the function (10) and its star relative (11) is

$$f * g(z) = \sum_{n=0}^{\infty} n a_n^2 z^n .$$

For nonnegative real numbers a and b, the nonnegative number c is the geometric mean of a and b provided $c^2 = ab$. By analogy, we might say that

$$\psi(z) = \sum_{n=0}^{\infty} c_n z^n$$

is an *Hadamard mean* of f(z) and $\phi(z)$ provided

(15) $$|c_n|^2 = |a_n b_n| \qquad (n = 0, 1, 2, \cdots).$$

Thus the Hadamard means of the function (10) and its star relative (11) are the functions

(16) $$h(z) = \sum_{n=0}^{\infty} \sqrt{n}\, a_n e^{i\theta_n} z^n \qquad (\theta_n \text{ arbitrary, real}).$$

One might ask, for example, whether or not there are analogues of the inequalities (3) and (4) with g(z) replaced by h(z), and whether or not, if f(z) is univalent and convex, there necessarily exists a univalent, starshaped Hadamard mean of f(z) and its star relative g(z). Similar questions might be asked if, in (16), $\sqrt{n}$ is replaced by some other mean-value function of 1 and n (see [5]).

4. ISOPERIMETRIC INEQUALITIES AND HADAMARD MEANS

If

(17) $$s = h(z)$$

is an Hadamard mean of f(z) and its star relative g(z) in $|z| < 1$, then between the length L(r; h) of the image of $|z| = r$ and the area A(r; h) of the image of $|z| \leq r$ $(0 < r < 1)$ under the transformation (17), the isoperimetric-inequality relationship is

(18) $$A(r; h) \leq \frac{1}{4\pi}[L(r; h)]^2.$$

We shall now extend the result of Theorem 1 in terms of A(r; h) and L(r; h), as follows:

THEOREM 2. *Let* f(z) *be analytic for* $|z| < 1$, *denote by* g(z) *the star relative of* f(z), *and let* h(z) *be an Hadamard mean of* f(z) *and* g(z). *For any* $\phi(z)$, *analytic in* $|z| < 1$, *let* $L(r; \phi)$ *denote the length of the image of* $|z| = r$, *and* $A(r; \phi)$ *the area of the image of* $|z| \leq r$ $(0 < r < 1)$, *under the transformation* $w = \phi(z)$. *Then*

(19) $$\frac{1}{4\pi}[L(r; f)]^2 \leq A(r; h)$$

and

(20) $$\frac{1}{4\pi}[L(r; h)]^2 \leq A(r; g).$$

For each of the inequalities (19) *and* (20), *the sign of equality holds if and only if* f(z) *is of the form*

$$f(z) = a_0 + a_k z^k. \tag{21}$$

Proof. Let f(z), g(z), and h(z) be given by (10), (11), and (16), respectively. By (9) and (12),

$$\frac{1}{4\pi}[L(r;\, f)]^2 \leq \pi \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}, \tag{22}$$

with equality if and only if f(z) is of the form (21). Also, by (13),

$$A(r;\, g) = \pi \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n}. \tag{23}$$

Similar computations yield

$$\frac{1}{4\pi}[L(r;\, h)]^2 \leq \pi \sum_{n=1}^{\infty} n^3 |a_n|^2 r^{2n}, \tag{24}$$

with equality if and only if f(z) is of the form (21), and

$$A(r;\, h) = \pi \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}. \tag{25}$$

Now (19) follows from (22) and (25), and (20) follows from (23) and (24). In each case, the sign of equality holds if and only if f(z) is of the form (21).

It might be noted that, in addition to holding for Hadamard means, (19) holds more generally for all mean-value functions of the form

$$\sum_{n=0}^{\infty} \gamma_n z^n, \quad \text{where } |\gamma_n| = n^{p_n} |a_n| \text{ and } 0 \leq p_n \leq \tfrac{1}{2} \ (n = 1, 2, \cdots), \tag{26}$$

and (20) holds more generally for all mean-value functions of the form

$$\sum_{n=0}^{\infty} \delta_n z^n, \quad \text{where } |\delta_n| = n^{q_n} |a_n| \text{ and } \tfrac{1}{2} \leq q_n \leq 1 \ (n = 1, 2, \cdots). \tag{27}$$

The set of Hadamard means, for which both inequalities hold, is the intersection of the sets (26) and (27).

The inequalities (5), (19), (18), (20), and (6) combine to yield

$$A(r;\, f) \leq \frac{1}{4\pi}[L(r;\, f)]^2 \leq A(r;\, h) \leq \frac{1}{4\pi}[L(r;\, h)]^2 \leq A(r;\, g) \leq \frac{1}{4\pi}[L(r;\, g)]^2, \tag{28}$$

the sign of equality holding throughout if and only if f(z) is of the form (8). The sequence of inequalities (28) can be continued indefinitely to the right and to the left by considering, respectively, the star relative of h(z) and the function of which h(z) is the star relative, and so forth.

5. IMPLICATIONS

If the function f(z), analytic for $|z| < 1$, is continuous for $|z| \leq 1$, then clearly $A(r; f)$ is a nondecreasing function of r for $r \leq 1$. It is also true, by the Bieberbach-Csillag theorem [1], that $L(r; f)$ is a nondecreasing function of r for $r \leq 1$. Further, if f(z) is analytic for $|z| < 1$, and $L(r; f) \leq L < \infty$ for $r < 1$, then f(z) can be defined [6] on $|z| = 1$ in such a way as to be continuous for $|z| \leq 1$ and of bounded variation on $|z| = 1$.

It follows from the foregoing results that if, for example, the function (11) is analytic for $|z| < 1$, and $L(r; g) \leq L < \infty$ for $r < 1$, then the functions (10), (11), (16), (26), and (27) can be defined on $|z| = 1$ in such a way as to be continuous for $|z| \leq 1$, and consequently for them the area function $A(r; \phi)$ and the length function $L(r; \phi)$ are finite for $r \leq 1$.

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