ON NONLINEAR PROJECTIONS IN BANACH SPACES

Joram Lindenstrauss

1. INTRODUCTION

Let Y be a Banach space, and let X be a closed linear subspace of Y. By a *projection* from Y onto X we mean a mapping P from Y onto X such that $Px = x$ for all $x \in X$. Being Banach spaces, Y and X have in particular a linear structure, a topology, and a uniformity assigned to them. It is therefore natural to consider many types of projections between Banach spaces—linear, continuous, bounded linear, uniformly continuous, and so forth. It is well known that a linear projection from Y onto X always exists. It is also known that a (norm-) continuous projection from Y onto X always exists (see Michael [21] for more general results and references to earlier papers). Another well-known result is that a projection which is both linear and continuous (that is, bounded linear) does not in general exist. In Section 2 we shall mention some of the known results in this direction.

In the present note we intend to study the question of the existence of projections that lie "between" the continuous and the bounded linear projections. We shall be interested in uniformly continuous projections generally, and particularly in those that satisfy a Lipschitz condition. (Unless it is stated otherwise, the metric—and hence the topology and uniformity—of a Banach space will be those determined by its norm.)

Section 2 contains the notation and necessary background material. In Section 3 we prove first some theorems which state that in many situations the existence of a uniformly continuous projection from Y onto X (where X is a closed linear subspace of the Banach space Y) implies the existence of a bounded linear projection. This is for example the case if X is a conjugate Banach space (that is, $X = Z^*$ for some Z). Combining these results with known results concerning the nonexistence of bounded linear projections, we get many examples of Banach spaces $Y \supset X$ such that there exists no uniformly continuous projection from Y onto X. Dual results concerning uniformly continuous liftings are obtained. The rest of Section 3 is devoted to the study of some projection constants of metric spaces, particularly of Banach spaces.

In Section 4 we consider the existence of uniformly continuous projections onto some special classes of spaces—C(K)-spaces and uniformly convex spaces. These questions have already been considered by Isbell [14]. We obtain stronger versions of his theorems; for L_p-spaces our results are in a sense the best possible. The study of the L_p-spaces enables us to give an example of a space such that neither it nor its unit cell are absolute neighborhood retracts uniformly (in the terminology of Isbell). This solves a problem raised in [14].

In Section 5 we apply the results of Sections 3 and 4 to the problem of the existence of uniform homeomorphisms between certain Banach spaces or their respective unit cells. The question of the topological classification of Banach spaces and their convex sets has in recent years been discussed by several authors. One of the outstanding problems in this direction is the question, raised by Fréchet and Banach,

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whether every two separable infinite-dimensional Banach spaces are homeomorphic. Many recently obtained strong results suggest that the answer to this problem is in the affirmative. For example, it has been proved that every separable Banach space having a closed infinite-dimensional linear subspace with an unconditional basis is homeomorphic to ℓ_2. This implies that all known examples of separable infinite-dimensional Banach spaces are homeomorphic to ℓ_2. For further results concerning this and related problems we refer the reader to Bessaga and Pełczyński [4], Corson and Klee [6], and a forthcoming paper of Bessaga [3], which has an extensive list of references.

In [3] and [4] the authors ask whether every two homeomorphic infinite-dimensional Banach spaces are also uniformly homeomorphic. Our results concerning projections imply that the answer is negative. We show for example that no infinite-dimensional C(K)-space is uniformly homeomorphic to a reflexive Banach space (see Theorem 11 for more details). Concerning the classification of bounded convex sets we give examples of two separable Banach spaces whose unit cells are homeomorphic but not uniformly homeomorphic, and also of two separable Banach spaces whose unit cells are uniformly homeomorphic but not Lipschitz-equivalent. (Two metric spaces are called Lipschitz-equivalent if there exists a one-to-one mapping T from one space onto the second such that T and T^{-1} satisfy a Lipschitz condition.)

Our next result gives a partial negative answer to the question whether $C[0, 1]$ is uniformly homeomorphic to a subset of ℓ_2. Finally we prove that the space c_0 has an uncountable number of mutually nonisomorphic (even not uniformly homeomorphic) closed linear subspaces.

Statements of some open problems occur throughout the paper.

2. NOTATION AND PRELIMINARIES

All Banach spaces are assumed to be over the reals. This is only a matter of convenience, and all our results hold also in the complex case, except that Theorems 6 and 7 require slight modifications. Let K be a topological space. By C(K) we denote the space of all bounded, real-valued continuous functions on K with the sup norm. For every set I, m(I) will denote the space of all bounded real-valued functions on I with the sup norm. For $1 \leq p \leq \infty$ and $n = 1, 2, \cdots$, we denote by $\ell_{p,n}$ the Banach space of all n-tuples $x = (x_1, x_2, \cdots, x_n)$ of real numbers with

$$\|x\| = (\sum |x_i|^p)^{1/p}$$

if $p < \infty$ and $\|x\| = \max |x_i|$ if $p = \infty$. The usual meaning is attached to c_0, ℓ_p, m, and $L_p(\mu)$, where μ is a measure on some measure space. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of Banach spaces, and let $p = 0$ or $1 \leq p \leq \infty$. The Banach space

$$X = (X_1 \oplus X_2 \oplus \cdots \oplus X_k \oplus \cdots)_p$$

is the space of all sequences

$$x = (x_1, x_2, \cdots, x_k, \cdots)$$

with $x_k \in X_k$ and with

$$(\|x_1\|, \|x_2\|, \cdots, \|x_k\|, \cdots) \in \begin{cases} \ell_p & \text{if } 1 \leq p \leq \infty, \\ c_0 & \text{if } p = 0. \end{cases}$$

$\|x\|$ is the norm of the latter sequence in ℓ_p (or in c_0 if $p = 0$).

The *cell* $S_X(x_0, r)$ in a metric space X is the set $\{x; d(x, x_0) \leq r\}$. The *distance* between two sets B and C in a metric space X is denoted by d(B, C). The *Hausdorff distance* between B and C (that is, the least upper bound of the distance of a point in one of these sets to the other set) is denoted by H(B, C).

Let T be a map from a metric space X into a metric space Y. The function

$$\phi(\varepsilon) = \sup_{d(x_1, x_2) \leq \varepsilon} d(Tx_1, Tx_2) \qquad (\varepsilon > 0),$$

is called the *modulus of continuity* of T. Clearly, T is uniformly continuous if and only if $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$. If X is a convex subset of a Banach space, then the modulus of continuity of every map T defined on X is subadditive, that is,

$$\phi(\varepsilon_1 + \varepsilon_2) \leq \phi(\varepsilon_1) + \phi(\varepsilon_2).$$

A special kind of uniformly continuous map is the class of Lipschitz maps, that is, the class of maps with $\|T\| < \infty$, where

$$\|T\| = \sup_{x_1 \neq x_2} d(Tx_1, Tx_2)/d(x_1, x_2). \tag{2.1}$$

We shall often make use of the following lemma, which states that a uniformly continuous map defined on a convex set "has the Lipschitz property for large distances." The lemma follows immediately from the subaddivity of the modulus of continuity, and it is a simple special case of a result of Corson and Klee [6, p. 48].

LEMMA 1. *Let* X *be a convex subset of a Banach space, and let* T *be a uniformly continuous map from* X *into a metric space* Y. *Then for every* $\eta > 0$ *there exists a* $\lambda < \infty$ *such that* $d(Tx_1, Tx_2) \leq \lambda d(x_1, x_2)$ *if* $d(x_1, x_2) \geq \eta$.

Let X be a Banach space. Along with its usual conjugate space X^* we shall consider also its Lipschitz conjugate $X^\#$, the space of all Lipschitz maps from X into the reals that vanish at the origin of X. The norm in $X^\#$ is that given by (2.1), and the linear operations are defined as usual for function spaces. It is easy to verify that $X^\#$ is a Banach space. X^* is a closed linear subspace of $X^\#$. An analogue to the Hahn-Banach Theorem holds for Lipschitz maps (see [7, p. 154] for references): *A Lipschitz map from a subset of a metric space* Y *to the reals can be extended to the whole of* Y *without increase of its norm.* This extension result is a special case of the following lemma due to Aronszajn and Panitchpakdi [2].

LEMMA 2. *Let* Y *be a metric space, and* X *a subset of* Y. *Let* T *be a map from* X *into* m(I) *for some set* I. *Suppose that the modulus of continuity of* T *does not exceed a given subadditive function* $\phi(\varepsilon)$. *Then* T *can be extended to a map from* Y *into* m(I) *having a modulus of continuity that does not exceed* $\phi(\varepsilon)$.

This lemma, combined with the well-known fact that any metric space X can be embedded isometrically in C(X) and hence in m(X), has many useful consequences. For example, if X admits a uniformly continuous projection from every metric space Y containing it, then there exists a function $\phi(\varepsilon)$ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that from every $Y \supset X$ there is a projection onto X with a modulus of continuity not exceeding $\phi(\varepsilon)$. Indeed, take any (isometric) embedding of X in a space $m(I)$, and let $\phi(\varepsilon)$ be the modulus of continuity of any uniformly continuous projection from $m(I)$ onto X.

Let X be a Banach space. We denote by $H(X)$ the space of all nonempty bounded closed convex subsets of X, metrized by Hausdorff distance. In the consideration of projections onto a Banach space X, it is sometimes more convenient to use the natural embedding of X in $H(X)$ instead of an imbedding in some $m(I)$. This was observed by Isbell, to whom the following useful lemma is due [14, Lemma 3.2].

LEMMA 3. *Let* X *be a Banach space, and let* Y *be a metric space containing* $H(X)$ *as a subspace. Then there exists a projection* P *from* Y *onto* $H(X)$ *with* $\|P\| \leq \eta_0$, *where* η_0 *is an absolute constant* (*that is, a constant independent of* X *and* Y).

In [14], Isbell asserted only the existence of a uniformly continuous projection from Y onto $H(X)$. However, simple computations show that the projection which he constructed is a Lipschitz projection of norm less than 12. The following is an easy consequence of Lemmas 2 and 3. Let $Z_2 \supset Z_1$ be metric spaces, and let T be a map from Z_1 into $H(X)$ (for some Banach space X) with a modulus of continuity not exceeding $\phi(\varepsilon)$, where ϕ is a subadditive function. Then there exists an extension of T from Z_2 into $H(X)$ with a modulus of continuity not exceeding $\eta_0\,\phi(\varepsilon)$.

A Banach space X is called a $\mathfrak{P}_\lambda$-space if from every Banach space Y containing X as a linear subspace there exists a linear projection from Y onto X with norm at most λ. A Banach space is called a $\mathfrak{P}$-space if it is a $\mathfrak{P}_\lambda$-space for some finite λ. If X is a $\mathfrak{P}_\lambda$-space, then X^{**} is a $\mathfrak{P}_\lambda$-space, but not conversely. Finite-dimensional spaces are $\mathfrak{P}$-spaces. An infinite-dimensional $\mathfrak{P}$ space cannot be separable, and it must have a linear subspace isomorphic to c_0 (thus it cannot be reflexive or w-sequentially complete). For proof of these and related results and for further references see [8, pp. 94-96], [18], and [22].

Let X be a metric space, and let n be an integer. By $P_n(X)$ we denote the greatest lower bound of the numbers λ with the following property. From every metric space $Y \supset X$, with $Y \sim X$ consisting of at most n points, there exists a projection onto X with norm at most λ. If no such λ exists, we put $P_n(X) = \infty$. $P_1(X)$ (and hence $P_n(X)$ for $n > 1$) is infinite if X is not complete, while for complete X, $P_1(X) \leq 2$ and hence $P_n(X) \leq 2^n$ (see Grünbaum [11]). For Banach spaces X, $P_1(X) = 1$ if and only if X is a $\mathfrak{P}_1$-space, and this is the case if and only if $X = C(K)$, where K is a compact Hausdorff space which is extremally disconnected (that is, in which the closure of an open set is again open). This is a result due to Nachbin, Goodner, and Kelley (see [8, p. 95]).

Some final remarks concerning our terminology: When we speak of a Banach space X as a subspace of X^{**}, we shall assume, unless it is stated otherwise, that X is canonically embedded in X^{**}. Similarly for inclusions like $X^{\#} \supset X^*$, $H(X) \supset X$, $m(K) \supset C(K)$, and $Y^* \supset X^*$ (or $Y^{\#} \supset X^{\#}$) in case X is a quotient Banach space of Y. If Y and X are metric spaces and we write that X is embedded in Y or $Y \supset X$, we always mean that the embedding of X in Y is an isometry. Mappings between Banach spaces will not be assumed to be linear, unless this is stated explicitly. However, the word "isomorphism" will mean "linear isomorphism."

3. GENERAL RESULTS

We prove first that under certain circumstances, the existence of a uniformly continuous projection implies the existence of a Lipschitz projection.

THEOREM 1. *Let* Y *be a Banach space, and let* X *be a closed linear subspace of* Y. *Suppose that there exists a uniformly continuous projection* P *from* Y *onto* X *and a Lipschitz projection* Q *from* X^{**} *onto* X. *Then there exists a Lipschitz projection from* Y *onto* X.

Proof. By Lemma 1 there exists a number $\lambda < \infty$ such that

$$\|Py_1 - Py_2\| \leq \lambda \|y_1 - y_2\|$$

if $\|y_1 - y_2\| \geq 1$. Consider the sequence of projections $P_n(y) = P(ny)/n$ from Y onto X as mappings from Y into X^{**}. For every fixed y the sequence $\{P_n(y)\}_{n=1}^{\infty}$ is bounded (indeed, $\|P_n(y)\| \leq \lambda \|y\|$ for $n > \|y\|^{-1}$). Since the cells in X^{**} are w*-compact, it follows from Tychonoff's Theorem that the sequence $\{P_n\}$ has a limit point P_0 in the topology of pointwise convergence (taking in X^{**} the w*-topology). $P_n x = x$ for every integer n and every $x \in X$, and hence $P_0 x = x$ for $x \in X$. Further, $\|P_n y_1 - P_n y_2\| \leq \lambda \|y_1 - y_2\|$ for $n > \|y_1 - y_2\|^{-1}$, consequently $\|P_0\| \leq \lambda$. It follows that the mapping QP_0 from Y into X is a Lipschitz projection from Y onto X.

Remark. It is well known (see Dixmier [9]) that if X is a conjugate space or an L_1-space, then there always exists a linear projection of norm 1 from X^{**} onto X. In the next section we shall show (see Theorem 6) that there may exist a Lipschitz projection from X^{**} onto X even if there exists no bounded linear projection ($X = c_0$ is a typical example). We do not know whether there exists a Banach space X for which there exists no Lipschitz projection from X^{**} onto X.

THEOREM 2. *Let* Y *be a Banach space, and let* X *be a closed linear subspace of* Y. *Then there exist linear projections of norm* 1, P_Y *from* $Y^{\#}$ *onto* Y^* *and* P_X *from* $X^{\#}$ *onto* X^*, *such that* $P_X R_1 = R_2 P_Y$, *where* R_1 [*respectively* R_2] *is the natural restriction map from* $Y^{\#}$ *onto* $X^{\#}$ [*respectively* Y^* *onto* X^*].

Proof. We shall first assume that Y is finite-dimensional and construct a linear projection from $Y^{\#}$ onto Y^*. Let $\{e_i\}_{i=1}^m$ be a basis of Y, and let $\psi(y)$ be a non-negative C^1 function on Y having a compact support and satisfying the condition $\int_Y \psi(y)\, dy = 1$ (here $y = \Sigma y_i e_i$, $dy = dy_1 \cdots dy_m$, and C^1 denotes the class of the continuously differentiable functions). We define a projection P from $Y^{\#}$ onto Y^* as follows:

$$PF\left(\sum_{i=1}^{m} \alpha_i e_i\right) = -\sum_{i=1}^{m} \alpha_i \int_Y F(y) \frac{\partial \psi(y)}{\partial y_i} dy \qquad (F \in Y^{\#}). \tag{3.1}$$

Clearly P is a linear mapping from $Y^{\#}$ into Y^*. We shall show that $PF = F$ for $F \in Y^*$ and $\|PF\| \leq \|F\|$ for every $F \in Y^{\#}$. To this end we assume first that $F \in C^1$. In this case (since ψ has a compact support)

$$-\int_Y F(y) \frac{\partial \psi(y)}{\partial y_i} dy = \int_Y \frac{\partial F(y)}{\partial y_i} \psi(y)\, dy \qquad (i = 1, \cdots, m).$$

That $PF = F$ for $F \in Y^*$ is now obvious. As for the norm of PF, let $\|\Sigma\, \alpha_i e_i\| = 1$ and $\lambda > 0$. Then

$$
\begin{aligned}
PF\left(\sum \alpha_i e_i\right) &= \lambda^{-1} \int_Y \psi(y) \left(\sum \lambda\alpha_i \frac{\partial F(y)}{\partial y_i}\right) dy \\
&= \lambda^{-1} \int_Y \psi(y)\,(F(y + \sum \lambda\alpha_i e_i) - F(y) + \lambda\theta(\lambda, y))dy,
\end{aligned}
\tag{3.2}
$$

where $\theta(\lambda, y)$ tends to 0 uniformly on the support of ψ as $\lambda \to 0$. Since

$$\| F(y + \sum \lambda\, \alpha_i e_i) - F(y)\| \leq \lambda \| F \|$$

and $PF(\Sigma\, \alpha_i\, e_i)$ does not depend on λ, we see from (3.2) that

$$\left| PF\left(\sum \alpha_i\, e_i\right)\right| \leq \| F \| \int_Y \psi(y)\, dy = \| F \|$$

and hence $\|PF\| \leq \|F\|$.

Let now F be a general element in $Y^{\#}$ (not necessarily in C^1). Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of nonnegative C^1 functions on Y such that $\int_Y \phi_n(y)\, dy = 1$ and the support of $\phi_n(y)$ is compact and shrinks to the origin as $n \to \infty$. Put $F_n = F * \phi_n$ (where $*$ denotes the usual convolution). Then $F_n(y) \to F(y)$ uniformly on Y as $n \to \infty$, $F_n \in Y^{\#} \cap C^1$, and $\|F_n\| \leq \|F\|$. It follows by (3.1) that $\|PF_n - PF\| \to 0$, and since we have already shown that $\|PF_n\| \leq \|F_n\|$, we conclude that $\|PF\| \leq \|F\|$ and hence $\|P\| = 1$.

Let now X be a linear subspace of Y (Y is still assumed to be finite-dimensional). We may assume that the basis in Y is chosen so that $\{e_i\}_{i=1}^{k}$ is a basis of X (for some $k < m$). We choose two nonnegative C^1 functions ψ_1 and ψ_2 of k, and $m - k$ variables respectively, each with compact support and with its integral over the respective space equal to 1. For every integer n, let P_n be the linear projection of norm 1 from $Y^{\#}$ onto Y^*, defined (for $F \in Y^{\#}$ and $i = 1, \cdots, m$) by

$$P_n F(e_i) = -n^{m-k} \int_Y F(y) \frac{\partial}{\partial y_i} [\psi_1(y_1, \cdots, y_k)\, \psi_2(ny_{k+1}, \cdots, ny_m)]\, dy .$$

Since $\|P_n\| \leq 1$ and Y^* is finite-dimensional, it follows that the sequence $\{P_n\}_{n=1}^{\infty}$ has a limiting point P_Y in the strong operator topology for linear operators from $Y^{\#}$ to Y^* (this assertion is an immediate consequence of the w*-compactness of the unit cell of $(Y^{\#})^*$). Clearly, P_Y is a linear projection of norm 1 from $Y^{\#}$ onto Y^*. For $i \leq k$, it follows from the uniform continuity of F that

$$
\begin{aligned}
P_Y F(e_i) &= \lim_{n \to \infty} P_n F(e_i) \\
&= -\int_X F(y_1, y_2, \cdots, y_k, 0, \cdots, 0) \frac{\partial}{\partial y_i} \psi_1(y_1, \cdots, y_k)\, dy_1 \cdots dy_k .
\end{aligned}
$$

Hence, if we define P_X from $X^{\#}$ onto X^* by

$$P_X G(e_i) = -\int_X G(x) \frac{\partial \psi_1(x)}{\partial x_i} dx \qquad (i = 1, \cdots, k;\ G \in X^{\#}),$$

then $R_2 P_Y = P_X R_1$, where the R_i are the restriction operators appearing in the statement of the theorem. This concludes the proof of the theorem for the case where Y is finite-dimensional.

Let now Y be an infinite-dimensional Banach space, and let X be a closed linear subspace of Y. Let B be a finite-dimensional subspace of Y, and put $C = B \cap X$. Denote by $R_{1,B}$ [respectively, $R_{2,B}$] the restriction map from $B^\#$ onto $C^\#$ [respectively, from B^* onto C^*]. By what we have already proved, there exist linear projections of norm 1, P_B from $B^\#$ onto B^* and P_C^B from $C^\#$ onto C^*, such that

$$R_{2,B} P_B = P_C^B R_{1,B}.$$

(We denoted the projection from $C^\#$ onto C^* by P_C^B rather than P_C, since $P_C^{B_1}$ may be different from $P_C^{B_2}$ if $C = B_1 \cap X = B_2 \cap X$.) Further, denote by R_1^B [respectively R_1^C] the restriction maps from $Y^\#$ onto $B^\#$ [respectively, from $X^\#$ onto $C^\#$]. We now assign to B a function f_B from $Y^\# \times Y$ into the reals and a function g_B from $X^\# \times X$ into the reals by putting

$$f_B(F, y) = \begin{cases} 0 & \text{if } y \notin B, \\ P_B R_1^B F(y) & \text{if } y \in B, \end{cases} \qquad (F \in Y^\#,\ y \in Y),$$

$$g_B(G, X) = \begin{cases} 0 & \text{if } x \notin B, \\ P_C^B R_1^C G(x) & \text{if } x \in B, \end{cases} \qquad (G \in X^\#,\ x \in X).$$

For every $F \in Y^\#$, $y \in Y$, and every B, $|f_B(F, y)| \leq \|F\|\,\|y\|$, and similarly $|g_B(G, x)| \leq \|G\|\,\|x\|$ for all G, x and B. We order the finite-dimensional linear subspaces of Y be inclusion, and get thus a directed set τ. By Tychonoff's Theorem there exist subnets

$$\{f_{B'}\}_{B' \in \tau'} \text{ of } \{f_B\}_{B \in \tau} \quad \text{and} \quad \{g_{B'}\}_{B' \in \tau'} \text{ of } \{g_B\}_{B \in \tau}$$

(where τ' is a certain directed set) that converge pointwise to mappings f and g from $Y^\# \times Y$, respectively $X^\# \times X$, into the reals. $|f(F, y)| \leq \|F\|\,\|y\|$ for all F and y, and f is a bilinear map. Indeed, $f_B(F, y)$ is linear in F for every B, and

$$\alpha f_B(F, y_1) + \beta f_B(F, y_2) = f_B(F, \alpha y_1 + \beta y_2)$$

for all B that contain y_1 and y_2. Hence $f(F, y) = P_Y F(y)$, where P_Y is a linear operator of norm at most 1 from $Y^\#$ into Y^*. Since $f_B(F, y) = F(y)$ for $F \in Y^*$ and B containing y, it follows that $f(F, y) = F(y)$ for all $y \in Y$ and $F \in Y^*$. Therefore P_Y is a projection from $Y^\#$ onto Y^*. Similarly there exists a linear projection P_X from $X^\#$ onto X^* such that $g(G, x) = P_X G(x)$. Let $F \in Y^\#$ and $x \in X \cap B = C$; then

$$f_B(F, x) = R_{2,B} P_B R_1^B F(x) = P_C^B R_{1,B} R_1^B F(x) = P_C^B R_{1,C} R_1 F(x) = g_B(R_1 F, x)$$

($R_{1,B} R_1^B = R_{1,C} R_1$ since each is the restriction map from $Y^\#$ onto $C^\#$). Hence,

$$R_2 P_Y F(x) = f(F, x) = g(R_1 F, x) = P_X R_1 F(x) \qquad (F \in Y^\#,\ x \in X).$$

We have thus shown that $R_2 P_Y = P_X R_1$, and this concludes the proof of the theorem.

Our next theorem is an easy consequence of Theorem 2.

THEOREM 3. (a) *Let* X *be a closed linear subspace of the Banach space* Y. *Suppose there exists a Lipschitz projection with norm* λ *from* Y *onto* X. *Then there exists a linear operator* T *of norm at most* λ *from* X^* *into* Y^* *such that* $R_2 T$ *is the identity map of* X^* (R_2 *denotes the restriction map from* Y^* *onto* X^*).

(b) *Let* X *be a quotient Banach space of the Banach space* Y *and let* R *be the quotient map. Suppose there exists a Lipschitz map* T *of norm* λ *from* X *into* Y *such that* RT *is the identity of* X. *Then there exists a linear projection of norm at most* λ *from* Y^* *onto* X^*.

Proof of (a). Let P_Y and P_X satisfy the condition $R_2 P_Y = P_X R_1$ (the notation being that of Theorem 2). Let P be a projection from Y onto X with $\|P\| = \lambda$, and $P^\#$ be the linear mapping from $X^\#$ into $Y^\#$ defined by $P^\# G(y) = G(Py)$ $(G \in X^\#,\ y \in Y)$. Clearly $\|P^\#\| \leq \lambda$, and $R_1 P^\#$ is the identity of $X^\#$. Put $T = P_Y P^\#$. Then T is a linear map from $X^\#$ into Y^* with $\|T\| \leq \lambda$. For $x^* \in X^*$,

$$R_2 Tx^* = R_2 P_Y P^\# x^* = P_X R_1 P^\# x^* = P_X x^* = x^* .$$

Hence the restriction of T to X^* has the required properties.

Proof of (b). We may assume that $T(0) = 0$ (otherwise, replace T by $T - T(0)$). Define $T^\#$ from $Y^\#$ onto $X^\#$ by $T^\# F(x) = F(Tx)$ $(x \in X,\ F \in Y^\#)$. Then $T^\#$ is a linear projection of norm at most λ. We recall that we identify an element $G \in X^\#$ with the element $\hat{G}$ in $Y^\#$ defined by $\hat{G}(y) = G(Ry)$. Let P_X be a linear projection of norm 1 from $X^\#$ onto X^*. Then the restriction of $P_X T^\#$ to Y^* is a linear projection of norm at most λ from Y^* onto X^* (again, we identify $x^* \in X^*$ with the element $\hat{x}^*$ in Y^* determined by $\hat{x}^*(y) = x^*(Ry)$).

COROLLARY 1. *Let* Y *be a Banach space and let* X *be a closed linear subspace of* Y. *Suppose there exists a Lipschitz projection of norm* λ *from* Y *onto* X. *Then there exists a linear projection of norm at most* λ *from* Y^{**} *onto* X^{**}.

Proof. Let T be the operator from X^* into Y^* whose existence is shown in Theorem 3(a). Then T^* is a linear projection of norm at most λ from Y^{**} onto X^{**}.

In general we cannot assert that, under the assumptions of Corollary 1, there exists a bounded linear projection from Y onto X (see Theorem 6 below). We have however the following proposition.

COROLLARY 2. *Let* Y *be a Banach space, and let* X *be a closed linear subspace of* Y. *Suppose there exists a bounded linear projection from* X^{**} *onto* X *and a uniformly continuous projection from* Y *onto* X. *Then there exists a bounded linear projection from* Y *onto* X.

Proof. By Theorem 1 and the preceding corollary, there exists a bounded linear projection P from Y^{**} onto X^{**}. Let Q be a bounded linear projection from X^{**} onto X. The restriction of QP to Y has the required properties.

COROLLARY 3. *Let* X *be a quotient Banach space of the Banach space* Y. *Let* R *denote the quotient map and let* Z *be the kernel of* R. *Suppose there exist a bounded linear projection from* Z^{**} *onto* Z *and a uniformly continuous map* T *from* X *into* Y *such that* RT *is the identity of* X. *Then there also exists a bounded linear operator* T_0 *from* X *into* Y *such that* RT_0 *is the identity of* X.

Proof. Without loss of generality we may assume that $T(0) = 0$. $I - TR$ is a uniformly continuous projection from Y onto Z (I denotes the identity operator of Y).

By Corollary 2 there exists a bounded linear projection P from Y onto Z. Take as T_0 the inverse of the restriction of R to the subspace (I - P)Y of Y. This operator T_0 has the required properties.

COROLLARY 4. *Let* X *be a Banach space that is not a* $\mathfrak{P}$*-space and such that there exists a bounded linear projection from* X^{**} *onto* X. *Let* T *be a uniformly continuous map from* X *into a space* m(I), *for some* I, *that has a uniformly continuous inverse* T^{-1}. *Then there exists no uniformly continuous projection from* m(I) *onto* TX.

Observe that TX is not necessarily a linear subspace of m(I).

Proof. Use Lemma 2 and Corollary 2 of Theorem 3.

Similarly, Lemma 3 yields the following proposition.

COROLLARY 5. *Let* X *be a Banach space that is not a* $\mathfrak{P}$*-space and such that there exists a bounded linear projection from* X^{**} *onto* X. *Then there exists no uniformly continuous projection from* H(X) *onto* X.

We now pass to some results concerning projection constants. We shall need the following elementary lemma.

LEMMA 4. *Let* $\{S_\alpha\}_{\alpha \in A}$ *be a collection of closed intervals on the real line. Let* ρ *be a metric on* A *such that* $d(S_\alpha, S_\beta) \leq \rho(\alpha, \beta)$ *for all* $\alpha, \beta \in A$. *Then there exist* $y_\alpha \in S_\alpha$ *such that* $|y_\alpha - y_\beta| \leq \rho(\alpha, \beta)$ *for all* $\alpha, \beta \in A$.

Proof. For a finite set A, Lemma 4 is a special case of Lemma 5 (take a K consisting of a single point, in Lemma 5), which will be proved in the next section. The general case follows from that with a finite A by a simple compactness argument.

COROLLARY 1. *Let* X *and* A *be metric spaces with metrics* d *and* ρ, *respectively. For every* $\alpha \in A$, *let*

$$\{S_X(x_{\alpha,\beta}, r_{\alpha,\beta})\}_{\beta \in B_\alpha} \tag{3.3}$$

be a collection of cells in X (*the* B_α *are sets depending on* α). *Suppose that for all* $\alpha_1, \alpha_2 \in A$

$$d(x_{\alpha_1,\beta_2}, x_{\alpha_2,\beta_2}) \leq r_{\alpha_1,\beta_1} + \rho(\alpha_1, \alpha_2) + r_{\alpha_2,\beta_2} \qquad (\beta_1 \in B_{\alpha_1}, \ \beta_2 \in B_{\alpha_2}). \tag{3.4}$$

Then there exists a metric space Y *containing* X (*the metric in* Y *will also be denoted by* d) *and points* y_α *in* Y ($\alpha \in A$) *such that*

$$d(y_{\alpha_1}, y_{\alpha_2}) \leq \rho(\alpha_1, \alpha_2); \quad y_\alpha \in S_Y(x_{\alpha,\beta}, r_{\alpha,\beta}) \quad (\beta \in B_\alpha). \tag{3.5}$$

Proof. Embed X in m(I) for some I. If $y \in m(I)$ and $i \in I$, we shall call y(i) the i-th coordinate of y. Consider the intervals

$$S^i_{\alpha,\beta} = [x_{\alpha,\beta}(i) - r_{\alpha,\beta}, \ x_{\alpha,\beta}(i) + r_{\alpha,\beta}] \qquad (\alpha \in A, \ \beta \in B_\alpha, \ i \in I).$$

By (3.4), $S^i_{\alpha,\beta_1} \cap S^i_{\alpha,\beta_2} \neq \emptyset$ for every $\beta_1, \beta_2 \in B_\alpha$ and $i \in I$. Hence, by Helly's theorem,

$$S^i_\alpha = \bigcap_{\beta \in B_\alpha} S^i_{\alpha,\beta} \neq \emptyset .$$

By (3.4), we see that also $d(S^i_{\alpha_1}, S^i_{\alpha_2}) \leq \rho(\alpha_1, \alpha_2)$. By Lemma 4, there exist $y_\alpha(i) \in S^i_\alpha$ such that $|y_{\alpha_1}(i) - y_{\alpha_2}(i)| \leq \rho(\alpha_1, \alpha_2)$. Let $\alpha \in A$. The set $\{y_\alpha(i)\}_{i \in I}$ is bounded (by $\|x_{\alpha,\beta}\| + r_{\alpha,\beta}$ for any $\beta \in B_\alpha$), and hence the i-th coordinate of some point y_α in m(I) is $y_\alpha(i)$. The points $\{y_\alpha\}_{\alpha \in A}$ chosen in this manner satisfy (3.5). As Y we may take m(I) or, of course, the subset $X \cup \{y_\alpha\}_{\alpha \in A}$ of m(I) (with the metric induced on it by m(I)).

Our next aim is to give an intrinsic characterization of the projection constants $P_n(X)$ defined in Section 2. To this end we consider for each integer n and each metric space X the number $E_n(X)$ which is the g.l.b. of the numbers λ with the following property. For every set A of cardinality at most n and every collection of cells (3.3) for which (3.4) holds, there exist points x_α ($\alpha \in A$) in X such that

$$x_\alpha \in S(x_{\alpha,\beta}, \lambda r_{\alpha,\beta}) \quad (\alpha \in A,\ \beta \in B_\alpha), \tag{3.6}$$

$$d(x_{\alpha_1}, x_{\alpha_2}) \leq \lambda \rho(\alpha_1, \alpha_2) \quad (\alpha_1, \alpha_2 \in A). \tag{3.7}$$

(If no such λ exists, we put $E_n(X) = \infty$.)

We are now ready to prove the following theorem, which generalizes one of the main results of Grünbaum [11].

THEOREM 4. *For every metric space* X *and every integer* n, $E_n(x) = P_n(X)$.

Proof. If X is not complete, then $E_n(X) = P_n(X) = \infty$ for $n = 1$ and hence for every n (Grünbaum [11]). Hence we may assume that X is complete and consequently $P_n(X) < \infty$. Let $\lambda > P_n(X)$, let A consist of at most n points, and for every $\alpha \in A$ let a collection of cells (3.3) be given such that (3.4) holds (the sets B_α may be of any finite or infinite cardinality). Let $Y = X \cup \{y_\alpha\}_{\alpha \in A}$ be the metric space constructed in Corollary 1 of Lemma 4. The cardinality of $Y \sim X$ does not exceed n, and hence there exists a projection P of norm at most λ from Y onto X. The points $x_\alpha = Py_\alpha$ satisfy (3.6) and (3.7). This proves that $P_n(X) \geq E_n(X)$. Now let $\lambda > E_n(X)$, and let Y be a metric space containing X with $Y \sim X = \{y_\alpha\}_{\alpha \in A}$, where the cardinality of A does not exceed n. Put

$$\rho(\alpha_1, \alpha_2) = d(y_{\alpha_1}, y_{\alpha_2}) \quad (\alpha_1, \alpha_2 \in A).$$

For $\alpha \in A$, consider the collection of cells $\{S_X(x, d(x, y_\alpha))\}_{x \in X}$. Inequality (3.4) holds for these cells, since

$$d(x_1, x_2) \leq d(x_1, y_{\alpha_1}) + \rho(\alpha_1, \alpha_2) + d(x_2, y_{\alpha_2}).$$

Hence there are x_α in X ($\alpha \in A$), for which

$$d(x_{\alpha_1}, x_{\alpha_2}) \leq \lambda \rho(\alpha_1, \alpha_2), \quad d(x_\alpha, x) \leq \lambda d(y_\alpha, x).$$

The projection P defined by $Py_\alpha = x_\alpha$ ($\alpha \in A$)—and, of course, $Px = x$ ($x \in X$)—is of norm at most λ. Thus $P_n(X) \leq E_n(X)$, and this concludes the proof of the theorem.

REMARK. The constants $E_\mathfrak{m}(X)$ and $P_\mathfrak{m}(X)$ may be defined also for infinite cardinals $\mathfrak{m}$ in an obvious way. Theorem 4 and its proof generalize easily to this situation.

We now examine the behavior of $\lim_{n\to\infty} P_n(X)$ for Banach spaces X. Let X be a Banach space. The g.l.b. of the λ such that X is a $\mathfrak{P}_\lambda$-space is called the *projection constant* P(X) of X (if X is not a $\mathfrak{P}$-space, we put $P(X) = \infty$).

THEOREM 5. *Let* X *be a Banach space, and suppose that there exists a bounded linear projection of norm* λ_0 *from* X^{**} *onto* X. *Then*

$$P(X)/\lambda_0^2 \le \lim_{n\to\infty} P_n(X) \le P(X).$$

Proof. Embed X linearly (and isometrically) in some m(I). Suppose $P(X) < \infty$. For every $\lambda > P(X)$ there exists a linear projection of norm at most λ from m(I) onto X. Hence, by Lemma 2, $P_n(X) \le \lambda$ for every n, and thus $\lim_{n\to\infty} P_n(X) \le P(X)$. This holds of course also if $P(X) = \infty$. Suppose now that $\lambda > P_n(X)$ for every n. Then for every finite set of points $\{y_i\}_{i=1}^n$ in m(I) ~ X, there exists a Lipschitz projection of norm at most λ from $X \cup \{y_i\}_{i=1}^n$ onto X. By the w*-compactness of the cells in X^{**} and by Tychonoff's Theorem we deduce that there exists a mapping of norm at most λ from m(I) into X^{**} whose restriction to X is the identity. Hence there exists a projection of norm at most $\lambda\lambda_0$ from m(I) onto X. From Corollary 2 of Theorem 3 (and its proof) we see that there exists a linear projection from m(I) onto X of norm at most $\lambda\lambda_0^2$, and hence $P(X) \le \lambda\lambda_0^2$. This concludes the proof of the theorem.

If in particular X is a conjugate Banach space, it follows from Theorem 5 that $P(X) = \lim_{n\to\infty} P_n(X)$. Grünbaum [12] has computed P(X) for certain finite-dimensional Banach spaces.

4. PROJECTIONS ON SOME SPECIAL SPACES

We first consider projections onto C(K)-spaces. We conjecture that for every K and for every metric space Y containing C(K) there exists a Lipschitz projection from Y onto C(K). Our next two theorems show that this is the case in many special situations.

THEOREM 6. (a) *Let* K *be a topological space, let* $k_0 \in K$, *and let* $X = C_0(K)$ *be the space of all bounded real-valued functions* f *on* K *for which* $f(k) \to 0$ *as* $k \to k_0$ (*with the* sup *norm*). *From every metric space* Y *containing* X *there is a projection onto* X *with norm at most* 2.

(b) *Let* K *be a metric space, and let* $X = C_u(K)$ *be the space of all bounded, uniformly continuous, real-valued functions on* K (*with the* sup *norm*). *From every metric space* Y *containing* X *there is a projection onto* X *with norm at most* η_1, *where* η_1 *is an absolute constant.*

Proof. (a) To every subset E of the line we assign a number $\psi(E)$ in the following manner. If E consists of negative numbers only, then

$$\psi(E) = \sup\{t;\ t \in E\};$$

otherwise, we take

$$\psi(E) = \inf\{|t|;\ t \in E\}.$$

For every two bounded subsets E_1 and E_2 of the line, $|\psi(E_1) - \psi(E_2)| \le H(E_1, E_2)$. In order to prove (a) it is sufficient to show that there exists a projection of norm at most 2 from $m(K)$ onto $C_0(K)$ (see Lemma 2). Let $f \in m(K)$, and let E_f be the set of all the limiting points of $f(k)$ as $k \to k_0$. Now put $Pf(k) = \psi(f(k) - E_f)$. For $f \in C_0(K)$, E_f consists only of the number 0, and hence in this case $Pf = f$. For every $f \in m(K)$, $Pf \in C_0(K)$. Indeed, let $k_\alpha \to k_0$; then $d(f(k_\alpha), E_f) \to 0$, and therefore $Pf(k_\alpha) \to 0$. We also have

$$H(f_1(k) - E_{f_1}, f_2(k) - E_{f_2}) \leq 2\|f_1 - f_2\| \qquad (f_1, f_2 \in m(K),\ k \in K).$$

Hence $\|P\| \leq 2$.

We turn to the proof of part (b). As above, we need only prove the existence of a projection with norm at most η_1 from $m(K)$ onto $C_u(K)$. We may clearly assume that the given metric ρ on K is bounded. For every $f \in m(K)$, denote by $\alpha(f)$ its distance from $C_u(K)$. Let t be a positive number, and put

$$\mu_f(t) = \sup\{r;\ \rho(k_1, k_2) \leq r \Rightarrow |f(k_1) - f(k_2)| \leq t\}.$$

Clearly, $\mu_f(t) > 0$ for $t > 2\alpha(f)$. We now define $\psi(f, k, t)$ for $f \in m(K)$, $k \in K$ and $t \geq 0$, as follows:

$$\psi(f, k, t) = \sup\{f(h);\ \rho(h, k) \leq 3^{-1} t\mu_f((3 + t)\alpha(f))\}.$$

Clearly,

$$f(k) \leq \psi(f, k, t) \leq f(k) + (3 + t)\alpha(f)$$

if $t \leq 3$, and hence $f(k) = \psi(f, k, t)$ if $f \in C_u(K)$. Put

$$Pf(k) = \int_0^1 \psi(f, k, t)\,dt \qquad (k \in K,\ f \in m(K)).$$

(The integral exists, since $\psi(f, k, t)$ is a bounded, nondecreasing function of t.) $Pf = f$ if $f \in C_u(K)$. For every $f \in m(K)$, $Pf \in C_u(K)$. Indeed, let $f \in m(K) \sim C_u(K)$, let $0 < \varepsilon < 1/2$, and suppose that $\rho(h, k) \leq 3^{-1}\varepsilon\mu_f(3\alpha(f))$. Then

$$\psi(f, k, t) \leq \psi(f, h, t + \varepsilon),$$

and thus

$$Pf(k) \leq \int_\varepsilon^{1+\varepsilon} \psi(f, h, t)\,dt \leq Pf(h) + 2\varepsilon(\|f\| + 5\alpha(f)).$$

By symmetry, $|Pf(k) - Pf(h)| \leq 2\varepsilon(\|f\| + 5\alpha(f))$, and hence $Pf \in C_u(K)$.

We shall now show that P is a Lipschitz map. Let $f, g \in m(K)$, and put $\varepsilon = \|f - g\|$, $a = \alpha(f)$, and $b = \alpha(g)$. Clearly $|a - b| \leq \varepsilon$. If $a < 3\varepsilon$, then $b \leq 4\varepsilon$ and hence $\|Pf - f\| \leq 4a \leq 12\varepsilon$, $\|Pg - g\| \leq 4b \leq 16\varepsilon$. Therefore

$$\|Pf - Pg\| \leq 29\|f - g\|,$$

and this proves our assertion if $a < 3\varepsilon$. Hence we may assume that $\min(a, b) \geq 3\varepsilon$. Since $(3 + t)a \leq (3 + t + 4\varepsilon/b)b$ for $t \in [0, 1]$ and $\mu_f(s) \leq \mu_g(s + 2\varepsilon)$ for all s,

$$\mathrm{Pf}(k) \le \varepsilon + \int_{6\varepsilon/b}^{1+6\varepsilon/b} \psi(g, k, t)\,dt = \varepsilon + g(k) + \int_{6\varepsilon/b}^{1+6\varepsilon/b} [\psi(g, k, t) - g(k)]\,dt$$

$$\le \varepsilon + g(k) + \int_{0}^{1+6\varepsilon/b} [\psi(g, k, t) - g(k)]\,dt$$

$$= \varepsilon + \mathrm{Pg}(k) + \int_{1}^{1+6\varepsilon/b} [\psi(g, k, t) - g(k)]\,dt$$

$$\le \varepsilon + \mathrm{Pg}(k) + (6\varepsilon/b)(4 + 6\varepsilon/b)b \le \mathrm{Pg}(k) + 37\varepsilon .$$

By symmetry, $|\mathrm{Pf}(k) - \mathrm{Pg}(k)| \le 37\varepsilon$, and hence $\|P\| \le 37$.

REMARKS. 1. Theorem 6(a) shows, in particular, the existence of a Lipschitz projection in the situation discussed in [23] (where the nonexistence of a bounded linear projection was proved). Part (b) shows, in particular, that a separable $C(K)$-space admits a Lipschitz projection from every metric space containing it.

2. Isbell [14, Theorem 3.1 (b)] showed that from every $Y \supset C_u(K)$ there is a uniformly continuous projection onto $C_u(K)$. The proof of part (b) given here is a modification of the argument of Isbell.

Our next result is on general $C(K)$-spaces.

THEOREM 7. *Let* $X = C(K)$, *where* K *is a compact Hausdorff space. Then* $P_n(X) \le 2$ *for all integers* n.

Proof. Let Y be a metric space containing X, with $Y \sim X = \{y_i\}_{i=1}^n$. We have to show that there exist $g_i \in X$ such that

$$\|g_i - g_j\| \le 2d(y_i, y_j) \qquad (1 \le i, j \le n) \tag{4.1}$$

and

$$\|g_i - f\| \le 2d(y_i, f) \qquad (1 \le i \le n,\ f \in C(K)). \tag{4.2}$$

For every i, put

$$F_i(k) = \liminf_{h \to k} \{ \inf_{f \in C(K)} [f(h) + 2d(y_i, f)]\} ,$$

$$G_i(k) = \limsup_{h \to k} \{ \sup_{f \in C(K)} [f(h) - 2d(y_i, f)]\} .$$

F_i is lower-semicontinuous, and G_i is upper-semicontinuous. Next we shall show that

$$G_i(k) \le F_j(k) + 2d(y_i, y_j) \qquad (1 \le i, j \le n,\ k \in K). \tag{4.3}$$

In order to establish (4.3), we have to prove that for all nets $\{k_\alpha\}$ and $\{k_\beta\}$ that converge to k and all nets of elements $\{f_\alpha\}$ and $\{f_\beta\}$ in $C(K)$,

$$(4.4)\quad \limsup_{k_\alpha \to k}[f_\alpha(k_\alpha) - 2d(y_i, f_\alpha)] \leq \liminf_{k_\beta \to k}[f_\beta(k_\beta) + 2d(y_j, f_\beta)] + 2d(y_i, y_j).$$

Let $\varepsilon > 0$, and let $f_{i,\varepsilon} \in C(K)$ satisfy the condition

$$d(y_i, f_{i,\varepsilon}) \leq \varepsilon + \inf_{f \in C(K)} d(y_i, f).$$

Then

$$\begin{aligned} f_\alpha(k_\alpha) - 2d(y_i, f_\alpha) &\leq f_\alpha(k_\alpha) - d(y_i, f_\alpha) - d(y_i, f_{i,\varepsilon}) + \varepsilon \\ &\leq f_\alpha(k_\alpha) - \|f_\alpha - f_{i,\varepsilon}\| + \varepsilon \leq f_{i,\varepsilon}(k_\alpha) + \varepsilon \\ &\leq |f_{i,\varepsilon}(k_\alpha) - f_{i,\varepsilon}(k_\beta)| + f_\beta(k_\beta) + \|f_\beta - f_{i,\varepsilon}\| + \varepsilon \\ &\leq |f_{i,\varepsilon}(k_\alpha) - f_{i,\varepsilon}(k_\beta)| + f_\beta(k_\beta) + 2d(y_j, f_\beta) + 2d(y_j, y_i) + 2\varepsilon. \end{aligned}$$

Thus since $f_{i,\varepsilon} \in C(K)$,

$$\limsup_{k_\alpha \to k}[f_\alpha(k_\alpha) - 2d(y_i, f_\alpha)] \leq \liminf_{k_\beta \to k}[f_\beta(k_\beta) + 2d(y_j, f_\beta)] + 2d(y_i, y_j) + 2\varepsilon.$$

Since ε was arbitrary, (4.4) and hence (4.3) follows. By Lemma 5 below, there exist $g_i \in C(K)$ that satisfy (4.1) and the requirement that $G_i(k) \leq g_i(k) \leq F_i(k)$ for all i and k. From the definition of the G_i and F_i it follows that also (4.2) holds for these g_i. This concludes the proof of the theorem.

REMARK. Combining Theorem 7 with a result of Amir [1], we see that $P_n[C(K)] = 2$ for all n and all compact Hausdorff spaces K that are not extremally disconnected. For extremally disconnected K, $P_n(C(K)) = 1$ for all n.

We now prove the lemma that was used in the proof of Theorem 7.

LEMMA 5. *Let* K *be a compact Hausdorff space, and let* $\{F_i\}_{i=1}^n$ *and* $\{G_i\}_{i=1}^n$ *be* $2n$ *functions on* K *such that*

$$(4.5)\qquad G_i(k) - F_j(k) \leq \rho(i, j) \qquad (1 \leq i, j \leq n, k \in K),$$

where ρ *is a metric on* $\{1, \cdots, n\}$. *Suppose further that each* G_i *is upper-semicontinuous and each* F_i *is lower-semicontinuous. Then there exist* n *continuous functions* $\{g_i\}_{i=1}^n$ *on* K *such that*

$$(4.6)\qquad G_i(k) \leq g_i(k) \leq F_i(k) \qquad (1 \leq i \leq n, k \in K)$$

and

$$(4.7)\qquad \|g_i - g_j\| \leq \rho(i, j) \qquad (1 \leq i, j \leq n).$$

Proof. For $n = 1$ the lemma is well known. We proceed by induction. Put

$$G'(k) = \sup_{1 \leq i \leq n} [G_i(k) - \rho(i, 1)], \qquad F'(k) = \inf_{1 \leq i \leq n} (F_i(k) + \rho(i, 1)).$$

G' is upper-semicontinuous and F' is lower-semicontinuous, and by (4.5), $G'(k) \leq F'(k)$ for all k. Hence there exists a $g_1 \in C(K)$ satisfying the condition $G'(k) \leq g_1(k) \leq F'(k)$. For $2 \leq i, j \leq n$ and $k \in K$,

$$\sup [G_i(k), g_1(k) - \rho(i, 1)] \leq \inf [F_j(k), g_1(k) + \rho(j, 1)] + \rho(i, j).$$

By the induction hypothesis there exist functions g_i in $C(K)$ such that

$$\|g_i - g_j\| \leq \rho(i, j) \qquad (2 \leq i, j \leq n)$$

and

$$\sup [G_i(K), g_1(k) - \rho(i, 1)] \leq g_i(k) \leq \inf [F_i(K), g_1(k) + \rho(i, 1)] \quad (2 \leq i \leq n).$$

These g_i $(2 \leq i \leq n)$ together with g_1 satisfy (4.6) and (4.7).

We pass now to another class of Banach spaces—the uniformly convex spaces (Clarkson [5]). For a Banach space X the modulus of convexity $\delta(\varepsilon)$ will be defined by

$$\delta(\varepsilon) = \inf_{\|x\|=\|y\| \leq 1,\ \|x-y\|=\varepsilon} (\|x\| + \|y\| - \|x+y\|) \qquad (2 \geq \varepsilon \geq 0).$$

For the sake of convenience we put $\delta(2\lambda) = \lambda\delta(2)$ for $\lambda > 1$. It is easy to verify that for all $\lambda > 1$ and all $\varepsilon > 0$, $\delta(\lambda\varepsilon) \geq \lambda\delta(\varepsilon)$, and $\varepsilon \geq \delta(\varepsilon) \geq 0$. X is uniformly convex if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. A uniformly convex space is reflexive, and for such a space, $\delta^{-1}(\varepsilon)$ is well-defined for every $\varepsilon > 0$ and satisfies the condition $\lambda\delta^{-1}(\varepsilon) \geq \delta^{-1}(\lambda\varepsilon)$ for $\lambda > 1$ and $\delta^{-1}(\varepsilon) \geq \varepsilon$.

Isbell [14, Theorem 3.1 (c)] asserted that for every uniformly convex Banach space X and every metric space $Y \supset X$ there exists a uniformly continuous projection from Y onto X. His computations contained an error, and in fact, as we saw in Section 3 (see Corollaries 4 and 5 of Theorem 3), the assertion itself is false, for example, if X is infinite-dimensional and $Y = H(X)$. However, by modifying the reasoning in [14], we get the following result.

THEOREM 8. *There exist constants* η_2 *and* η_3 *such that for any uniformly convex Banach space* X *with modulus of convexity* $\delta(\varepsilon)$ *the following holds.*

(a) *From every metric space* $Y \supset X$, *with* $d(y, X) \leq r$ *for all* $y \in Y$, *there is a projection onto* X *whose modulus of continuity* $\phi(\varepsilon)$ *satisfies the condition* $\phi(\varepsilon) \leq \eta_2 r \delta^{-1}(\varepsilon/r)$.

(b) *From every metric space* $Y \supset S_X(0, 1)$ *there is a projection onto* $S_X(0, 1)$ *whose modulus of continuity* $\phi(\varepsilon)$ *satisfies the condition* $\phi(\varepsilon) \leq \eta_3 \delta^{-1}(\varepsilon)$.

Proof. (a) Let C be a closed bounded convex set in X, and for $x \in X$ let $\rho(x, C) = \sup \{\|z - x\|, z \in C\}$. There is a unique point $x_C \in X$ at which $\rho(x, C)$ attains its minimum (the existence of a point x_C is a consequence of the conditional w-compactness of bounded sets in X; its uniqueness follows from the uniform convexity of X, by a special case of the continuity argument that will be given below). We call x_C the *center* of C and $\rho_C = \rho(x_C, C)$ the *radius* of C. We shall show that

$$\|x_1 - x_2\| \leq 3r\,\delta^{-1}(H(C_1, C_2)/r) \tag{4.8}$$

if $r \geq \max(\rho_1, \rho_2)$ (and $r \neq 0$), where $\rho_i = \rho_{C_i}$ and $x_i = x_{C_i}$ $(i = 1, 2)$. Let the Hausdorff distance $H(C_1, C_2)$ between C_1 and C_2 be equal to ε, and assume that $\rho_2 \leq \rho_1$. Since $\rho((x_1 + x_2)/2, C_1) \geq \rho_1$, there exists a $z \in C_1$ such that

$$\|x_1 + x_2 - 2z\| \geq 2\rho_1 .$$

Put $a = \|x_1 - z\|$ and $b = \|x_2 - z\|$. We see that $a \leq \rho_1$, $b \leq \rho_2 + \varepsilon$, $a + b \geq 2\rho_1$, and hence that $\rho_1 - \varepsilon \leq a \leq \rho_1 \leq b \leq \rho_1 + \varepsilon$. Inequality (4.8) holds if $a = 0$, since in this case

$$\|x_1 - x_2\| = b \leq 2\varepsilon = 2r(\varepsilon/r) \leq 2r\delta^{-1}(\varepsilon/r).$$

We assume from now on that $a \neq 0$. By the definition of the modulus of convexity,

$$\begin{aligned} 2\rho_1 &\leq \|(x_1 - z) + (x_2 - z)\| \leq b - a + a\left\| \frac{x_1 - z}{a} + \frac{x_2 - z}{b} \right\| \\ &\leq b - a + a\left[2 - \delta\left(\left\| \frac{x_1 - z}{a} - \frac{x_2 - z}{b} \right\|\right)\right] \\ &\leq 2\rho_1 + \varepsilon - a\delta\left(\left\| \frac{x_1 - x_2}{a} + \frac{(x_2 - z)(b - a)}{ab} \right\|\right) . \end{aligned}$$

Hence

$$\|x_1 - x_2\| \leq a\,\delta^{-1}(\varepsilon/a) + 2\varepsilon \leq 3a\delta^{-1}(\varepsilon/a) \leq 3r\,\delta^{-1}(\varepsilon/r),$$

and this proves (4.8).

Now let Y be a metric space containing X with $d(y, X) \leq r$ $(y \in Y)$. By Lemma 3 there exists a map T with $\|T\| \leq \eta_0$ from Y into $H(X)$ whose restriction to X is the identity. All the convex sets Ty $(y \in Y)$ have a radius not exceeding $\eta_0 r$. Let Py be the center of Ty $(y \in Y)$. Then P is a projection from Y onto X whose modulus of continuity $\phi(\varepsilon)$ satisfies the condition $\phi(\varepsilon) \leq 3\eta_0 r\,\delta^{-1}(\varepsilon/r)$, and this concludes the proof of (a).

Next we prove (b). For every $C \in H(X)$, let $p(C)$ be the point of C nearest to the origin of X. We define a projection P from $H(X)$ onto $S_X(0, 1)$ by putting

$$PC = \begin{cases} p(C)/\|p(C)\| & \text{if } \|p(C)\| \geq 1, \\ p(C) & \text{if } \|p(C)\| \leq 1. \end{cases}$$

By an argument similar to that used in the proof of (4.8), it can be shown that P is uniformly continuous and that its modulus of continuity is at most $\eta_4\,\delta^{-1}(\varepsilon)$, for some constant η_4. Theorem 8(b) follows now with the help of Lemma 3.

Our next aim is to show that Theorem 8 gives the best possible result if $X = L_p(\mu)$ $(1 < p < \infty)$.

LEMMA 6. *Let* $X = \ell_{p,2n-1}$ *with* $1 < p < \infty$, $n = 2, 3, \cdots$, *and let* $\varepsilon, r > 0$. *Then there exists a metric space* $Y \supset X$ *with* $d(y, X) \leq r$ *for all* $y \in Y$ *and such that for every projection* P *from* Y *onto* X *the modulus of continuity* ϕ *of* P *satisfies the condition*

(4.9) $$\phi(r^{1-p}\varepsilon^p) + 2\phi(r)n^{-1/p} \geq \varepsilon .$$

Proof. For every subset α of $\{1, 2, \cdots, 2n-1\}$, let x_α be the point in $\ell_{p,2n-1}$ given by

$$x_\alpha(i) = \begin{cases} \varepsilon & \text{if } i \in \alpha, \\ 0 & \text{if } i \notin \alpha. \end{cases}$$

Then $\|x_\alpha - x_\beta\|^p = \Delta(\alpha, \beta)\varepsilon^p$, where $\Delta(\alpha, \beta)$ denotes the number of points in the symmetric difference of α and β. Clearly, $\Delta(\alpha, \beta)$ is a metric on the set A consisting of the 2^{2n-1} subsets of $\{1, 2, \cdots, 2n-1\}$. Since $t \leq 1 + t^p$ for every $t \geq 0$,

$$\|x_\alpha - x_\beta\| \leq r + \varepsilon^p r^{1-p}\Delta(\alpha, \beta) \qquad (\alpha, \beta \in A).$$

By Corollary 1 of Lemma 4, there exists a metric space $Y \supset X$ such that $Y = X \cup \{y_\alpha\}_{\alpha \in A}$ and

$$d(y_\alpha, x_\alpha) \leq r, \qquad d(y_\alpha, y_\beta) \leq \varepsilon^p r^{1-p}\Delta(\alpha, \beta)$$

for all α and β in A. (We do not assert that $y_\alpha \neq y_\beta$ if $\alpha \neq \beta$, or that $y_\alpha \notin X$; thus the cardinality of $Y \sim X$ may be less than 2^{2n-1}.) Let P be a projection from Y onto X, and let ϕ be its modulus of continuity. Let Π be the group of all the permutations of $\{1, 2, \cdots, 2n-1\}$. For every $\sigma \in \Pi$, let U_σ be the isometric operator on $\ell_{p,2n-1}$ defined by $U_\sigma x(i) = x(\sigma i)$. Clearly $U_{\sigma\tau} = U_\tau U_\sigma$ and $x_{\sigma^{-1}\alpha} = U_\sigma x_\alpha$. Put

$$z_\alpha = \frac{1}{(2n-1)!} \sum_{\sigma \in \Pi} U_\sigma P y_{\sigma\alpha} \qquad (\alpha \in A).$$

Then

$$U_\tau z_\alpha = \frac{1}{(2n-1)!} \sum_{\sigma \in \Pi} U_{\sigma\tau} P y_{\sigma\alpha} = \frac{1}{(2n-1)!} \sum_{\sigma \in \Pi} U_\sigma P y_{\sigma\tau^{-1}\alpha} = z_{\tau^{-1}\alpha},$$

and in particular

(4.10) $$\tau\alpha = \alpha \Rightarrow U_\tau z_\alpha = z_\alpha \qquad (\alpha \in A, \tau \in \Pi).$$

Further,

(4.11) $$\|z_\alpha - x_\alpha\| \leq \frac{1}{(2n-1)!} \sum_{\sigma \in \Pi} \|U_\sigma P y_{\sigma\alpha} - x_\alpha\|$$
$$\leq \frac{1}{(2n-1)!} \sum_{\sigma \in \Pi} \|P y_{\sigma\alpha} - x_{\sigma\alpha}\| \leq \phi(r) \qquad (\alpha \in A).$$

Since $\Delta(\alpha, \beta) = \Delta(\sigma\alpha, \sigma\beta)$ for every α, β, and σ, we deduce similarly that

(4.12) $$\|z_\alpha - z_\beta\| \leq \phi(\varepsilon^p r^{1-p}\Delta(\alpha, \beta)) \qquad (\alpha, \beta \in A).$$

Consider the set $\alpha_0 = \{1, 2, \cdots, n\}$. By (4.10),

$$z_{\alpha_0}(i) = \begin{cases} z_{\alpha_0}(n) & (1 \le i \le n), \\ z_{\alpha_0}(n+1) & (n+1 \le i \le 2n-1). \end{cases}$$

Hence, by (4.11),

$$n[z_{\alpha_0}(n) - \varepsilon]^p + (n-1) z_{\alpha_0}(n+1)^p \le \phi(r)^p,$$

and thus $|z_{\alpha_0}(n) - \varepsilon| \le \phi(r) n^{-1/p}$. Similarly, by taking $\beta_0 = \{1, \cdots, n-1\}$, we see that $|z_{\beta_0}(n)| \le \phi(r) n^{-1/p}$. By (4.12),

$$\phi(\varepsilon^p r^{1-p}) \ge \|z_{\alpha_0} - z_{\beta_0}\| \ge z_{\alpha_0}(n) - z_{\beta_0}(n) \ge \varepsilon - 2\phi(r) n^{-1/p},$$

and this is (4.9).

REMARKS. 1. We have actually shown here that if we embed X in some m(I) and take $Y = \{y;\ y \in m(I),\ d(y, X) \le r\}$, then (4.9) holds for every projection from Y onto X. Since this Y depends only on r, and not on ε, (4.9) holds (with this Y) for every $\varepsilon > 0$.

2. In the proof we did not use the full strength of the assumption that P is a projection from Y onto X, but merely the facts that P maps Y into X and that $Px_\alpha = x_\alpha$ for all α.

Combining these remarks and letting $n \to \infty$ in Lemma 6, we get the following consequence.

COROLLARY 1. *Let* $1 < p < \infty$, *and embed* ℓ_p *in* m. *Let* $\varepsilon > 0$, *and let* A *denote the set consisting of all the finite subsets of the integers. For every* $\alpha \in A$, *denote by* x_α *the point in* ℓ_p *defined by*

$$x_\alpha(i) = \begin{cases} \varepsilon & (i \in \alpha), \\ 0 & (i \notin \alpha). \end{cases}$$

Then for every $r > 0$ *there exist* $y_\alpha = y_\alpha(r) \in m$ *with* $\|y_\alpha - x_\alpha\| \le r$ $(\alpha \in A)$ *such that, for every map* P *from* $\ell_p \cup \{y_\alpha\}_{\alpha \in A}$ *into* ℓ_p *for which* $Px_\alpha = x_\alpha$ $(\alpha \in A)$, *the modulus of continuity* ϕ *of* P *satisfies the condition* $\phi(\varepsilon^p r^{1-p}) \ge \varepsilon$.

Proof. Observe that if we identify $\ell_{p,n}$ with the subspace of ℓ_p consisting of all the elements whose coordinates vanish beyond the n-th place, then there exists a linear projection of norm 1 from ℓ_p onto $\ell_{p,n}$. Combining this observation with the proof of Lemma 6, we get the desired result.

COROLLARY 2. $P_{2^n}(\ell_{p,n}) \ge 5^{-1} n^{(p-1)/p^2}$ *for every integer* n *and every* p $(1 < p < \infty)$.

Proof. In the proof of Lemma 6 the cardinality of $Y \sim X$ does not exceed 2^{2n-1}. Suppose there exists a projection of norm λ from Y onto $X = \ell_{p,2n-1}$. By (4.9)

$$\lambda r^{1-p} \varepsilon^p + 2\lambda r n^{-1/p} \ge \varepsilon.$$

Taking $r = 1$ and $\varepsilon^p = 2n^{-1/p}$, we see that $4P_{2^{2n-1}}(\ell_{p,2n-1}) \geq n^{(p-1)/p^2}$. From this the assertion of the corollary follows immediately.

THEOREM 9. *Let* $\gamma(p) = 1/p$ *if* $2 \leq p < \infty$ *and* $\gamma(p) = 1/2$ *if* $1 < p \leq 2$. *There exist positive constants* B_p *and* C_p $(1 < p < \infty)$ *such that with* $X = L_p(\mu)$ *(for some measure* μ*) the following holds.*

(a) *From every metric space* $Y \supset X$ *for which* $d(y, X) \leq r$ *for all* $y \in Y$ *there is a projection onto* X *whose modulus of continuity* ϕ *satisfies the condition*

$$\phi(\varepsilon) \leq B_p r^{1-\gamma(p)} \varepsilon^{\gamma(p)} \qquad (0 \leq \varepsilon \leq r).$$

(b) *If* X *is infinite-dimensional, then for every* $r > 0$ *there exists a metric space* $Y \supset X$, *with* $d(y, X) \leq r$ *for all* $y \in Y$, *such that the modulus of continuity* ϕ *of any projection from* Y *onto* X *satisfies the condition* $\phi(\varepsilon) \geq C_p r^{1-\gamma(p)} \varepsilon^{\gamma(p)}$ *for all* ε.

Proof. Part (a) is a consequence of Theorem 8(a) and the fact that the modulus of convexity of X satisfies the condition $\delta(\varepsilon) \geq \alpha_p \varepsilon^{1/\gamma(p)}$ for some $\alpha_p > 0$ and all ε in $[0, 1]$ (see Hanner [13]).

To prove (b), embed X in $m(I)$ for some I, and let

$$Y = \{y;\ y \in m(I),\ d(y, X) \leq r\}.$$

Since X has a linear subspace isometric to ℓ_p on which there is a linear projection of norm 1, it follows from Corollary 1 of Lemma 6 that the modulus of continuity of any projection from Y onto X satisfies the condition $\phi(\varepsilon) \geq \varepsilon^{1/p} r^{1-1/p}$. This proves part (b) with $C_p = 1$ if $2 \leq p < \infty$. Part (b) for $1 < p < 2$ follows from Lemma 6 (for $p = 2$) and from the fact that for every p $(1 < p < \infty)$ the Banach space ℓ_p is isomorphic to

$$(\ell_{2,1} \oplus \ell_{2,2} \oplus \cdots \oplus \ell_{2,n} \oplus \cdots)_p.$$

The isomorphism of these spaces was established by Pełczyński [22, Proposition 7].

For the unit cells of L_p-space we prove a similar proposition.

THEOREM 10. *Let* $\gamma(p)$ *be as in Theorem* 9. *There exist positive constants* b_p, c_p, *and* τ_p $(1 < p < \infty)$ *such that with* $X = L_p(\mu)$ *the following holds.*

(a) *From every metric space* $Y \supset S_X(0, 1)$ *there is a projection onto* $S_X(0, 1)$ *whose modulus of continuity* ϕ *satisfies the condition* $\phi(\varepsilon) \leq b_p \varepsilon^{\gamma(p)}$ $(0 \leq \varepsilon \leq 1)$.

(b) *If* X *is infinite-dimensional, then for every* $r > 0$ *there exists a metric space* $Y \supset S_X(0, 1)$, *with* $d(y, S_X(0, 1)) \leq r$ *for all* $y \in Y$, *such that the modulus of continuity* ϕ *of any projection from* Y *onto* X *satisfies the condition*

$$\phi(\varepsilon) \geq c_p \rho^{1-\gamma(p)} \varepsilon^{\gamma(p)}$$

for $0 \leq \varepsilon \leq 1$, *if* $\rho \leq r$ *and* $\phi(\rho) \leq \tau_p$.

Proof. Part (a) follows from Theorem 8(b). We turn to the proof of part (b). Embed X in $m(I)$ and take

$$Y = \{y;\ y \in m(I),\ d(y, S_X(0, 1)) \leq r\}.$$

For every n, X has a linear subspace isometric to $\ell_{p,n}$ onto which there is a linear projection P_n from X with norm 1. Let P be a projection from Y onto $S_X(0, 1)$, and let ϕ be its modulus of continuity. Then $P_{2n-1}P$ is a projection from Y onto the unit cell of a space isometric to $\ell_{p,2n-1}$, and the modulus of continuity of $P_{2n-1}P$ does not exceed ϕ. The points x_α used in the proof of Lemma 6 are of norm at most $(2n)^{1/p}\cdot\varepsilon$, and thus if $(2n)^{1/p}\cdot\varepsilon \leq 1$, they are in the unit cell of $\ell_{p,2n-1}$. If $\delta_n = \frac{1}{2}n^{-1/p}$, $\rho \leq r$, and $\phi(\rho) \leq 1/8$, then by Lemma 6 (see also Remark 1 and 2 after it)

$$\phi(\rho^{1-p}\delta_n^p) + \delta_n/2 \geq \delta_n \qquad (n = 2, 3, \cdots),$$

that is,

$$\phi(\varepsilon_n) \geq \frac{1}{4}\rho^{1-1/p}\cdot\varepsilon_n^{1/p} \qquad (\varepsilon_n = \rho^{1-p}\cdot n^{-1}\cdot 2^{-p}).$$

Since ϕ is decreasing and $\rho \leq \phi(\rho) \leq 1/8$, it follows that $\phi(\varepsilon) > \frac{1}{8}\rho^{1-1/p}\cdot\varepsilon^{1/p}$ for $0 \leq \varepsilon \leq 1$. This proves part (b) (with $c_p = \tau_p = 1/8$) for $2 \leq p < \infty$. For $1 < p \leq 2$, part (b) follows from the fact that ℓ_p is isomorphic to

$$(\ell_{2,1} \oplus \cdots \oplus \ell_{2,n} \oplus \cdots)_p.$$

We omit the details.

COROLLARY 1. *Let*

$$X = (\ell_{p_1} \oplus \cdots \oplus \ell_{p_k} \oplus \cdots)_p,$$

with $p_k \to \infty$ *and* $p = 0$ *or* $1 \leq p \leq \infty$, *and embed* X *in* m. *Then* X [*respectively,* $S_X(0, 1)$] *is not a uniformly continuous retract of any of its uniform neighborhoods in* m.

Proof. Since there exists a Lipschitz projection from X onto $S_X(0, 1)$ (map x with $\|x\| > 1$ to the point $x/\|x\|$), it is sufficient to prove the assertion concerning $S_X(0, 1)$. For every k, $S_X(0, 1)$ has a subset isometric to the unit cell of ℓ_{p_k}, on which there is a projection (from $S_X(0, 1)$) with norm 1. Let

$$Y = \{y;\ y \in m,\ d(y, S_X(0, 1)) \leq r\}$$

for some $r > 0$, and let P be a projection from Y onto X with modulus of continuity ϕ. By Theorem 10 (b), either $\phi(\varepsilon) \geq 1/8$ for all $\varepsilon > 0$, or

$$\phi(\varepsilon) \geq \frac{1}{8}\rho^{1-1/p_k}\cdot\varepsilon^{1/p_k}$$

if $\rho \leq r$, $\phi(\rho) \leq 1/8$, and $p_k \geq 2$. Letting $k \to \infty$ in the latter case we see that $\phi(\varepsilon) \geq \rho/8$ for all $\varepsilon > 0$, and therefore P is not uniformly continuous.

REMARKS. 1. In the terminology of Isbell [14], the corollary asserts that neither X nor $S_X(0, 1)$ is ANRU, and therefore it solves a problem raised by Isbell.

2. An examination of the argument used in the proof of Theorem 10 and its corollary shows that Corollary 1 still holds if we take as X spaces of the form

$$(\ell_{p_1,n_1} \oplus \cdots \oplus \ell_{p_k,n_k} \oplus \cdots)_p,$$

where $p_k \to \infty$ and $\log n_k/p_k \to \infty$ as $k \to \infty$.

5. APPLICATIONS

Our first application of the results of Sections 3 and 4 is to the problem of the existence of uniform homeomorphisms between certain Banach spaces. All the Banach spaces appearing in Theorem 11 will be assumed to be infinite-dimensional.

THEOREM 11. (a) *A* $C(K)$*-space is not uniformly homeomorphic to any Banach space* X *such that* (i) *there exists a bounded linear projection from* X^{**} *onto* X, *and* (ii) X *is not a* $\mathfrak{P}$*-space.*

(b) *Let* $X_1 = L_{p_1}(\mu_1)$ *and* $X_2 = L_{p_2}(\mu_2)$. *Suppose that* $p_1 > \max(p_2, 2)$. *Then* X_1 *is not uniformly homeomorphic to* X_2, *and* $S_{X_1}(0, 1)$ *is not Lipschitz-equivalent to* $S_{X_2}(0, 1)$.

(c) *Let* $X = (\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_k} \oplus \cdots)_p$, *with* $p_k \to \infty$ *and* $p = 0$ *or* $1 \leq p \leq \infty$. *Then* X *is not uniformly homeomorphic to a uniformly convex Banach space, and* $S_X(0, 1)$ *is not uniformly homeomorphic to the unit cell of a uniformly convex Banach space.*

Proof of (a). Let X satisfy (i) and (ii), and suppose that T is a uniform homeomorphism from X onto $C(K)$. Let ϕ and ψ be the moduli of continuity of T and T^{-1}, respectively. Let Y be a metric space containing X and such that $Y \sim X$ is finite. By Lemma 2, the map T from X into $C(K)$ has an extension $\tilde{T}$ from Y into $m(K)$ such that ϕ is also the modulus of continuity of $\tilde{T}$. By Theorem 7, there exists a projection Q from $\tilde{T}Y$ onto $C(K)$ with $\|Q\| \leq 2$. Hence $T^{-1}Q\tilde{T}$ is a projection from Y onto X whose modulus of continuity ϕ_Y satisfies $\phi_Y(\varepsilon) \leq \psi(2\phi(\varepsilon))$. Since this estimate is independent of Y, and since we assume that (i) holds, it follows (see the proof of Theorem 5) that from every metric space containing X there is a projection onto X with a modulus of continuity at most $\lambda_0\, \psi(2\phi(\varepsilon))$ (where λ_0 is the norm of any bounded linear projection from X^{**} onto X). This however contradicts our assumptions on X (see Corollary 4 of Theorem 3).

Proof of (b). Suppose there exists a mapping T from X_1 onto X_2 such that T and T^{-1} are uniformly continuous. By Lemma 1 there exists a λ such that $\|Tx - Tx'\| \leq \lambda\|x - x'\|$ if $\|x - x'\| \geq 1$. Embed X_1 in $m(I_1)$ and X_2 in $m(I_2)$. Let $r > 0$, and put

$$Y_j(r) = \{y;\, y \in m(I_j);\, d(y, X_j) \leq r\} \qquad (j = 1, 2).$$

X_1 contains a linear subspace Z_1 which is isometric to ℓ_{p_1} and on which there is a linear projection P (from X_1) with $\|P\| = 1$. Let $\{x_\alpha\}_{\alpha \in A}$ be the points of Z_1 appearing in the statement of Corollary 1 of Lemma 6 (taking there $\varepsilon = 1$), and let $y_\alpha(r)$ be the suitable points in $Y_1(r)$. $\|x_\alpha - x_\beta\| \geq 1$ for $\alpha \neq \beta$, and hence, by our choice of λ and by Lemma 2, there exists a map $\tilde{T}$, with $\|\tilde{T}\| \leq \lambda$, from $m(I_1)$ into

$m(I_2)$ such that $\tilde{T}x_\alpha = Tx_\alpha$ $(\alpha \in A)$. Since $\|y_\alpha(r) - x_\alpha\| \leq r$, we see that $\tilde{T}y_\alpha(r) \in Y_2(\lambda r)$ for all α and r. By Theorem 9 (a) there exists a projection Q_r from $Y_2(\lambda r)$ onto X_2 with a modulus of continuity ψ_r satisfying the inequality

$$\psi_r(\varepsilon) \leq B_{p_2}(\lambda r)^{1-\gamma(p_2)} \cdot \varepsilon^{\gamma(p_2)} \qquad (0 \leq \varepsilon \leq \lambda r).$$

For every $r > 0$, denote by P_r the restriction of $PT^{-1}Q_r\tilde{T}$ to the set

$$Z_1(r) = Z_1 \cup \{y_\alpha(r)\}_{\alpha \in A}.$$

Then P_r maps $Z_1(r)$ into Z_1, $P_r x_\alpha = x_\alpha$ for all r and α, and the modulus of continuity ϕ_r of P_r satisfies

$$\phi_r(\varepsilon) \leq \theta\left(B_{p_2}(\lambda r)^{1-\gamma(p_2)} \cdot (\lambda\varepsilon)^{\gamma(p_2)}\right) \qquad (\varepsilon \leq \lambda r), \tag{5.1}$$

where θ is the modulus of continuity of T^{-1}. By Corollary 1 of Lemma 6 (taking $\varepsilon = 1$ there), we see that $\phi_r(r^{1-p_1}) \geq 1$. Combining this with (5.1), we deduce that

$$\theta\left(B_{p_2}\lambda \cdot r^{1-p_1\gamma(p_2)}\right) \geq 1 \qquad (r \geq 1).$$

Since $p_1\gamma(p_2) > 1$, it follows that $\theta(\varepsilon) \geq 1$ for every $\varepsilon > 0$, and this contradicts the uniform continuity of T^{-1}. This concludes the proof of the assertion that X_1 and X_2 are not uniformly homeomorphic. That $S_{X_1}(0, 1)$ and $S_{X_2}(0, 1)$ are not Lipschitz-equivalent follows from Theorem 10.

Part (c) follows from Theorem 8 and Corollary 1 of Theorem 10.

REMARKS. 1. If $X_1 = L_{p_1}(\mu_1)$ and $X_2 = L_{p_2}(\mu_2)$ have the same density character, then $S_{X_1}(0, 1)$ and $S_{X_2}(0, 1)$ are uniformly homeomorphic (Mazur [20]).

2. The method of proof of Theorem 11 can be used to prove more than the nonexistence of uniform homeomorphisms between certain Banach spaces. It can be shown, for example, that if $X = (\ell_{p_1} \oplus \cdots \oplus \ell_{p_k} \oplus \cdots)_p$ with $p_k \to \infty$ and $Y = C(K)$, then for every map T from X onto Y there exist sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ in X such that either $\|u_n - v_n\| \to 0$ while $\|Tu_n - Tv_n\| \to \infty$, or $\|u_n - v_n\| \to \infty$ while $\|Tu_n - Tv_n\| \to 0$.

Theorem 11 gives rise to many problems, first of all the question whether there exist any nonisomorphic Banach spaces that are uniformly homeomorphic (this question was raised also in [3]). We think, though we have not worked out the details, that a study of the lifting problem (similar to that in Section 4 concerning projections) will establish the nonexistence of a uniform homeomorphism between $L_{p_1}(\mu_1)$ and $L_{p_2}(\mu_2)$, if $p_1 \neq p_2$ with $1 \leq p_1, p_2 \leq 2$. However, it does not seem to us that the consideration of questions concerning the existence of liftings or projections of different kinds will yield a classification of all Banach spaces from the point of view of uniform equivalence. We do not see a way of distinguishing between ℓ_p and $L_p(0, 1)$ $(1 < p < \infty,\ p \neq 2)$, by a projection or lifting property. Are these spaces uniformly homeomorphic for any such p?

We shall now prove a result that is related to Ju. Smirnov's question whether every separable metric space is uniformly homeomorphic to a subset of ℓ_2. (For a discussion of this problem and for further references, see Gorin [10].)

THEOREM 12. *There exists no invertible uniformly continuous map* T *from* C[0, 1] *into* ℓ_2 *such that* T^{-1} *is a Lipschitz map for large distances (that is, satisfies the conclusion of Lemma* 1).

Proof. Suppose that some map T from C[0, 1] into ℓ_2 has the properties appearing in the statement of the theorem. Embed ℓ_2 in m, and let $\{x_\alpha\}_{\alpha \in A}$ be the points in ℓ_2 defined in Corollary 1 of Lemma 6 (taking $\varepsilon = 1$ there). It follows from that corollary that there exists no uniformly continuous map P from m into ℓ_2 such that $Px_\alpha = x_\alpha$ $(\alpha \in A)$. It is well known that there exists an isometry T_0 from ℓ_2 into C[0, 1]. Put $u_\alpha = T T_0 x_\alpha$ $(\alpha \in A)$. Since $\|x_\alpha - x_\beta\| \geq 1$ for $\alpha \neq \beta$, it follows from our assumption on T^{-1} that there is a $\lambda < \infty$ such that

$$\|u_\alpha - u_\beta\| \leq \lambda \|x_\alpha - x_\beta\|$$

for all α and β in A. By a theorem of Kirszbraun [14] (see also [7, p. 154]) there exists a map Q from ℓ_2 into itself such that $\|Q\| \leq \lambda$ and $Qu_\alpha = x_\alpha$ $(\alpha \in A)$. By Theorem 6 (b), T_0 has a Lipschitz extension $\tilde{T}_0$ from $m \supset \ell_2$ into C[0, 1]. $QT\tilde{T}_0$ is a uniformly continuous map from m into ℓ_2 such that $QT\tilde{T}_0 x_\alpha = x_\alpha$ $(\alpha \in A)$. This is impossible, as was remarked at the beginning of the proof.

REMARK. The theorem implies, in particular, that C[0, 1] is not uniformly homeomorphic to a subset B of ℓ_2 on which there is a uniformly continuous projection from ℓ_2 (or even only from a convex set in ℓ_2 containing B). However, we do not know the answer to the problem of Smirnov.

Our last application of the results of Sections 3 and 4 concerns essentially a problem in the linear theory of Banach spaces. We shall prove that c_0 has an uncountable number of mutually nonisomorphic closed linear subspaces. The only previously known result in this direction seems to be that c_0 has a closed infinite-dimensional linear subspace that is not isomorphic to c_0. The first to prove this was Sobczyk [24] (see also Köthe [16]). His proof is based on the "only if" part of the following result. *A closed infinite-dimensional linear subspace* X *of* c_0 *is isomorphic to* c_0 *if and only if there exists a bounded linear projection from* c_0 *onto* X (the "if" part is due to Pełczyński [22], and the "only if" part to Sobczyk [24]). It is well known and easy to see that if $\{X_n\}_{n=1}^\infty$ is a sequence of finite-dimensional Banach spaces, then

$$X = (X_1 \oplus X_2 \oplus \cdots \oplus X_n \oplus \cdots)_0 \tag{5.2}$$

is isomorphic to a linear subspace of c_0. The space X given by (5.2) is isomorphic to c_0 itself if and only if $X_n \in \mathfrak{P}_\lambda$ for some $\lambda < \infty$ (independent of n). A criterion for determining the existence of such a λ was given in [17]. One way for classifying the spaces of the form (5.2) is by computing the index of convergence, α_{X^*}, of the conjugate of X (see [19] for its definition). It seems quite probable that a computation of α_{X^*} for various choices of $\{X_n\}_{n=1}^\infty$ will provide a proof of Theorem 13 below (if we replace uniform homeomorphism by isomorphism). The general problem of classifying all the spaces of type (5.2) is very far from a solution. Is $X \oplus X$ isomorphic to X if X is of the type (5.2)? Is every closed infinite-dimensional linear subspace of c_0 isomorphic to a space of the type (5.2)?

THEOREM 13. *The space* c_0 *has an uncountable number of mutually not uniformly homeomorphic closed linear subspaces.*

Proof. Put $X_p = (\ell_{p,1} \oplus \cdots \oplus \ell_{p,n} \oplus \cdots)_0$ $(1 \leq p \leq \infty)$. By the preceding remarks it is sufficient to show that X_{p_1} is not uniformly homeomorphic to X_{p_2} if $p_1 > \max(2, p_2)$. The proof of this is much the same as that of Theorem 11 (b). We show here only that, like the spaces L_p, the X_p also have the property that from every metric space $Y \supset X_p$ with $d(y, X_p) \leq r$ $(y \in Y)$, there is a projection onto X_p whose modulus of continuity ϕ satisfies the condition

$$\phi(\varepsilon) \leq M_p r^{1-\gamma(p)} \cdot \varepsilon^{\gamma(p)} \qquad (\varepsilon \leq r),$$

for a suitable constant M_p. To show this, we embed X in a natural way in $Z = (m \oplus m \oplus \cdots \oplus m \oplus \cdots)_0$. The desired result will follow from the following two observations. (i) Let Y be a metric space containing X; then there exists a map of norm at most 2 from Y into Z which is the identity on X (see Theorem 6 (a)). (ii) Put $Z_r = \{z;\ z \in Z,\ d(z, X_p) \leq r\}$. Then there exists a projection from Z_r onto X_p whose modulus of continuity ψ satisfies the condition

$$\psi(\varepsilon) \leq b_p r^{1-\gamma(p)} \cdot \varepsilon^{\gamma(p)}$$

(apply Theorem 9 (a) separately to each of the direct summands $\{\ell_{p,n}\}_{n=1}^{\infty}$ of X).

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Yale University