ON EXTREMAL MEASURES AND SUBSPACE DENSITY

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The purpose of this note is to investigate the relation between a measure's property of being an extreme point of a certain convex set of probability measures and the denseness of a certain space of functions in the L_p-space of this measure. This problem is associated with certain questions raised in [2], and the results obtained were strongly influenced by a classical theorem of M. Riesz on the undetermined moment problem.

After defining our convex set of measures, we state as Theorem 1 our result on the relation between extremal measures and subspace density in L_1. By an example we show that the same proposition cannot hold in general when L_1 is replaced by L_p $(p > 1)$, and we obtain a result for L_p, under an additional hypothesis.

Our problem is also related to a problem studied by Choquet [1]; in particular, one of Choquet's questions is answered completely by Theorem 1, another partly by Theorem 2.

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Let X be a locally compact Hausdorff space, and let $M^+(X)$ denote the space of finite, nonnegative regular Borel measures defined on X. Let F be a linear space of real-valued (not necessarily bounded) Borel functions defined on X that contains the constant functions. For each positive measure $\mu \in M(X)$ having the property that $\int_X |f|\, d\mu < \infty$ for every $f \in F$, set

$$E_\mu = \left\{ \nu \,\middle|\, \nu \in M^+(X),\ \int_X |f|\, d\nu < \infty \text{ and } \int_X f\, d\nu = \int_X f\, d\mu \ \forall f \in F \right\}.$$

The space of functions F can be identified (in a canonical way) as a subspace of $L_1(\mu)$ (this correspondence need not be one-to-one). The following theorem describes the relation between the extremality in E_μ of a measure and the density of F in $L_1(\mu)$.

THEOREM 1. *The subspace* F *is dense in* $L_1(\mu)$ *if and only if* μ *is an extreme point of* E_μ.

Proof. Assume that μ is not an extreme point of E_μ; then there exist measures μ_1 and μ_2 in E_μ such that $\mu = (\mu_1 + \mu_2)/2$ and $\mu_1 \neq \mu_2$. This implies $2\mu \geq \mu_1 \geq 0$, and by the Radon-Nikodym Theorem there thus exists a function $h \in L_\infty(\mu)$ such that $d\mu_1 = h\, d\mu$ and $1 - h \neq 0$. The function $1 - h$ is orthogonal to F, that is,

$$\int_X f(1 - h)\, d\mu = \int_X f\, d\mu - \int_X fh\, d\mu = \int_X f\, d\mu - \int_X f\, d\mu_1 = 0$$

for every $f \in F$. Therefore, F is not dense in $L_1(\mu)$.

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Assume that F is not dense in $L_1(\mu)$; then it follows from the Hahn-Banach Theorem and the identification $L_1^*(\mu) = L_\infty(\mu)$ that there exists a nonzero function $h \in L_\infty(\mu)$ that is orthogonal to F. Set

$$\nu = \frac{1}{\|h\|_\infty} \int h\, d\mu, \qquad \mu_1 = \mu + \nu, \qquad \mu_2 = \mu - \nu.$$

Then the measures μ_1 and μ_2 are positive because $1 \pm h/\|h\|_\infty \geq 0$. Moreover, each of μ_1 and μ_2 is in E_μ, because

$$\int_X f\, d(\mu \pm \nu) = \int_X f\, d\mu \pm \int_X f\, d\nu = \int_X f\, d\mu \pm \frac{1}{\|h\|_\infty} \int_X fh\, d\mu = \int_X f\, d\mu.$$

Therefore, μ is not an extreme point of E_μ, because $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ and $\mu_1 \neq \mu_2$.

Naĭmark proved this theorem [3, Theorem 4, p. 342] for the special case where X is the space of reals and F the linear space of all polynomials. Our proof of the "only if" statement is similar to his; but his proof of the "if" statement is based on a result on the extension of symmetric operators.

The context of Naĭmark's theorem is the same as that of the theorem of Riesz. If for some positive measure μ, E_μ consists of more than μ, then the moments $c_n = \int_{-\infty}^{\infty} x^n\, d\mu(x)$ of μ constitute an undetermined moment problem [4]. Nevanlinna proved a remarkable theorem characterizing the solutions of such a moment problem, and in particular he described a certain class of extremal solutions. The cited theorem of M. Riesz states that a solution is extremal if and only if the polynomials are dense in the L_2-space of the measure. A natural question is whether a solution is extremal if and only if the measure is an extreme point [see 1].

In one direction the answer is clear; if μ is an extremal solution, then F is dense in $L_2(\mu)$ (by the theorem of Riesz), and hence in $L_1(\mu)$. Thus μ is an extreme point of E_μ, by Theorem 1. Therefore an extremal measure (in the sense of Nevanlinna) is necessarily an extreme point of E_μ. (This also follows from properties of $I(z; \psi)$ established in [4].) The converse is not true, however; D. Greenstein has informed the author that he has recently shown that for some measures there exist extreme points of E_μ that are not extremal solutions.

Assume now that X, F, and μ satisfy the original hypothesis as well as the additional hypothesis that $\int_X |f|^p\, d\mu < \infty$ for $f \in F$, where p is some number greater than 1. Further, set

$$E_\mu^{(p)} = \left\{ \nu \in E_\mu \,\middle|\, \int_X |f|^p\, d\nu < \infty \ \forall f \in F \right\}.$$

It is easy to see that a measure $\nu \in E_\mu^{(p)}$ is an extreme point of $E_\mu^{(p)}$ if and only if it is an extreme point of E_μ.

The relation between extremality of a measure in $E_\mu^{(p)}$ and denseness of the subspace F in L_p is more complex, in case $p > 1$. Consider the following example.

If μ is a measure that does not consist of a finite number of atoms, then there exists an unbounded function $f \in L_q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let F be the space of

Borel functions such that $\int_X |h|^p \, d\mu < \infty$ and $\int_X hf \, d\mu = 0$. Then the only summable Borel function on X that is orthogonal to F is unbounded, and hence $(0) = F^{\perp} \subset L_\infty(\mu)$. Therefore F is dense in $L_1(\mu)$, and thus μ is an extreme point of E_μ, but F is not dense in $L_p(\mu)$.

A result such as appears in Theorem 1 thus does not hold for $p > 1$ without some further hypothesis. One adequate additional hypothesis is that F be a vector lattice.

THEOREM 2. *If* F *is also a vector lattice, then* F *is dense in* $L_p(\mu)$ *if and only if* μ *is an extreme point of* $E_\mu^{(p)}$.

Proof. It is clear that if μ is not an extreme point of $E_\mu^{(p)}$, then F is not dense in $L_p(\mu)$. Suppose μ is an extreme point of $E_\mu^{(p)}$; then μ is also an extreme point of E_μ, and therefore F is dense in $L_1(\mu)$, by Theorem 1. If h is a bounded function in $L_p(\mu)$, then there exists a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in F such that $\lim_{n\to\infty} \|h - f_n\|_1 = 0$. But, because F is a vector lattice and $1 \in F$, the functions

$$h_n = (f_n \wedge \|h\|_\infty \cdot 1) \vee (-\|h\|_\infty \cdot 1)$$

are also in F, and $\lim_{n\to\infty} \|h - h_n\|_p = 0$. Therefore F is dense in $L_p(\mu)$, and the theorem is proved.

COROLLARY. *If* A *is a subalgebra of bounded real-valued Borel functions on* X *that contains the constants, and* $1 \leq p < \infty$, *then* A *is dense in* $L_p(\mu)$ *if and only if* μ *is an extreme point of* E_μ.

Proof. Since the uniform closure of A is a vector lattice, Theorem 2 yields the result.

In [1] Choquet considers a subspace F consisting of continuous (not necessarily bounded) functions defined on an X, where F is assumed to have certain additional properties. (More precisely, F is assumed to be *adapté* in his terminology.) Discussing a uniqueness question, in his concluding paragraph, Choquet observes that a necessary condition for F to be dense in $L_1(\mu)$ is that μ be an extreme point of E_μ, and he asks under what circumstances this is also sufficient. Theorem 1 provides a complete answer. Choquet also raises the analogous question for L_p $(p > 1)$, and Theorem 2 provides an answer in the case where F is a vector lattice.

Finally, observe that nowhere in the statement of either Theorems 1 or 2 is any hint given as to whether a particular E_μ has an extreme point. If F consists of continuous functions that vanish at infinity, then the Riesz-Kakutani Representation Theorem enables us to show that E_μ is an ω^*-compact and convex subset of $M^+(X)$. Thus it follows from the Kreĭn-Mil′man Theorem that the ω^*-closed convex hull of the set of extreme points of E_μ is equal to E_μ. Alternately, if F is a subspace of continuous functions that is *adapté* in the sense of Choquet, then the same conclusion holds [1, Proposition 4]. Although other hypotheses also imply the existence of extreme points in E_μ, the problem of deciding their existence in general seems to be difficult.

If F is a space of complex-valued functions, then the conclusion of Theorem 1 is valid when denseness of F is replaced by that of $F + \overline{F}$.

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