EXTENSIONS OF IRREDUCIBLE MODULES

Irving Reiner

Let G be a finite group of order g, let R be the ring of all algebraic integers in an algebraic number field K, and let RG be the group ring consisting of the R-linear combinations of the elements of G. We consider left RG-modules which are finitely generated over R. For two such modules M and N, let $\mathrm{Ext}^1_{RG}(M, N)$ denote the R-module whose elements are the classes of extensions of M by N. When there is no danger of confusion, we shall write Ext instead of Ext^1_{RG}.

A well-known form of Maschke's Theorem (see [1], which is a general reference for results cited in this note) asserts that

$$g \cdot \mathrm{Ext}(M, N) = 0$$

for all RG-modules M and N. As has been shown by D. G. Higman [3], it is useful to associate with each fixed RG-module M, the collection d(M) of all elements $a \in R$ such that

$$a \cdot \mathrm{Ext}(M, N) = 0 \qquad \text{for all } N.$$

Of course, d(M) is an ideal in R, and $g \in d(M)$.

Now let M be an RG-module which is torsion-free over R, and set

$$KM = K \otimes_R M.$$

We shall call M *absolutely irreducible* if the KG-module KM is absolutely irreducible, that is, if $L \otimes_K KM$ is irreducible for each extension field L of K. For such a module M, it is well known that the dimension (KM: K) divides g. Let us set

$$d_M = \frac{g}{(KM: K)}.$$

The purpose of this note is to establish the following surprising result.

THEOREM. *Let* M *be an absolutely irreducible* R-*torsion-free* RG-*module. Then* d(M) *is the principal ideal generated by* d_M. *In other words,* d_M *annihilates* $\mathrm{Ext}^1_{RG}(M, N)$ *for all* N, *and any element of* R *with this property must be a multiple of* d_M.

In proving the theorem, we may first reduce the problem to the case where the underlying ring R is a principal ideal ring. For P a prime ideal in R, let R_P denote the valuation ring in K belonging to the P-adic valuation of K. Then

$$\mathrm{Ext}^1_{RG}(M, N) \cong (\text{direct sum}) \sum_{P \supset gR} \mathrm{Ext}^1_{R_P G}(R_P M, R_P N),$$

Received January 21, 1963.

This research was supported by the Guggenheim Foundation and the Office of Naval Research.

and so it suffices to work over the principal ideal rings R_P instead of over R. To avoid changing notation, we assume for the remainder of the proof that R is a principal ideal ring containing all the algebraic integers of K.

Next we need a lemma which is due to D. G. Higman [2]. For the convenience of the reader, we shall sketch a proof thereof.

LEMMA. *Let* M *be an arbitrary* R-*torsion-free* RG-*module. An element* $a \in R$ *lies in* d(M) *if and only if there exists an* R-*endomorphism* u *of* M *such that*

$$\sum_{x \in G} x^{-1} u(xm) = am \qquad (m \in M). \tag{1}$$

Proof. Define the RG-module $F = RG \otimes_R M$, with the action of G given by

$$y(\sum b_i \otimes m_i) = \sum y b_i \otimes y m_i \quad (b_i \in RG,\ m_i \in M,\ y \in G).$$

It is easily seen (Swan [4]) that an R-basis of M is also an RG-basis for F, so F is a free RG-module. Let $G = \{x_1, \cdots, x_g\}$, where $x_1 = 1$. Then $F = \Sigma x_i \otimes M$, and the map

$$\sum x_i \otimes m_i \to \sum m_i$$

is an RG-homomorphism of F onto M. The kernel M' of this homomorphism is the RG-submodule of F given by

$$M' = \sum_{i=2}^{g} (x_i - 1) \otimes M.$$

Now let N be any RG-module. The exact sequence

$$0 \to M' \to F \to M \to 0$$

gives rise to the exact sequence

$$\mathrm{Hom}_{RG}(F, N) \xrightarrow{\theta} \mathrm{Hom}_{RG}(M', N) \to \mathrm{Ext}(M, N) \to 0,$$

since $\mathrm{Ext}(F, N) = 0$. Thus for $a \in R$, $a \cdot \mathrm{Ext}(M, N) = 0$ if and only if

$$a \cdot \mathrm{Hom}_{RG}(M', N) \subset \theta\{\mathrm{Hom}_{RG}(F, N)\}. \tag{2}$$

Let $a \in R$, and suppose that (1) holds for some $u \in \mathrm{Hom}_R(M, M)$. Define $v \in \mathrm{Hom}_R(M', M')$ by

$$v\{\sum (x_i - 1) \otimes m_i\} = \sum (x_i - 1) \otimes u(m_i).$$

From (1) we find readily that

$$\sum_{y \in G} y^{-1} v y = a \cdot 1_{M'},$$

where $1_{M'}$ denotes the identity map on M'. To prove that (2) holds, let h be any element of $\mathrm{Hom}_{RG}(M', N)$. Then $hv \in \mathrm{Hom}_R(M', N)$. Since M is R-torsion-free, $F \cong M \oplus M'$ as R-modules, and so we can choose $t \in \mathrm{Hom}_R(F, N)$ such that $t(m') = hv(m')$, $m' \in M'$. If we set

$$w = \sum_{y \in G} y^{-1} t y,$$

it is easy to see that $w \in \mathrm{Hom}_{RG}(F, N)$ and that $\theta(w) = a \cdot h$. Hence (2) is true, and therefore $a \in d(M)$.

Conversely, let $a \in d(M)$. Then in particular,

$$a \cdot \mathrm{Hom}_{RG}(M', M') \subset \theta\{\mathrm{Hom}_{RG}(F, M')\},$$

and so $a \cdot 1_{M'} = \theta(w)$ for some $w \in \mathrm{Hom}_{RG}(F, M')$. Set

$$w(1 \otimes m) = \sum_{i=2}^{g} (x_i - 1) \otimes u_i(m) \qquad (m \in M).$$

Then each $u_i \in \mathrm{Hom}_R(M', M')$, and from the relation $a \cdot 1_{M'} = \theta(w)$, one finds that (1) holds with u chosen as $-u_2$. This completes the proof of the lemma.

Assume now that M is absolutely irreducible, and suppose that relative to some R-basis, M affords the matrix representation

$$x \to A(x) = (a_{ij}(x)) \qquad (x \in G),$$

where the indices i and j range from 1 to (KM: K). One of the standard orthogonality relations states that

$$\sum_{x \in G} a_{ij}(x)\, a_{rs}(x^{-1}) = d_M \delta_{is} \delta_{jr}. \tag{3}$$

Let U be the diagonal matrix with diagonal entries 0, ⋯, 0, 1, of the same size as each A(x), and let I be the identity matrix of that size. Then (3) yields

$$\sum_{x \in G} A(x^{-1}) U A(x) = d_M I.$$

If u is the R-endomorphism of M defined by U, the above may be written as

$$\sum_{x \in G} x^{-1} u\,(xm) = d_M m \qquad (m \in M).$$

By virtue of the preceding lemma, we conclude from this that $d_M \in d(M)$.

Furthermore, let $a \in d(M)$. Then there exists a matrix $V = (v_{ij})$ with entries in R, such that

$$\sum_{x \in G} A(x^{-1}) V A(x) = aI.$$

By using (3), this gives

$$a = d_M \cdot \sum v_{ii},$$

and so a is a multiple of d_M. This completes the proof of the theorem.

COROLLARY. *Let* N *be an absolutely irreducible* R-*torsion-free* RG-*module. Then*

$$d_N \cdot \mathrm{Ext}(M, N) = 0 \qquad \textit{for all } M,$$

and any element of R *with this property must be a multiple of* d_N.

Proof. Let N* denote the contragredient of N, and M* that of M. Then N* is also absolutely irreducible, and $d_{N^*} = d_N$. The result now follows from the isomorphism $\mathrm{Ext}(M, N) \cong \mathrm{Ext}(N^*, M^*)$.

REFERENCES

1. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Wiley, New York, 1962.

2. D. G. Higman, *Relative cohomology*, Canad. J. Math. 9 (1957), 19-34.

3. ———, *On representations of orders over Dedekind domains*, Canad. J. Math. 12 (1960), 107-125.

4. R. G. Swan, *Induced representations and projective modules*, Ann. of Math. (2) 71 (1960), 552-578.

University of Illinois
Urbana, Illinois