RELATIVE INVERSES IN BAER *-SEMIGROUPS

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1. INTRODUCTION

In this note we give some results on relative inverses in Baer *-semigroups which seem to have interesting consequences when interpreted in terms of the Moore-Penrose generalized inverse for (square) matrices. In [7] Penrose shows that if A is an n-by-m matrix over the real or the complex field, then there exists a unique m-by-n matrix A^+, called the *generalized inverse* of A, such that $A = AA^+A$, $A^+ = A^+AA^+$, $(AA^+)^* = AA^+$, and $(A^+A)^* = A^+A$ (where A^* is the transposed conjugate of a matrix A). Our methods apply only to square matrices, but it seems plausible that they can be extended to rectangular matrices by some device such as the adjunction of rows or columns of zeros.

We begin with several definitions. An *involution semigroup* is a semigroup S together with a mapping $*: S \to S$ such that (i) $(xy)^* = y^*x^*$ and (ii) $x^{**} = x$ for all $x, y \in S$. A *projection* in such an S is an element $e \in S$ with $e = e^2 = e^*$. We denote by $P = P(S)$ the partially ordered set of all projections in S, the partial order being defined by the condition that $e \leq f$ if and only if $e = ef$ $(e, f \in P)$.

A *Baer *-semigroup* is an involution semigroup S with a two sided zero 0 having the following property: For each element $s \in S$ there exists a projection $s' \in P$ such that $\{x \in S \mid sx = 0\} = s'S$. It is clear that the projection s' is uniquely determined by s, since two principal right ideals generated by projections in an involution semigroup S are equal if and only if the projections are equal. The notion of a Baer *-semigroup was introduced (in a slightly more general form) in [1].

If a is an element of the Baer *-semigroup S, then we say that a is **-regular* in S if there exists a (necessarily unique) element a^+ in S such that $a = aa^+a$, $a^+ = a^+aa^+$, $aa^+ = (a^*)''$, and $a^+a = a''$. (Actually, the concept of *-regularity will be given a slightly different, but equivalent, working definition in Section 4.) The element a^+ will be called the *relative inverse* of a in S; it clearly specializes to the Moore-Penrose generalized inverse of a if S is the Baer *-semigroup of all n-by-n matrices over the real or the complex field.

If a and b are *-regular elements of the Baer *-semigroup S, it is natural to ask for conditions which will guarantee that ab is also *-regular in S. In Section 7 we show that *-regularity of the product $a''(b^*)''$ is a necessary and sufficient condition for *-regularity of the product ab; and we obtain, in this case, the formula

$$(ab)^+ = (ab)''\, b^+ (a''(b^*)'')^+ a^+ (b^* a^*)''$$

for the relative inverse of ab. This formula reduces the problem of computing the relative inverse of the product of two (square) matrices to the problem of computing the relative inverse of the product of two projection matrices.

If a is an invertible element in the Baer *-semigroup S, then clearly $a^{-1} = a^+$. If both a and b are invertible, then $(ab)^{-1} = b^{-1}a^{-1}$. Hence it is natural to ask for conditions equivalent to $(ab)^+ = b^+a^+$ if a and b are not necessarily invertible, but

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only *-regular. Such conditions are obtained in Section 7. If we specialize to (square) matrices, the conditions are as stated in the following theorem: *Let* A *and* B *be two* n-*by*-n *matrices over the real or the complex field. Let* M *be the row-space of* A, *and let* N *be the column space of* B. *Put* $K = M \cap N$. *Then* $(AB)^+ = B^+A^+$ *if and only if the following conditions hold:* (i) $M \cap K^\perp$ *is orthogonal to* $N \cap K^\perp$. (ii) K *reduces both* A^*A *and* BB^*.

2. BASIC THEOREMS ON BAER *-SEMIGROUPS

In what follows, the symbol S will always denote a Baer *-semigroup. In the present section we collect some facts about S that will be useful in the sequel.

We define $P'(S) = P$ by the condition $P' = \{s' \mid s \in S\}$. Note that $P' \subset P$. A projection e in P is said to be *closed* provided $e = e''$. We show (Theorem 1) that a projection e in P is closed if and only if it belongs to P' and that the set P' of all closed projections in S forms an orthomodular lattice. For $e, f \in P'$, $e \perp f$ means $e \leq f'$.

We denote the join and meet operations in a lattice L by $\vee$ and $\wedge$, respectively. An *orthocomplemented lattice* is a lattice L with a zero and a unit together with a mapping $': L \to L$ (called the *orthocomplementation*) such that (i) $e \vee e' = 1$, (ii) $e \wedge e' = 0$, (iii) $e \leq f \Rightarrow f' \leq e'$, and (iv) $e = e''$ for all $e, f \in L$. If L is an orthocomplemented lattice and if $e, f \in L$ with $e \leq f'$, then we say that e is *orthogonal* to f, and we write $e \perp f$. An ordered pair (e, f) of elements $e, f \in L$ is called a *modular pair* if

$$g \in L,\ g \leq f \Rightarrow g \vee (e \wedge f) = (g \vee e) \wedge f.$$

An *orthomodular lattice* is an orthocomplemented lattice in which every orthogonal pair is a modular pair. The latter condition is easily shown to be equivalent to the condition

$$e \leq f \Rightarrow f = e \vee (f \wedge e') \qquad (e, f \in L).$$

If L is an orthomodular lattice and if $e, f \in L$, then we say that e *commutes with* f (in symbols e C f) provided $(e \vee f') \wedge f = e \wedge f$. (Alternate formulations of the notion of commutativity are discussed in [3].) If M is a non-empty subset of L, we define

$$C(M) = \{e \in L \mid e\ C\ f \text{ for every } f \in L\}.$$

The set C(L) is called the *center* of L, and L is called *irreducible* in case $C(L) = \{0, 1\}$.

The facts stated in the next theorem are either proved in [1] or follow immediately from the results in [1].

THEOREM 1.

(i) *For* $e, f \in P(S)$, $e \leq f \Rightarrow f' \leq e'$.

(ii) *For* $e \in P(S)$, $e \leq e''$.

(iii) *For* $e \in P(S)$, $e' = e'''$.

(iv) *If* e *is a closed projection in* P(S), *and if* $a \in S$, *then* $ae = a$ *if and only if* $a'' \leq e$.

(v) *For* $a \in S$, $a' = a'''$.

(vi) $e \in P'(S)$ *if and only if* e *is a closed projection in* $P(S)$.

(vii) *For* $a, b \in S$, $ab = 0$ *if and only if* $a'' \perp (b^*)''$.

(viii) $0'$ (*which we write henceforth as* 1) *is a unit for* S.

(ix) *If* $a, b \in S$, *if* $b = b^*$, *and if* $ab = ba$, *then* $ab' = b'a$.

(x) $P'(S)$ *is an orthomodular lattice under the partial order inherited from* $P(S)$, *with* $e \to e'$ *as orthocomplementation.*

(xi) *For* $e, f \in P'(S)$, $e \wedge f = (e'f)'f$ *and* $e \vee f = [(ef')'f']'$.

(xii) *For* $a, b \in S$, $(ab)'' = [(a'')b]''$.

(xiii) *For* $a \in S$, $(a^*a)'' = a''$.

(xiv) *For* $a, b \in S$, $(ab)'' \leq b''$.

(xv) *For* $e, f \in P'(S)$, $(ef)'' = (e \vee f') \wedge f$.

(xvi) *For* $e \in P'(S)$ *and* $a \in S$, $[(ea^*)'a]'' \leq e' \wedge a''$.

(xvii) *If* $a \in S$, *and if* $\{e_\alpha\}$ *is a collection of elements in* $P'(S)$ *such that* $e = \bigvee_\alpha e_\alpha$ *exists in* $P'(S)$, *then* $\bigvee_\alpha (e_\alpha a)''$ *exists in* $P'(S)$ *and equals* $(ea)''$.

THEOREM 2.

(i) *If* $e \in P'(S)$, $a \in S$, *then* $ea = ae$ *if and only if* $(ea)'' \vee (ea^*)'' \leq e$.

(ii) *For* $e, f \in P'(S)$, $ef = fe$ *if and only if* $e\, C\, f$.

(iii) *For* $e, f, g \in P'(S)$, *if any two of the three relations* $e\, C\, f$, $f\, C\, g$ *or* $e\, C\, g$ *hold, then* $(e \vee f) \wedge g = (e \wedge g) \vee (f \wedge g)$ *and* $(e \wedge f) \vee g = (e \vee g) \wedge (f \vee g)$.

(iv) *If* $\{e_\alpha\}$ *is a collection of elements in* $P'(S)$, *if* $e = \bigvee_\alpha e_\alpha$ *exists in* $P'(S)$ *and if* $a \in S$ *is such that* $e_\alpha a = ae_\alpha$ *for every* α, *then* $ea = ae$.

Proof. (i): Note that by part (iv) of Theorem 1, $(ea)'' \leq e$ if and only if $ea = eae$ and $(ea^*)'' \leq e$ if and only if $ae = eae$, from which the result follows. (ii): By the definition of e commutes with f, and by part (xv) of Theorem 1, e commutes with f if and only if $(ef)'' = e \wedge f$. Also, by part (xv) of Theorem 1, it is plain that $e \wedge f \leq (ef)'' \leq f$ always; hence, e commutes with f if and only if $(ef)'' \leq e$. An application of part (i) of the present theorem now completes the proof that $ef = fe$ if and only if e commutes with f. Part (iii) is proved in [3]. (iv): Since $e_\alpha a = ae_\alpha$, $(e_\alpha a)'' \leq e_\alpha$ and $(e_\alpha a^*)'' \leq e_\alpha$ by part (i) of the present theorem. Taking suprema on both sides of these inequalities and using part (xvii) of Theorem 1, we see that $(ea)'' \leq e$ and $(ea^*)'' \leq e$. It follows by part (i) of the present theorem that $ea = ae$.

Part (iii) of Theorem 2 is very useful when making calculations involving lattice polynomials over $P'(S)$. For example, with its use one easily derives the following corollary.

COROLLARY. *Let* $e \in P'(S)$.

(i) $C(e) = \{(e \vee g) \wedge (e' \vee g) \mid g \in P'(S)\}$.

(ii) *The set of all elements in* $P'(S)$ *that are complements of* e *is*

$$\{[(e \wedge f)' \wedge f] \vee (e \vee f)' \mid f \in P'(S)\}.$$

(iii) $e \in C(P'(S))$ *if and only if* e *has a unique complement in* $P'(S)$.

(iv) $C(P'(S))$ *is a Boolean lattice.*

The following lemma will be useful:

LEMMA 3. *For* $e, f \in P'(S)$, e commutes with f *if and only if*

$$e \wedge (e \wedge f)' \perp f \wedge (e \wedge f)'.$$

Proof. If the relation $e\, C\, f$ holds, then the relations $f'\, C\, e$ and $e'\, C\, f$ obtain; so

$$e \wedge (e \wedge f)' = e \wedge f', \quad f \wedge (e \wedge f)' = f \wedge e',$$

$$\text{and } e \wedge (e \wedge f)' \perp f \wedge (e \wedge f)'.$$

Conversely, if $e \wedge (e \wedge f)' \leq f' \vee (e \wedge f)$, then

$$(f' \vee e') \wedge e \leq [f' \vee (e \wedge f)] \wedge (e \wedge f)' = f'.$$

By part (xv) of Theorem 1, $(f'e)'' \leq f'$; hence, f' commutes with e by part (i) of Theorem 2. Therefore e commutes with f and the proof is complete.

In dealing with S, the following notation will be useful: If M is a non-empty subset of S, then

$$Z(M) = \{s \in S \mid sx = xs \text{ for all } x \in M\}.$$

Note, in particular, that if $M \subset P'(S)$, then $C(M) = Z(M) \cap P'(S)$.

A subset S_1 of S is called a *Baer *-subsemigroup* of S provided (i) S_1 is a subsemigroup of S, (ii) $S_1^* = S_1$, and (iii) $s \in S_1 \Rightarrow s' \in S_1$. Evidently, such an S_1 is a Baer *-semigroup in its own right with $P'(S_1) = S_1 \cap P'(S)$.

As in [1] we call S a *complete* Baer *-semigroup if the right annihilating ideal of any non-empty subset M of S is a principal right ideal generated by a projection. It is clear from [1, Lemma 2, p. 650] that S *is a complete Baer *-semigroup if and only if* $P'(S)$ *is a complete lattice.*

LEMMA 4. *Let* M *and* N *be non-empty subsets of a Baer *-semigroup* S.

(i) $M \subset N \Rightarrow Z(N) \subset Z(M)$.

(ii) $M \subset Z(Z(M))$.

(iii) $Z(M) = Z(Z(Z(M)))$.

(iv) *If* $M = M^*$, *then* $Z(M)$ *is a Baer *-subsemigroup of* S, *and*

$$P'(Z(M)) = Z(M) \cap P'(S)$$

is closed under the computation of arbitrary suprema and infima in $P'(S)$ *provided these exist in* $P'(S)$. *Consequently, if* S *is complete, so is* $Z(M)$.

Proof. Conclusions (i), (ii) and (iii) are clear. From part (ix) of Theorem 1 and from part (iv) of Theorem 2, (iv) follows immediately.

If L is any orthomodular lattice, we follow [1] by saying that S *coordinatizes* L if there exists an orthocomplementation preserving lattice isomorphism from L onto $P'(S)$. In [1] we proved that *every orthomodular lattice* L *can be coordinatized by* *some (not necessarily unique) Baer *-semigroup* S. It follows that *any general theorem about* P'(S) *is a theorem about orthomodular lattices in general.*

LEMMA 5. *If* $e \in P'(S)$, *if* $a \in S$ *and if the relation* $e \, C \, (a^*)''$ *holds, then* $ea = (e \wedge (a^*)'')a$.

Proof. Since the relation $e \, C \, (a^*)''$ holds, part (ii) of Theorem 2 yields the equality $e(a^*)'' = (a^*)''e$. If two projections in P(S) commute, it is plain that their product is their infimum in P(S). Noting that $e, (a^*)'' \in P'(S)$ and invoking [1, Lemma 2, p. 650], we conclude that $e \wedge (a^*)'' = e(a^*)''$; whence,

$$ea = e(a^*)''a = (e \wedge (a^*)'')a.$$

According to part (xvi) of Theorem 1, $[(ea^*)'a]'' \leq e' \wedge a''$ always for all $e \in P'(S)$ and for all $a \in S$. It is natural to inquire under what conditions equality holds. One answer is given by the next theorem, which involves the notion of a *range-closed* element in S. Following [2, p. 890], we say that the element $a \in S$ is *range-closed* if the condition $g \in P'(S)$ with $g \leq a''$ and $(ga^*)'' = (a^*)''$, necessarily implies $g = a''$.

THEOREM 6. *Let* a *be an element of a Baer *-semigroup* S. *Then the following conditions are mutually equivalent:*

(i) $[(ea^*)'a]'' = e' \wedge a''$ *for all* $e \in P'(S)$.

(ii) a *is range-closed in* S.

(iii) *For* $e, f \in P'(S)$, $(ea^*)'' = (fa^*)'' \Rightarrow e \vee a' = f \vee a'$.

(iv) *If* $e \in P'(S)$ *with* $e \leq a''$, *there exists an* $f \in P'(S)$ *with* $f \leq (a^*)''$ *such that* $(fa)'' = e$.

(v) *For* $e, f \in P'(S)$ *with* $e, f \leq a''$, $(ea^*)'' = (fa^*)'' \Rightarrow e = f$.

Proof. Because of the remark directly preceding Lemma 1 in [2, p. 891], the equivalence of (i), (ii) and (iii) follows from [2, Theorem 2, p. 891]. To prove that (i) implies (iv), put $f = (e'a^*)' \wedge (a^*)''$. Since $(e'a^*)' \geq (a^*)'$ by part (xiv) of Theorem 1, the relation $(e'a^*)' \, C \, (a^*)''$ obtains; hence, by Lemma 5, $(fa)'' = [(e'a^*)'a]''$. Applying (i) to the latter equality, we see that $(fa)'' = e \wedge a'' = e$. To prove (iv) implies (ii), suppose that $g \leq a''$ and that $(ga^*)'' = (a^*)''$. Put $e = a'' \wedge g'$ and note that it suffices to prove $e = 0$. By (iv), there exists an $f \in P'(S)$ with $f \leq (a^*)''$ such that $(fa)'' = e$, so $(fa)' = a' \vee g$. From the latter equation and part (xvii) of Theorem 1, we obtain the relations

$$[(fa)'a^*]'' = (a'a^*)'' \vee (ga^*)'' = 0 \vee (a^*)''.$$

It follows that

$$(a^*)'' = [(fa)'a^*]'' \leq f', \quad \text{so } f \leq (a^*)'.$$

Since also $f \leq (a^*)''$, $f = 0$; so $e = (fa)'' = 0$. The equivalence of (iii) and (v) is trivial.

3. EXAMPLES

In this section we give two nontrivial examples of Baer *-semigroups.

Example 1. Let H be a Hilbert space, and let $\mathscr{B}(H)$ represent the multiplicative semigroup of all bounded operators on H. Then $S = \mathscr{B}(H)$ is a Baer *-semigroup where $*: S \to S$ is taken, as usual, to represent the passage from a bounded operator to its adjoint. In this case, the projections are the orthogonal projections onto closed linear subspaces of H, so P(S) is isomorphic to the lattice of closed linear subspaces of H. If we regard the operators in $\mathscr{B}(H)$ as operating on the *right*, then for $T \in \mathscr{B}(H)$, T' is the projection onto the orthogonal complement of the range of T, and T'' is the projection onto the closure of the range of T. Also, $(T^*)'$ is the projection onto the null space of T.

If E is any projection in $\mathscr{B}(H)$, then $E' = 1 - E$. Thus $E = E''$; hence all projections are closed, and $P(S) = P'(S)$. We showed in [2, p. 890] that for $T \in \mathscr{B}(H)$, T is range-closed if and only if the range of T is a closed linear subspace of H. We remark that if $T \in \mathscr{B}(H)$, then the operator T is determined up to a nonzero scalar factor by the mapping $E \to (ET)''$ from $P'(\mathscr{B}(H))$ into $P'(\mathscr{B}(H))$.

Various Baer *-subsemigroups of $\mathscr{B}(H)$ are of interest, for example, any weakly closed self-adjoint subalgebra of $\mathscr{B}(H)$ or the subset of $\mathscr{B}(H)$ consisting of all those operators whose norm does not exceed 1.

Example 2. Let X be any non-empty set, and let $\mathscr{R}(X)$ denote the set of all subsets of $X \times Y$ (these subsets being regarded as binary relations). For $R, S \in \mathscr{R}(X)$, define

$$RS = \{(x, y) \in X \times X \mid \text{for some } z \in X,\ (x, z) \in R \text{ and } (z, y) \in S\},$$

and define $R^* = \{(y, x) \in X \times X \mid (x, y) \in R\}$. Then $S = \mathscr{R}(X)$ is a Baer *-semigroup. The projections in $\mathscr{R}(X)$ are the equivalence relations defined on subsets of X. If $R \in \mathscr{R}(X)$, then R' is the identity relation restricted to the complement of the range of R, and R'' is the identity relation restricted to the range of R. Thus, in this case, $P(S) \neq P'(S)$. Note that $P'(S)$ is isomorphic to the Boolean lattice of all subsets of X.

It is easy to show that a relation $R \in \mathscr{R}(X)$ is range-closed if and only if there exist relations R_1, R_2 such that $R = R_1 \cup R_2$, the domain of R_1 is disjoint from the domain of R_2, the range of R_1 equals the range of R, and R_1 is single-valued, (that is, R_1 is a function). In particular, every single-valued relation in $\mathscr{R}(X)$ is range-closed.

Various Baer *-subsemigroups of $\mathscr{R}(X)$ are of interest; for example Bij(X), which we define to be the subset of $\mathscr{R}(X)$ consisting of all those relations in $\mathscr{R}(X)$ that are bijective between their domains and ranges. In particular, *every* element in Bij(X) is range-closed. Of course, Bij(X) is an example of an *inverse semigroup* in the sense of Preston [8].

A second example is afforded by the situation in which X is the Stone space corresponding to a Boolean lattice B. Following Halmos [4], we say that $R \in \mathscr{R}(X)$ is *Boolean* in case $(x)R$ is closed for every $x \in X$ and $(M)R^*$ is closed and open for every closed and open $M \subset X$. The subset of $\mathscr{R}(X)$ consisting of all those relations R such that both R and R^* are Boolean relations is a Baer *-subsemigroup of $\mathscr{R}(X)$. Actually, this Baer *-subsemigroup is isomorphic to S(B). (For the notation S(B), see [1].)

4. *-REGULAR ELEMENTS IN A BAER *-SEMIGROUP

Taking [2, Section 3, p. 892] as our point of departure, we define an element $a \in S$ to be *right *-regular* in S if $aS = (a^*)'' S$, and we define $a \in S$ to be *left *-regular* in S if $Sa = Sa''$. If $a \in S$ is both right and left *-regular in S, then we say that a is **-regular* in S. If every element in S is *-regular in S, then we call S a **-regular* Baer *-semigroup. In [2, Theorem 10, p. 894] we showed that *an orthomodular lattice* L *is modular if and only if* L *can be coordinatized by a *-regular Baer *-semigroup;* hence, *-regular Baer *-semigroups exist in abundance.

LEMMA 7. *If* a *is an element of a Baer *-semigroup* S, *the following are mutually equivalent:* (i) a^* *is right *-regular.* (ii) a *is left *-regular.* (iii) *There exists a closed projection* e *in* S *such that* $Sa = Se$. (iv) *There exists an element* $x \in S$ *with* $xa = a''$. (v) *There exists an element* $x \in S$ *such that* $x = a'' s(a^*)''$, $(x^*)'' = a''$, *and* $xa = a'' = a^* x^*$.

Proof. That (i) is equivalent to (ii) is straightforward. That (ii) implies (iii) is obvious. To prove (iii) implies (iv), suppose that $Sa = Se$ and $e = e''$. Since $Sa = Se$, there exist elements $x, y \in S$ with $xa = e$, $ye = a$. Thus,

$$e = e'' = (xa)'' \leq a'', \quad \text{and } a'' = (ye)'' \leq e'' = e$$

by part (xiv) of Theorem 1. Hence $e = a''$, and $xa = a''$. To prove (iv) implies (v), suppose that $ya = a''$, and put $x = a'' y(a^*)''$. Note that

$$xa = a'' y(a^*)'' a = a'' ya = (a'')^2 = a'' \quad \text{and } x = a'' x(a^*)''.$$

Since a'' is a projection, $xa = a'' = (a'')^* = a^* x^*$. Hence

$$a'' = (a^* x^*)'' \leq (x^*)''$$

by part (xiv) of Theorem 1. Since $x = a'' x$, $x^* = x^* a''$. Therefore

$$(x^*)'' = (x^* a'')'' \leq (a'')'' = a''$$

by part (xiv) of Theorem 1 again. Consequently, $(x^*)'' = a''$, which completes the proof that (iv) implies (v). Finally, to prove (v) implies (ii), note that

$$xa = a'' \Rightarrow Sa'' \subset Sa.$$

Since $a = aa''$, $Sa \subset Sa''$ always; hence, $Sa = Sa''$.

We call an element $a \in S$ *left *-semiregular* if the statements $b, c \in S$ and $ab = ac$ imply $a'' b = a'' c$. We say that $a \in S$ is *right *-semiregular* if the statements $b, c \in S$ and $ba = ca$ imply $b(a^*)'' = c(a^*)''$; that is, a is right *-semiregular if and only if a^* is left *-semiregular. Of course, we call $a \in S$ **-semiregular* if it is both right and left *-semiregular, and we call S itself **-semiregular* if every element $a \in S$ is *-semiregular.

Note that the Baer *-semigroup $\mathscr{B}(H)$ of Example 1 of Section 3 is *-semiregular, but it is *-regular if and only if H is finite-dimensional. Also, an element R in the Baer *-semigroup $\mathscr{R}(X)$ of Example 2 of Section 3 is left *-semiregular if and only if R is left *-regular, and R is left *-regular if and only if R is range-closed.

LEMMA 8. *If* a *is left* *-*regular in a Baer* *-*semigroup* S, *then* a *is left* *-*semiregular and range-closed in* S.

Proof. Since a is left *-regular in S, there exists an $x \in S$ such that $xa = a''$. Thus, if $ab = ac$, then $xab = xac$, so $a''b = a''c$. Hence, a is left *-semiregular. Now suppose $g \in P'(S)$ with $g \leq a''$ and $(ga^*)'' = (a^*)''$. We want to prove that $g = a''$. By part (xii) of Theorem 1,

$$g = ga'' = (ga'')'' = (ga^* x^*)'' = [(ga^*)'' x^*]''.$$

Thus,

$$g = [(a^*)'' x^*]'' = (a^* x^*)'' = (a'')'' = a'',$$

and the lemma is proved.

If every element $a \in S$ is range-closed, we say that S itself is *range-closed.* We proved in [2, Theorem 10, p. 894] that *an orthomodular lattice* L *is modular if and only if* L *can be coordinatized by a range-closed Baer* *-*semigroup* S. A consequence of Lemma 8 is that *every* *-*regular Baer* *-*semigroup is* *-*semiregular and range-closed.*

LEMMA 9. *If* $a \in S$, *the necessary and sufficient condition for the existence of a (not necessarily closed) projection* e *with* $Sa = Se$ *is that* $aS = aa^*S$. *Hence, if all projections in* S *are closed, then* a *is left* *-*regular if and only if* $aS = aa^*S$.

Proof. If $aS = aa^*S$, then there exists an $x^* \in S$ with $a = aa^*x^*$. Then

$$xa = xaa^*x^* = (xa)(xa)^*, \quad \text{so } xa = (xa)^* = (xa)^2 = a^*x^*.$$

Put $e = xa$, and note that $ae = axa = aa^*x^* = a$. It follows that $Sa = Se$. Conversely, if $Sa = Se$, then $ae = a$ and $e = xa$ for some $x \in S$. Thus, $e = a^*x^*$ so that

$$a = ae = aa^*x^* \quad \text{and } aS = aa^*S.$$

LEMMA 10. *In a* *-*semiregular Baer* *-*semigroup* S, *all projections are closed.*

Proof. Let e be a projection in the *-semiregular Baer *-semigroup S. By part (ii) of Theorem 1, $e = ee = ee''$; hence, $e''e = e''e''$, that is, $e = e''$.

5. THE RELATIVE INVERSE OF A *-REGULAR ELEMENT

If $a \in S$ is *-regular, then there exists an element a^+ in S, called the *relative inverse* of a, which has many of the properties that a^{-1} would have if it existed. This statement is a loose version of the next theorem.

THEOREM 11. *If* a *is* *-*regular in a Baer* *-*semigroup* S, *then there exists a unique element* a^+ *in* S *such that* $a^+a = a''$ *and* $a^+(a^*)'' = a^+$. *The element* a^+, *(called the relative inverse of* a), *may also be characterized as the unique solution* x *of the simultaneous equations* $a = aa^*x^*$, $x = xx^*a^*$.

Proof. The existence of an element a^+ satisfying the conditions $a^+a = a''$ and $a^+(a^*)'' = a^+$ follows from part (v) of Lemma 7 and the fact that a is left *-regular. To prove the uniqueness of a^+, suppose that $xa = a''$ and $x(a^*)'' = x$ also. Since a is right *-regular, there exists by Lemma 7 an element $y \in S$ such that $ay = (a^*)''$ and $y = a''y$. Hence,

$$a^+ = a^+(a^*)'' = a^+ay = a''y = xay = x(a^*)'' = x.$$

This proves the uniqueness of a^+ and at the same time proves that $y = a^+$, so that $aa^+ = (a^*)''$ and $a''a^+ = a^+$. We now see that

$$aa^*(a^+)^* = a(a^+a)^* = a(a'')^* = aa'' = a, \quad \text{and } a^+(a^+)^*a^* = a^+(aa^+)^* = a^+(a^*)'' = a^+;$$

hence, a^+ is a solution of $a = aa^*x^*$, $x = xx^*a^*$. Finally, if $a = aa^*x^*$ and $x = xx^*a^*$,

$$a'' = a^+a = a^+aa^*x^* = a''a^*x^* = a^*x^*,$$

so $xa = a''$. Also,

$$x(a^*)'' = xx^*a^*(a^*)'' = xx^*a^* = x,$$

so $x = a^+$ by the uniqueness of a^+.

COROLLARY. *Let* $a \in S$ *be *-regular, and let* a^+ *be its relative inverse. Then* (i) $a^+a = a''$, (ii) $aa^+ = (a^*)''$, (iii) $a = aa^+a$, (iv) $a^+ = a^+aa^+$, (v) $(a^+)'' = (a^*)''$, (vi) $((a^+)^*)'' = a''$, (vii) a^+ *is *-regular with* $a^{++} = a$, *and* (viii) $a^{*+} = a^{+*}$.

Consider, for a moment, the Baer *-semigroup $\mathscr{B}(H)$ of Example 1, Section 3. If an operator $T \in \mathscr{B}(H)$ is *-regular, then by Lemma 8 its range is closed. Conversely, it is easy to show that if the range N of T is closed, then T is *-regular in $\mathscr{B}(H)$. In fact, if N is closed, T^+ can be constructed as follows: Let M be the orthogonal complement of the null space of T. Then, in a well-known way, T induces an isomorphism $T_1: M \simeq N$. Let E be the orthogonal projection of H onto the closed subspace N of H and define $T^+ = ET_1^{-1}$. One easily shows that T^+ is effective as the relative inverse of T. Hence, *an operator* T *in the Baer *-semigroup* $\mathscr{B}(H)$ *is *-regular if and only if it has a closed range.* Incidentally, this gives a proof of the well-known result that an operator in $\mathscr{B}(H)$ has a closed range if and only if its adjoint has a closed range.

We have already mentioned that a relation R in the Baer *-semigroup $\mathscr{R}(X)$ of Example 2, Section 3 is left *-regular if and only if it is range-closed. It follows easily from this that R *is *-regular if and only if* R *belongs to the Baer *-subsemigroup* Bij(X) *of* $\mathscr{R}(X)$, (and in this case $R^+ = R^* = R^{-1}$).

We say that an element $a \in S$ is *invertible* provided a is *-regular and $a'' = (a^*)'' = 1$. If a is invertible, we write a^+ as a^{-1}. We call $a \in S$ *unitary* if a is invertible and $a^{-1} = a^*$, and we call $a \in S$ *partially unitary* if a is *-regular and $a^+ = a^*$. In $\mathscr{B}(H)$ the partially unitary operators are the partial isometries of the Hilbert space H; while in $\mathscr{R}(X)$, every *-regular relation is partially unitary.

Note that any closed projection e in S is partially unitary with $e = e^* = e^+$.

THEOREM 12. *Let* a *be *-regular in a Baer *-semigroup* S *and let* $g \in P'(S)$. *Then*

$$(ga^+)'' = [(g' \wedge a'')a^*]' \wedge (a^*)''.$$

Proof. Put $h = [(g' \wedge a'')a^*]'$. Since $h' \leq (a^*)''$, $h \, C \, (a^*)''$; so by Lemma 5,

$$[(h \wedge (a^*)'')a]'' = (ha)'' .$$

Since a is *-regular, a is range-closed by Lemma 8; hence, by part (i) of Theorem 6 and part (xv) of Theorem 1,

$$(ha)'' = (g' \wedge a'')' \wedge a'' = (g \vee a') \wedge a'' = (ga'')'' .$$

On the other hand,

$$((ga^+)'' a)'' = (ga^+ a)'' = (ga'')'' .$$

It follows that

$$[(h \wedge (a^*)'')a]'' = ((ga^+)'' a)'' .$$

By Lemma 8 again, a^* is range-closed; so by part (v) of Theorem 6, the latter equation implies that $h \wedge (a^*)'' = (ga^+)''$ (since $h \wedge (a^*)'' \leq (a^*)''$ and

$$(ga^+)'' \leq (a^+)'' = (a^*)'') .$$

This completes the proof.

COROLLARY. *Let* T *be a bounded operator with a closed range on the Hilbert space* H, *let* N *be the range of* T, *let* M *be the orthogonal complement of the null space of* T, *and let* W *be any closed linear subspace of* H. *Then*

$$\overline{(W)T^+} = [(W^\perp \cap N)T^*]^\perp \cap M .$$

The above corollary seems to be of interest because, by a remark which we made in the discussion following Example 1 in Section 3, a knowledge of the mapping $W \to \overline{(W)T^+}$ determines T^+ up to a nonzero scalar factor. This should be of some use in constructing an algorithm for the computation of, say, the Moore-Penrose generalized inverse of a matrix.

6. THE *-CANCELLATION LAW

The Baer *-semigroup $\mathscr{B}(H)$ has an important property not possessed by Baer *-semigroups in general; namely, if $A, B \in \mathscr{B}(H)$ with $AA^* = BA^* = BB^*$, then $A = B$. We call this property the **-cancellation law*.

If S is any Baer *-semigroup, then we say that S is a **-cancellation semigroup* or that S satisfies the **-cancellation law* if the statements $a, b \in S$ and $aa^* = ba^* = bb^*$ imply $a = b$.

By a *Baer *-ring* we mean a ring R equipped with an anti-automorphic involution $*: R \to R$ whose multiplicative semigroup is a Baer *-semigroup. (This definition differs slightly from the definition given by Kaplansky [5, p. 17], where he defines what we would prefer to call a *complete* Baer *-ring.)

THEOREM 13. *If* R *is a Baer *-ring, then* R *satisfies the *-cancellation law,* R *is *-semiregular and all projections in* R *are closed.*

Proof. Suppose $a, b, c \in R$ with $ab = ac$. Then, $a(b - c) = 0$, so

$$(b - c) = a'(b - c) .$$

Hence

$$a''(b - c) = a'' a'(b - c) = 0, \quad \text{that is,} \quad a'' b = a'' c.$$

This proves that R is *-semiregular and, at the same time, (Lemma 10) that all projections in R are closed. Next, we remark that in any Baer *-semigroup S, if $aa^* = 0$, then by part (xiii) of Theorem 1, $(a^*)'' = (aa^*)'' = 0$. Consequently

$$a^* = a^*(a^*)'' = a^* 0 = 0;$$

hence that $aa^* = 0$ implies $a = 0$. Suppose now that

$$aa^* = ba^* = bb^* \quad \text{for } a, b \in R.$$

Then,

$$aa^* = (aa^*)^* = (ba^*)^* = ab^*.$$

But, that

$$(a - b)(a - b)^* = aa^* - ab^* - ba^* + bb^* = 0$$

implies $a - b = 0$ by the above remark. It follows that R satisfies the *-cancellation law.

THEOREM 14. *Let* S *be a *-semiregular Baer *-semigroup. Then* S *satisfies the *-cancellation law if and only if for* $e, f \in P'(S)$, *the equality* $e = efe$ implies that $e \leq f$.

Proof. If S satisfies the *-cancellation law and if $e = efe$ for $e, f \in P'(S)$, then

$$(ef)(ef)^* = e(ef)^* = ee^*;$$

hence, $ef = e$, that is, $e \leq f$. Conversely, suppose that $e = efe$ implies $e \leq f$ for $e, f \in P'(S)$. If $a, b \in S$ and $aa^* = ba^* = bb^*$, then $ab^* = bb^*$ and $aa^* = ba^*$; hence (by *-semiregularity), $ab'' = bb''$ and $aa'' = ba''$, that is, $b = ab''$ and $a = ba''$. It follows that $bb'' = ba''b''$ and $aa'' = ab''a''$. By *-semiregularity again, $b''b'' = b''a''b''$, and $a''a'' = a''b''a''$; that is, $b'' = b''a''b''$ and $a'' = a''b''a''$. By our hypothesis, $b'' \leq a''$ and $a'' \leq b''$; that is, $b'' = a''$. Hence,

$$a = ba'' = bb'' = b,$$

and the proof is complete.

It is interesting to note that *-cancellation can fail even in the presence of the condition of *-regularity. For example, let S be the Baer *-semigroup obtained from the Baer *-semigroup of all 2 by 2 matrices over the complex field by identifying those matrices that differ by a nonzero scalar factor. Then S is *-regular, but the *-cancellation law fails. The Baer *-semigroup $\mathscr{R}(X)$ does not satisfy the *-cancellation law unless X is trivial.

LEMMA 15. *If a Baer *-semigroup* S *satisfies the *-cancellation law, if* $e, f \in P'(S)$, *and if* fef *is a projection in* S, *then* e *commutes with* f.

Proof. Set $q = fef$. Then $qq^* = (fe)q^* = (fe)(fe)^*$; hence $q = fe$. Since $q = q^*$, $fe = ef$.

7. THE RELATIVE INVERSE OF A PRODUCT

In this section we consider the following question: If a and b are *-regular in S, is ab *-regular in S and what is the relationship between $(ab)^+$, a^+ and b^+?

For the purposes of the present section, we adopt the following hypotheses and notation once and for all: S *is a Baer *-semigroup,* a *and* b *are *-regular elements in* S, $e = a''$ *and* $f = (b^*)''$. Note that $b = fb$, $a = ae$, $e = a^+a$, and $f = bb^+$.

LEMMA 16. *If* eb *is *-regular in a Baer *-semigroup* S, *then* ab *is *-regular in* S *and* $(ab)^+ = (ab)^+ a^+(b^* a^*)''$.

Proof. Put $x = (eb)^+ a^+(b^* a^*)''$. Then

$$xab = (eb)^+ a^+((ab)^*)'' ab = (eb)^+ a^+ ab = (eb)^+ eb = (eb)''$$

$$= (a'' b)'' = (ab)''.$$

Also,

$$abx = ab(eb)^+ a^+(b^* a^*)'' = aeb(eb)^+ a^+ (b^* a^*)''$$

$$= a((eb)^*)'' a^+(b^* a^*)'' = a(b^* e)'' a^+(b^* a^*)''$$

$$= a((b^*)'' e)'' a^+ (b^* a^*)'' = a(fe)'' a^+(b^* a^*)''.$$

But, if we put

$$g = ((a^+(b^* a^*)'')^*)'' = ((b^* a^*)'' a^{+*})'' = (b^* a^* a^{*+})''$$

$$= (b^* a'')'' = ((b^*)'' a'')'' = (fe)'',$$

then

$$abx = aga^+(b^* a^*)'' = aa^+(b^* a^*)'' = (a^*)''(b^* a^*)''$$

$$= (b^* a^*)'' = ((ab)^*)''.$$

Thus, ab is *-regular. Since $x = x((ab)^*)''$, $x = (ab)^+$.

COROLLARY. *If* af *is *-regular in* S, *then* ab *is *-regular in* S, *and* $(ab)^+ = (ab)'' b^+(af)^+$.

Proof. In Lemma 16, replace a by b^*, b by a^* and e by f. Since af is *-regular, $(af)^* = fa^*$ is *-regular; so by Lemma 16 (with the indicated replacements), $(b^* a^*)^+ = (fa^*)^+ b^{*+}(ab)''$. Performing the operation * on both sides of the latter equation and using part (viii) of the corollary to Theorem 11, we obtain the desired result.

LEMMA 17. *If* $g \in P'(S)$ *with* gf **-regular in* S, *then* gb *is *-regular in* S, *and* $(gb)^+ = (gb)'' b^+(gf)^+$.

Proof. In the corollary to Lemma 16, replace a by g.

COROLLARY. *If* $g \in P'(S)$ *and* eg *is *-regular in* S, *then* ag *is *-regular in* S, *and* $(ag)^+ = (eg)^+ a^+(ga^*)''$.

Proof. In Lemma 17, replace b by a^* and f by e to conclude that

$$(ga^*)^+ = (ga^*)'' a^{*+}(ge)^+ .$$

Performing the operation * on both sides, we obtain the desired result.

THEOREM 18. *If* ef *is *-regular in a Baer *-semigroup* S, *then* ab *is *-regular in* S, *and*

$$(ab)^+ = (ab)'' b^+(ef)^+ a^+(b^* a^*)'' .$$

Proof. Since ef is *-regular in S, by Lemma 17, eb is *-regular in S and $(eb)^+ = (eb)'' b^+(ef)$. Applying Lemma 16, we see that ab *-regular in S and

$$(ab)^+ = (eb)^+ a^+(b^* a^*)'' = (eb)'' b^+(ef)^+ a^+(b^* a^*)'' .$$

Since $(eb)'' = (a''b)'' = (ab)''$, the theorem is proved.

LEMMA 19. *Let* x *and* x* *be range-closed in* S, *and let* $g \in P'(S)$. *Then*

$$(gx^*)'' = [(g' \wedge x'')x^*]' \wedge (x^*)''$$

if and only if $(gx^* x)'' = (gx'')''$.

Proof. Put $h = [(g' \wedge x'')x^*]$. Since $h' \leq (x^*)''$, h commutes with $(x^*)''$. Therefore by Lemma 5, part (i) of Theorem 6, and part (xv) of Theorem 1,

$$[(h \wedge (x^*)'')x]'' = (hx)'' = (g' \wedge x'')' \wedge x''$$
$$= (g \vee x') \wedge x'' = (gx'')'' .$$

Hence, if $(gx^*)'' = h \wedge (x^*)''$, then

$$(gx^* x)'' = ((gx^*)'' x)'' = (gx'')'' .$$

Conversely, suppose that $(gx^* x)'' = (gx'')''$. Applying the operation ' to both sides of the last equation and using part (xv) of Theorem 1, we find that

$$(gx^* x)' = (g' \wedge x'') \vee x' .$$

Multiplying both sides of this equation by x*, performing the operation " on both sides, and using part (xvii) of Theorem 1, we see that

$$[(gx^* x)' x^*]'' = [(g' \wedge x'')x^*]'' \vee (x' x^*)'' = [(g' \wedge x'')x^*]'' .$$

Since x* is range-closed, part (i) of Theorem 6 implies that

$$[(gx^* x)' x^*]'' = (gx^*)' \wedge (x^*)'' ;$$

therefore

$$(gx^*)' \wedge (x^*)'' = h', \quad \text{that is,} \quad (gx^*)'' \vee (x^*)' = h .$$

Forming the meet of both sides of the latter equation with $(x^*)''$ yields the desired result $(gx^*)'' = h \wedge (x^*)''$.

COROLLARY. *Let* x, x* *be range-closed, let* $g \in P'(S)$, *and let* $k = (gx'')''$. *Then*

$$(gx^*)'' = [(g' \wedge x'')x^*]' \wedge (x^*)''$$

if and only if $(kx^*x)'' = k$.

Proof. $(kx^*x)'' = ((gx'')''x^*x)'' = (gx''x^*x)'' = (gx^*x)''$.

LEMMA 20. *Let* ab *be *-regular, and suppose that* $(ab)^+ = b^+a^+$. *Put* $h = (fe)''$, $k = (ef)''$. *Then,* $(ha^*a)'' = h$ *and* $(kbb^*)'' = k$.

Proof. By Theorem 12,

$$\begin{aligned}(fa^*)'' &= ((b^*)''a^*)'' = (b^*a^*)'' = ((ab)^*)'' \\ &= ((ab)^+)'' = (b^+a^+)'' = ((b^+)''a^+)'' \\ &= ((b^*)''a^+)'' = (fa^+)'' \\ &= [(f' \wedge a'')a^*]' \wedge (a^*)''.\end{aligned}$$

By the corollary to Lemma 19, it follows that $(ha^*a)'' = h$. A similar argument gives the conclusion $(kbb^*)'' = k$.

LEMMA 21. *If* ef *is *-regular in* S, *then*

$$(b^+(ef)^+a^+)'' = [(f' \wedge e)a^*]' \wedge (a^*)'' \quad \textit{and} \quad [(b^+(ef)^+a^+)^*]'' = [(e' \wedge f)b]' \wedge b''.$$

Proof. Let $g = (b^+(ef)^+)''$. Then by Theorem 12,

$$\begin{aligned}[(b^+)''(ef)^+]'' &= [(b^*)''(ef)^+]'' = (f(ef)^+)'' \\ &= [(f' \wedge (ef)'')fe]' \wedge (fe)''.\end{aligned}$$

But, since $(ef)'' \leq f$, $f' \wedge (ef)'' = 0$, so $g = (fe)''$. Hence, by Theorem 12 again,

$$\begin{aligned}(b^+(ef)^+a^+)'' &= (ga^+)'' = (fea^+)'' = (fa^+)'' \\ &= [(f' \wedge e)a^*]' \wedge (a^*)''.\end{aligned}$$

This establishes the first equation of the lemma. The second equation is established similarly.

THEOREM 22. *If* ef *is *-regular in a Baer *-semigroup* S, *then* $(ab)^+ = b^+a^+$ *if and only if the following four conditions hold:* (i) $((fe)''a^*a)'' = (fe)''$, (ii) $((ef)''bb^*)'' = (ef)''$, (iii) $efe = (fe)''$, *and* (iv) $fef = (ef)''$.

Proof. Suppose first that $(ab)^+ = b^+a^+$. Then by Lemma 20, conditions (i) and (ii) hold. Since conditions (i) and (ii) hold, by the corollary to Lemma 19 and by Lemma 21,

$$(ab)''b^+(ef)^+a^+(b^*a^*)'' = b^+(ef)^+a^+;$$

hence, it follows from Theorem 18 and our hypothesis that $b^+a^+ = b^+(ef)^+a^+$. Premultiplication of both sides of the last equation by b and postmultiplication by a yield the equality $fe = f(ef)^+e$; hence

$$efe = ef(ef)^+e = (fe)''e = (fe)'',$$

establishing (iii). Similarly,

$$fef = f(ef)^+ ef = f(ef)'' = (ef)'' ,$$

establishing (iv).

Conversely, suppose that conditions (i)-(iv) hold. Just as above,

$$(ab)^+ = b^+(ef)^+ a^+ .$$

Put $x = fe$, and note that by condition (iii), $efx = (fe)'' = ((ef)^*)''$; while by condition (iv), $xef = (ef)''$. Since $x((ef)^*)'' = x(fe)'' = x$, Theorem 11 implies that $x = fe = (ef)^+$. It follows that $(ab)^+ = b^+ fea^+ = b^+ a^+$, and our theorem is proved.

THEOREM 23. *Let a Baer *-semigroup* S *satisfy the *-cancellation law. Then* ab *is *-regular in* S *with* $(ab)^+ = b^+ a^+$ *if and only if the following three conditions hold:*

(i) $((e \wedge f)a^* a)'' = e \wedge f$,

(ii) $((e \wedge f)bb^*)'' = e \wedge f$, *and*

(iii) e *commutes with* f.

Proof. Suppose that (i)-(iii) hold. Since e commutes with f

$$ef = fe = (ef)'' = (fe)'' = e \wedge f = (ef)^+ .$$

Hence, all four conditions (i)-(iv) of Theorem 22 are satisfied, and ef is *-regular. It follows that $(ab)^+ = b^+ a^+$.

Conversely, suppose that ab is *-regular in S and $(ab)^+ = b^+ a^+$. Then

$$fe = bb^+ a^+ a = b(ab)^+ a \quad \text{and } efe = eb(ab)^+ a .$$

Hence,

$$(efe)^2 = eb(ab)^+ aeb(ab)^+ a = eb(ab)^+ ab(ab)^+ a = eb(ab)^+ a = efe = (efe)^* .$$

It follows that efe is a projection in S. By Lemma 15, e commutes with f. Thus, condition (iii) is established. Also $ef = e \wedge f = fe$; hence, by Lemma 20, conditions (i) and (ii) must obtain.

LEMMA 24. *Let* $g \in P'$. *Then* $(gb)'' b^+$ *is *-regular in* S, *and*

$$[(gb)'' b^+]^+ = (gf)'' b(gb)'' .$$

Similarly, $a^+(ga^*)''$ *is *-regular in* S, *and*

$$[a^+(ga^*)'']^+ = (ga^*)'' a(ge)'' .$$

Proof. We prove only the first assertion; the second follows analogously. The first assertion follows immediately from Lemma 17 if we replace g by $(gb)''$, b by b^+, and f by b''.

THEOREM 25. *Suppose that* ab *is *-regular in a Baer *-semigroup* S. *Then* af *is *-regular in* S, *and*

$$(af)^+ = (ef)'' b(ab)^+ .$$

also, eb *is *-regular in* S, *and*

$$(eb)^+ = (ab)^+ a(fe)'' .$$

Proof. We prove only the first assertion; the second follows analogously. Note first that

$$(ab)[(eb)'' b^+] = aeb(eb)'' b^+ = aebb^+ = aef = af .$$

By Lemma 24, $(eb)'' b^+$ is *-regular. Also,

$$[((eb)'' b^+)^*]'' = (b^{+*}(eb)'')'' = (b''(eb)'')'' = (eb)'' .$$

In Theorem 18, replace a by (ab), b by $(eb)'' b^+$, e by $(ab)'' = (eb)''$, and f by $(eb)''$. Since $(eb)''(eb)'' = (eb)''$ is *-regular in S, $af = (ab)[(eb)'' b^+]$ is *-regular in S, and

$$(af)^+ = (af)''[(eb)'' b^+]^+(eb)''(ab)^+(fa^*)'' .$$

Invoking Lemma 24 again, we see that

$$(af)^+ = (ef)'' b(eb)''(ab)^+(fa^*)'' .$$

Since $[(ab)^+]'' = (fa^*)''$ and $[(ab)^{+*}]'' = (eb)''$, we obtain the desired result

$$(af)^+ = (ef)'' b(ab)^+ .$$

THEOREM 26. *If* ab *is *-regular in a Baer *-semigroup* S, *then* ef *is *-regular in* S, *and*

$$(ef)^+ = (af)^+ a(fe)'' = (ef)'' b(eb)^+ = (ef)'' b(ab)^+ a(fe)'' .$$

Proof. Suppose that ab is *-regular in S. By Theorem 25, both af and eb are *-regular in S. Since eb is *-regular in S, Theorem 25 (with a replaced by e) implies the *-regularity of ef and the formula $(ef)^+ = (ef)'' b(eb)^+$. Invoking Theorem 25 again, we see that $(eb)^+ = (ab)^+ a(fe)''$; hence,

$$(ef)^+ = (ef)'' b(eb)^+ = (ef)'' b(ab)^+ a(fe)'' .$$

Finally, since $(af)^+ = (ef)'' b(ab)^+$, $(ef)^+ = (af)^+ a(fe)''$.

COROLLARY. *If* a *and* b *are *-regular elements of a Baer *-semigroup* S, *then* ab *is *-regular if and only if* $a''(b^*)''$ *is *-regular.*

Because of the above corollary, the question of *-regularity of the product ab of the *-regular elements a and b is reduced to the question of *-regularity of the product ef of the closed projections e and f. If L is an orthomodular lattice and if e, f $\in$ L, then *-regularity of the product ef in a coordinatizing Baer *-semigroup S for L seems to depend strongly on S; that is, we cannot decide whether ef is *-regular merely on the basis of our knowledge of L, e, and f. Of course, in view of Lemma 8, a necessary condition that ef be *-regular in S is that both ef and fe be range-closed in S. We shall show (in the next lemma and theorem) that ef is range-closed in S if and only if (e', f) is a modular pair in L.

LEMMA 27. *Let* e, f *and* g *be closed projections in a Baer *-semigroup* S, *and put* $h = (gf)''$. *Then the following two conditions are equivalent:*

(i) $[(gfe)'ef]'' = g' \wedge (ef)''$.

(ii) $(h \vee e') \wedge f = h \vee (e' \wedge f)$.

Proof. By part (xii) of Theorem 1, $(gfe)'' = (he)''$; hence, $(gfe)' = (he)'$. Using the latter equation and part (xv) of Theorem 1, we see that (i) is equivalent to the statement

$$[(he)'ef]'' = g' \wedge f \wedge (e \vee f').$$

By part (xi) of Theorem 1, $(he)'e = h' \wedge e$. Thus, using part (xv) of Theorem 1, we find that

$$[(he)'ef]'' = [(h' \wedge e)f]'' = [(h' \wedge e) \vee f'] \wedge f.$$

Consequently,

$$\text{(i)} \Leftrightarrow [(h' \wedge e) \vee f'] \wedge f = (g' \wedge f) \wedge (e \vee f'), \text{ that is,}$$

$$\text{(i)} \Leftrightarrow [(h \vee e') \wedge f] \vee f' = (g \vee f') \vee (e' \wedge f).$$

Taking the meet of both sides of the last equation with f and using the distributive law which is part (iii) of Theorem 2, we conclude that

$$(h \vee e') \wedge f = [(g \vee f') \wedge f] \vee (e' \wedge f).$$

Conversely, taking the join of both sides of the latter equation with f' and using part (iii) of Theorem 2, we obtain the result

$$[(h \vee e') \wedge f] \vee f' = (g \vee f') \vee (e' \wedge f)$$

again. It follows that

$$\text{(i)} \Leftrightarrow (h \vee e') \wedge f = [(g \vee f') \wedge f] \vee (e' \wedge f).$$

By part (xv) of Theorem 1, $(g \vee f') \wedge f = (gf)'' = h$. Consequently

$$\text{(i)} \Leftrightarrow (h \vee e') \wedge f = h \vee (e' \wedge f).$$

THEOREM 28. *Let* e *and* f *be closed projections in a Baer *-semigroup* S. *Then the element* ef *is range-closed in* S *if and only if* (e', f) *is a modular pair in the lattice* $P'(S)$.

Proof. By parts (i) and (ii) of Theorem 6, ef is range-closed in S if and only if condition (i) of Lemma 27 holds for every element $g \in P'$. Hence, by Lemma 27, ef is range-closed if and only if

$$(h \vee e') \wedge f = h \vee (e' \wedge f)$$

for every element h of the form $h = (gf)''$ with $g \in P'$. But the set of all sub-elements of f is the set of all elements

$$h = (gf)'' = (g \vee f') \wedge f$$

corresponding to the elements g in P'. It follows that ef is range-closed if and only if (e', f) is a modular pair.

8. CONCLUSION

In this final section, we set forth a few results which follow easily from the material developed in the previous sections and which are of some interest in connection with matrices, operators on a Hilbert space, and Baer *-rings.

THEOREM 29. *A bounded operator on a Hilbert space* H *is *-regular if and only if its range is a closed linear subspace of* H. *Let* A *and* B *be *-regular operators in* $\mathscr{B}(H)$. *Then the relative inverses* A^+ *and* B^+ *of* A *and* B *coincide with their Moore-Penrose generalized inverses* [7]. *Let* M *be the range of* A, *and let* N *be the orthogonal complement of the null space of* B. *Then* AB *is *-regular in* $\mathscr{B}(H)$ *if and only if* $(M^{\perp}, N)$ *is a modular pair in the lattice of closed linear subspaces of* H. *Finally, a necessary and sufficient condition that* AB *be *-regular in* $\mathscr{B}(H)$ *with* $(AB)^+ = B^+A^+$ *is the fulfillment of the following three conditions:*

(i) $\overline{(M \cap N)A^*A} = M \cap N$,

(ii) $\overline{(M \cap N)BB^*} = M \cap N$, *and*

(iii) $M \cap (M \cap N)^{\perp}$ *is orthogonal to* $N \cap (M \cap N)^{\perp}$.

Proof. The first assertion of the theorem has already been established in the remark following the corollary to Theorem 11. The second one follows directly from the definition of the Moore-Penrose generalized inverse. The third conclusion follows directly from Theorem 28 and the corollary to Theorem 26. The final assertion of the theorem is a direct translation of Theorem 23 in the situation where S is the *-cancellation Baer *-semigroup $\mathscr{B}(H)$. Condition (iii) of Theorem 23 becomes condition (iii) of the present theorem because of Lemma 3.

COROLLARY. *Let* A *and* B *be *-regular (that is, range-closed) operators in* $\mathscr{B}(H)$. *Let* M *be the range of* A, *and let* N *be the orthogonal complement of the null space of* B. *Put* $K = M \cap N$. *Then* AB *is *-regular in* $\mathscr{B}(H)$ *with* $(AB)^+ = B^+A^+$ *if and only if the following conditions hold:* (i) K *reduces* A^*A, (ii) K *reduces* BB^*, *and* (iii) $M \cap K^{\perp}$ *is orthogonal to* $N \cap K^{\perp}$.

Proof. Clearly, it will be sufficient to show that $KA^*A \subset K$ implies $KA^*A = K$. Put $A^*A = C$. In Theorem 30 (part (ii)) we shall prove that if A is *-regular then $C = A^*A$ is *-regular. Also,

$$C'' = (C^*)'' = (A^*A)'' = A'' = \text{the orthogonal projection of } H \text{ onto } M.$$

Let P be the orthogonal projection of H onto K, and note that $P \leq (C^*)''$. In Lemma 17, we replace g by P, b by C, and f by $(C^*)''$; thus, we conclude that PC is range-closed. But the range of PC is KC; hence, KC is closed. Now, suppose that K reduces C, and let C_0 be the restriction of C to the Hilbert space K. Since $K \subset M$ and since M is the orthogonal complement of the null space of C, $\{0\}$ is the null space of C_0. It follows (since C_0 is Hermitian) that the range of C_0 is dense in K. But the range of C_0 is the closed subspace KC of K; hence, $KC = K$. We conclude that the condition $KA^*A = K$ is equivalent to $KA^*A \subset K$, so our proof is complete.

THEOREM 30. *Let* A *be a bounded operator on the Hilbert space* H.

(i) *If* A *is* *-*regular, then* AA^* *is* *-*regular, and* $(AA^*)^+ = A^{*+}A^+$.

(ii) A *is* *-*regular if and only if* AA^* *is* *-*regular.*

(iii) *If* $A = A^*$, *if* $A \geq 0$, *and if* A *is* *-*regular, then* $A^+ \geq 0$.

(iv) *If* $A = A^* \geq 0$, *then that* A *is* *-*regular implies that* $A^{1/2}$ *is* *-*regular with* $(A^{1/2})^+ = (A^+)^{1/2}$.

(v) A *is a partial isometry, (that is,* AA^* *is a projection) if and only if* A *is* *-*regular with* $A^* = A^+$.

(vi) *If* A *is normal and* *-*regular, then* A^+ *is normal.*

Proof. Part (i) follows immediately from Theorem 29. (ii) One of the implications in (ii) follows directly from (i). In order to prove the converse implication, it will be sufficient to prove that if AA^* is range-closed, then A^* is range-closed. Suppose that E is a projection with $E \leq (A^*)''$ and $(EA)'' = A''$. Then,

$$E \leq (AA^*)'' = (A^*)'', \quad \text{and} \quad (EAA^*)'' = ((EA)''A^*)'' = (A''A^*)'' = (A^*)'' = (AA^*)''.$$

Since AA^* is supposed to be range-closed, $E = (AA^*)'' = (A^*)''$, so A^* is range-closed. (iii): Since $A = A^* \geq 0$, there exists a $B = B^*$ with $A = BB$. Since A is *-regular, B is *-regular by part (ii) of the present theorem. By part (i) of the present theorem, $A^+ = B^+B^+$, hence,

$$A^+ = A^{*+} = A^{+*} \geq 0.$$

Part (iv) follows immediately from (i), (ii) and (iii). (v): Suppose $AA^* = E$, a projection. It is well known that this implies $A^*A = F$, a projection. Then

$$E = E'' = (AA^*)'' = (A^*)''.$$

Similarly, $F = A''$, so $A^* = A^+$. The converse is clear. Part (vi) follows immediately from (i).

THEOREM 31. *Let* A *be a bounded operator on a Hilbert space* H, *and let* $A = BU$ *be the polar form for* A *so that* $B = (AA^*)^{1/2}$ *and* U *is a partial isometry with* $U'' = A''$, $(U^*)'' = (A^*)''$. *Then* A *is* *-*regular if and only if* B *is* *-*regular; and if* A *is* *-*regular, then* $A^+ = U^+B^+ = U^*B^+$.

Proof. That A is *-regular if and only if B is *-regular follows from parts (ii) and (iv) of Theorem 30. The remainder of the theorem is a direct consequence of Theorem 29.

LEMMA 32. *If* S *is any Baer* *-*semigroup, if* $a, b \in S$, *if* a *is* *-*regular, and if* a *commutes with both* b *and* b^*, *then* a^+ *commutes with both* b *and* b^*.

Proof. Clearly, $aba' = baa' = b0 = 0$, so $ba' = a'ba'$. Similarly, $b^*a' = a'b^*a'$, so $a'b = a'ba' = ba'$. By part (ix) of Theorem 1, $a''b = ba''$. By a similar argument, $(a^*)''b = b(a^*)''$. Thus,

$$aba^+ = baa^+ = b(a^*)'' = (a^*)''b = aa^+b,$$

so

$$a^+aba^+ = a^+aa^+b = a^+b,$$

that is, $a''ba^+ = a^+b$. Thus, $a^+b = ba''a^+ = ba^+$. A similar argument shows that b^* commutes with a^+.

The following theorem generalizes [6, Lemma 7, p. 526]:

THEOREM 33. *Let* S *be a Baer *-semigroup, let* M *be a non-empty subset of* S *with* $M = M^*$, *and let* a *be *-regular in* S. *Then, if* $a \in Z(M)$, $a^+ \in Z(M)$. *In particular, then, if* S *is *-regular, so is* Z(M).

Proof. The theorem follows directly from Lemma 32.

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